THE POINT SOURCE METHOD IN ACOUSTIC SCATTERING: NUMERICAL RECONSTRUCTION OF THE SCATTERED FIELD FROM FAR FIELD MEASUREMENTS OF INHOMOGENEOUS MEDIA

D. Russell Luke and Roland Potthast [1]

[1] Institute for Numerical and Applied Mathematics, University of Göttingen, Germany
http://www.scienceatlas.de/nfg

ABSTRACT

A fundamental problem in scattering theory is to reconstruct the scattered field on a given region from knowledge of the wave on a surface in the far field. A recent methodology which we call point source methods has been developed and applied to the reconstruction of *impenetrable* obstacles. In this work we present the application of point source methods to the reconstruction of the scattered acoustic field from far field measurements of an unknown *penetrable* inhomogeneous medium illuminated by a single plane wave.

1. INTRODUCTION

The problem of reconstructing a scattered acoustic wave from far field measurements of the total wave is fundamental to inverse scattering applications. The setting we consider is that of an inhomogeneous medium illuminated by a plane wave from a single incident direction. We assume that the medium is deterministic, isotropic and contains isotropic inhomogeneities. We further assume that the support of the inhomogeneities are contained in some compact domain $\mathbb{D}$. No other *a priori* information about the medium is assumed, including the precise location or refractive index of the inhomogeneities. Using a methodology we call point source methods described below, we demonstrate the successful reconstruction of the scattered field from far field measurements.

We assume that the reader is already initiated to the field of inverse scattering and is familiar with common deterministic scattering models. For background material see [1, 2, 3]. The method of point sources was first proposed in [4] and was later developed in [3, 5] where it is applied to inverse obstacle scattering problems. These methods formulate the reconstruction problem for the scattered field as a linear inverse problem which is easily solved computationally. However, the inverse problem is ill-posed, that is small errors in the data cause large errors in the reconstructions. Ill-posedness is addressed with standard filtering techniques often refered to in the mathematical literature as regularization.

This work is organized as follows. In Section 2 we provide a terse formulation of the forward scattering model and the corresponding inverse problem of constructing the scattered field from far field data. The point source methodology and algorithm for solving the inverse problem are presented in Section 3. Results from a computer simulation are detailed in Section 4. We conclude with directions for future research in Section 5.

2. SCATTERING BY INHOMOGENEOUS MEDIA

We restrict our discussion to the scattering of small amplitude, monochromatic ($\omega = const$), time-harmonic acoustic waves in an isotropic inhomogeneous medium. Since the medium is isotropic we can treat each of the spatial components of the wave as independent scalar waves, $u(x)e^{-i\omega t}$. For convenience we omit the time-dependent part of the wave and focus only on the scattering amplitude $u(x) = u^i(x) + u^s(x)$, where $u^i : \mathbb{R}^m \to \mathbb{C}$ denotes the *incident* field, and $u^s : \mathbb{R}^m \to \mathbb{C}$, denotes the *scattered* field for $m = 2$, or $m = 3$. The scattering medium is illuminated by an incident wave u^i which is assumed to be known. The inhomogeneity of the scattering medium is characterized by the refractive index

$$n(x) := \frac{c_0^2}{c^2(x)} + i\sigma(x),$$

where $c_0 \in \mathbb{R}_+ \setminus \{0\}$ denotes the sound speed of a homogeneous background medium, $c : \mathbb{R}^m \to \mathbb{R}_+ \setminus \{0\}$ is the sound speed, and $\sigma : \mathbb{R}^m \to \mathbb{R}_+$ is a function which models the influence of absorption. We assume the scatterer to be bounded and embedded in a background medium with unit index of refraction, i.e., $n(x) = 1$ for x in the open exterior of a bounded domain $\mathbb{D} \subset \mathbb{R}^m$. The boundary of the region $\mathbb{D}$ is denoted $\partial\mathbb{D}$. The closure of $\mathbb{D}$, that is the interior of $\mathbb{D}$ and its boundary, is denoted $\overline{\mathbb{D}}$.

2.1. Forward scattering

We state the *forward* acoustic scattering model that will serve as the basis for the *inverse* scattering problem. To do this we make use of the *volume potential*

$$(V\varphi)(x) := \int_{\mathbb{R}^m} \Phi(x,y)\widetilde{n}(y)\varphi(y)\, dy, \quad x \in \mathbb{R}^m, \quad (2.1)$$

where $\widetilde{n} : \mathbb{R}^m \to (-\infty, 1) + i(-\infty, 0)$ is given by

$$\widetilde{n} := 1 - n,$$

and $\Phi(\cdot, y) : \mathbb{R}^m \setminus \{y\} \to \mathbb{C}$ is the fundamental solution to the Helmholtz equation given explicitly by

$$\Phi(x,y) := \begin{cases} \frac{i}{4} H_0^{(1)}(\kappa|x-y|), & x \neq y, \text{ and } m = 2 \\ \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, & x \neq y, \text{ and } m = 3, \end{cases} \quad (2.2)$$

with $H_0^{(1)}$ denoting the zero-th order Hankel function of the first kind. Note that the function $\widetilde{n}$ has compact support with $\text{supp}\,(\widetilde{n}) = \mathbb{D}$, thus the integral in Eq.(2.1) need only be computed over $\mathbb{D}$ rather than all of $\mathbb{R}^m$.

MODEL 1 *For an incident field u^i with wave number $\kappa = \omega/c_0$ and an inhomogeneous penetrable scatterer contained in $\mathbb{D} \subset \mathbb{R}^m$ with compact support, the total field u satisfies*

$$(I + \kappa^2 V)u = u^i \quad (2.3)$$

where I denotes the identity operator.

The operator $I + \kappa^2 V$ is continuously invertible in $C(\mathbb{D})$, the space of continuous functions on $\mathbb{D}$. Indeed, by Theorem 8.7 of [1], for each $\kappa > 0$ there exists a *unique* field u satisfying Model 1 with u depending continuously on the incident field u^i. From Eq.(2.3) an the standard decomposition of the total field into the incident field and the scattered field we have

$$u^s(x) = -\kappa^2 (Vu)(x), \quad x \in \mathbb{R}^m. \quad (2.4)$$

2.2. Inverse Scattering

The forward model of the previous section allows one to calculate the total field or the scattered field from a given incident illumination u^i and a given index of refraction n. The inverse scattering problem that we study here arises when, rather than index of refraction n, only the wave on a sphere in the far field of the scattering medium is known. Thus we have the following *ill-posed inverse problem* for the scattered field:

INVERSE PROBLEM 2 *Let $\mathbb{S}^R \subset \mathbb{R}^m$ denote the sphere of radius R centered on the origin in the far field (i.e. R is large) of the scattering medium contained in $\mathbb{D} \subset \mathbb{R}^m$. Let $C^l(\Omega)$ denote the space of l-times continuously differentiable functions on some arbitrary domain Ω. Suppose the region $\mathbb{D}$ is illuminated by a known plane wave u^i with incident direction d. Given measurements of u on $\mathbb{S}^R$ due to a known u^i, determine the scattered field $u^s \in C^2(\mathbb{R}^m \setminus \partial\mathbb{D}) \cap C^1(\mathbb{R}^m)$, such that the total field u satisfies Model 1 on $\mathbb{R}^m \setminus \overline{\mathbb{D}}$.*

The source of the ill-posedness is addressed in the next section.

3. THE POINT-SOURCE METHOD

In this section we detail the point source methodology for the reconstruction of the scattered field. The algorithm is given in Alg. 5.

Point source methods for solving the inverse problem of the previous section are based on the following reciprocity relation relating the scattered field u^s on the exterior of the scatterer $\mathbb{R}^m \setminus \overline{\mathbb{D}}$ to the far field distribution of the fundamental solution Φ due to a point source $z \in \mathbb{R}^m \setminus \overline{\mathbb{D}}$. Before stating the reciprocity relation we introduce parameterizations of the fields with respect to directions. Let $\mathbb{S}^1$ denote the unit sphere in $\mathbb{R}^m$. We model the incident field as a unit magnitude plane wave with direction $d \in \mathbb{S}^1$; this is denoted $u^i(\cdot, d)$ where

$$u^i(x,d) = e^{i\kappa x\cdot d}.$$

Let $u^s(\cdot, d)$ and $u(\cdot, d)$ denote the scattered and total fields respectively parameterized by the direction d of the incident field. We write the far field in terms of the unit outward normals to the sphere $\mathbb{S}^R$ in the far field by writing

$$u^\infty(\hat{x}, d) = \lim_{R\to\infty} u^s(x,d)|x|^{(m-1)/2}, \; \hat{x} = x/|x|, \; x \in \mathbb{S}^R.$$

The set of unit normals to $\mathbb{S}^R$ is simply the unit sphere $\mathbb{S}^1$. As a further simplification we apply the approximation (see [1])

$$u^\infty(\hat{x}, d) \approx u^s(x,d)|x|^{(m-1)/2}, \quad \hat{x} = x/|x|, \quad x \in \mathbb{S}^R.$$

In a similar manner, the field on $\mathbb{S}^R$ due to a point source at $z \in \mathbb{R}^m$ is given by

$$\Phi^\infty(\hat{x}, z) := \Phi(x,z)|x|^{(m-1)/2}, \quad \hat{x} = x/|x|, \quad x \in \mathbb{S}^R.$$

.

THEOREM 3 (MIXED RECIPROCITY RELATION) *For acoustic scattering of plane waves $u^i(\cdot, d)$, $d \in \mathbb{S}^1$ and point-sources $\Phi(\cdot, z)$, $z \in \mathbb{R}^m \setminus \mathbb{D}$ from an inhomogeneous medium on the interior of $\mathbb{D}$ we have*

$$\Phi^\infty(\hat{x}, z) = \gamma_m\, u^s(z, -\hat{x}), \quad z \in \mathbb{R}^m \setminus \mathbb{D}, \; x \in \mathbb{S}^R \quad (3.1)$$

where $\hat{x} = x/|x|$ and the constant γ_m is defined by

$$\gamma_m = \begin{cases} \dfrac{e^{i\pi/4}}{\sqrt{8\pi\kappa}}, & m = 2 \\ \dfrac{1}{4\pi}, & m = 3, \end{cases} \tag{3.2}$$

depending on the dimension $m = 2, 3$.

REMARK 4 *Theorem 3.2 complements the standard far field reciprocity relation*

$$u^\infty(\hat{x}, d) = u^\infty(-d, -\hat{x}). \tag{3.3}$$

For details see [3, Theorem 2.2.4].

The idea of point source methods is to construct a mapping between the far field pattern Φ^∞ due to a point source at a point z, and the *measured data*, that is the wave u^∞ in the far field $\mathbb{S}^R$ due to the illumination of the region $\mathbb{D}$ by a plane wave with direction $d \in \mathbb{S}^1$. To do this we define an arbitrary domain of approximation $\mathbb{G}_0 \subset \mathbb{R}^m \setminus \{0\}$ distinguished by it's orientation relative to the origin. Denote the translated domain of approximation by $\mathbb{G}_z$,

$$\mathbb{G}_z := \mathcal{T}_z \mathbb{G}_0$$

where $\mathcal{T}_z$ denotes the translation mapping that maps the origin to the point $z \in \mathbb{R}^m$ (see Figure 1). Define the *region of illumination* $\mathbb{E} \subset \mathbb{R}^m$ by

$$\mathbb{E} := \{ z \in \mathbb{R}^m \mid \mathbb{D} \subset \mathbb{G}_z \}.$$

The goal is to construct $g(\cdot, z) : \mathbb{S}^1 \to \mathbb{C}$ for $z \in \mathbb{E}$ such that

$$\Phi^\infty(\hat{x}, z) \approx \int_{\mathbb{S}^1} u^\infty(\hat{x}, d) g(d, z) ds(d) \tag{3.4}$$

where $\hat{x} \in \mathbb{S}^1$. Using the reciprocity relations Eq.(3.1) and Eq.(3.3) together with Eq.(3.4) we obtain

$$u^s(z, -\hat{x}) \approx \int_{\mathbb{S}^1} u^\infty(d, -\hat{x}) \left[\frac{1}{\gamma_m} g(-d, z) \right] ds(d), \tag{3.5}$$

where $\hat{x} \in \mathbb{S}^1$, $z \in \mathbb{E}$, and γ_m is given by Eq.(3.2).

For $\mathbb{Q} \subset \mathbb{R}^m$ we define the operator

$$\mathcal{A} : L^2(\mathbb{S}^1) \to L^\infty(\mathbb{Q})$$

by

$$(\mathcal{A}w)(z) := \frac{1}{\gamma_m} \int_{\mathbb{S}^1} w(d) g(-d, z) \, ds(d), \tag{3.6}$$

$z \in \mathbb{Q}$. Using this notation we rewrite Eq.(3.5) as

$$u^s(z, -\hat{x}) \approx (\mathcal{A} u^\infty(\cdot, -\hat{x}))(z). \tag{3.7}$$

From this equation it is clear that $\mathcal{A}$ defines a *back projection operator* mapping the far field u^∞ on $\mathbb{S}^R$ to the scattered field u^s on $\mathbb{E}$. For details on above approximations see [3, Chapter 3].

Since the fields on arbitrary domains due to point sources can be calculated explicitly, Eq.(3.7) yields the following algorithm.

ALGORITHM 5 (POINT SOURCE ALGORITHM)

Step 0: *(Initialization) Choose a simply connected* domain of approximation $\mathbb{G}_0 \subset \mathbb{R}^m \setminus \{0\}$ *large enough that the translated domain $\mathbb{G}_z$ contains the unknown scatterer, that is $\mathbb{D} \subset int\mathbb{G}_z$,*

Step 1: *(Determine the* Herglotz density *g) Solve the equation*

$$Hg(\cdot, 0) = \Phi(\cdot, 0) \quad on\ \partial\mathbb{G}_0, \tag{3.8}$$

for the Herglotz density $g(\cdot, 0) : \mathbb{R}^m \to \mathbb{C}$ *where* $H : L^2(S_1) \to L^2(\partial\mathbb{G}_0)$ *is the* Herglotz integral operator*[1]*

$$(Hg(\cdot, 0))(x) = \int_{S_1} e^{i\kappa x \cdot d} g(d, 0) \, ds(d), \quad x \in \partial\mathbb{G}_0 \tag{3.9}$$

and $\Phi(\cdot, 0)$ is the field due to a point source at 0 given by Eq.(2.2). Equation (3.8) is ill-posed and must be regularized. See [1], [6] or [7] for thorough treatments of regularization techniques.

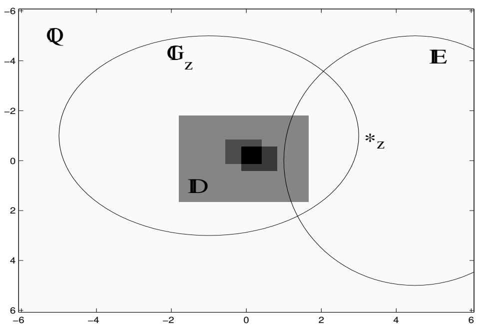

Fig. 1. Inhomogeneity $\widetilde{n}(x)$, domain of approximation $\mathbb{G}_0$, and reconstruction domain $\mathbb{Q}$.

Let $\mathbb{Q} \subset \mathbb{R}^m$ be some area where we would like to reconstruct the unknown scattered field u^s (see Fig.(1)). Using the Fourier Shift Theorem and the symmetry of Φ^∞ we solve Eq.(3.8) for all $z \in \mathbb{Q}$ by translations of $\mathbb{G}_0$:

$$g(\cdot, z) = e^{i\kappa s \cdot d} g(\cdot, 0).$$

Step 2: *(Back projection) Calculate the scattered field at the point $z \in \mathbb{Q}$ due to an incident wave with direction $-\hat{x}$ via Eq.(3.7).*

For the proof of convergence of Alg. 5 see [3, Theorem 5.1.2].

4. RESULTS

We demonstrate the algorithm applied to a two dimensional inhomogeneous medium with index of refraction given by the surface shown in Fig.(1). The theory in the previous sections is on the continuum. In practice, we sample the far field pattern u^∞ at discrete points evenly spaced on the interval $[0, 2\pi)$, that is on all of $\mathbb{S}^R$. Though the scattered field is calculated on the reconstruction domain $\mathbb{Q}$ shown in Fig.(1), the domain over which the reconstruction is reliable is limited to the illumination domain $\mathbb{E}$. To obtain a reliable reconstruction of the scattered field on most of $\mathbb{Q}$ we rotate $\mathbb{G}_0$ around the inhomogeneity. We show the reconstructed total field for a single rotation orientation that coincides with the direction from which the incident field enters the medium. Reconstructions for the rest of the domain are obtained in a similar manner by rotating the domain of approximation $\mathbb{G}_0$.

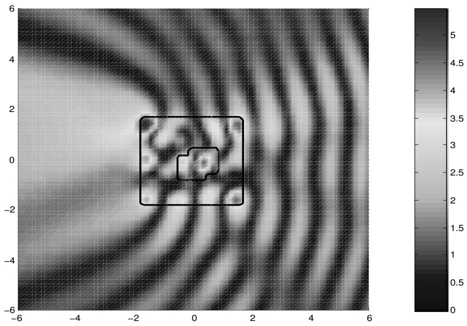

Fig. 2. True total field with inhomogeneity

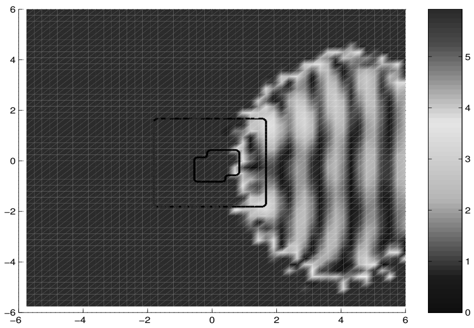

Fig. 3. Reconstructed total field with inhomogeneity

5. CONCLUSION

Reconstruction of the scattered field from far field measurements is a fundamental problem in inverse scattering. Our results indicate that the point source methodology behind the algorithm described in Alg.(5) is a promising technique for inverse scattering problems. We have demonstrated the successful reconstruction of the scattered and total fields from far field measurements of an acoustic field resulting from scattering of a slowly varying, isotropic, inhomogeneous medium illuminated by a plane wave from a single direction. This is the first application of point source methods to the reconstruction of scattering from inhomogeneous media.

This study serves as a qualitative “concept proof”. The reconstruction shown in Fig.(3) took less than 10 seconds for a 50×50 pixelated reconstruction domain, thus the point source methodology is computationally practical. The data required for these reconstructions, however, require measurements of the field u^∞ on the entire sphere $\mathbb{S}^R$. In most realistic settings, such as objects buried in the earth, this information is impossible to obtain. This opens the door to research on limited aperture point source methods in order to extend these techniques to practical applications. Further directions for research include quantitative numerical studies and techniques for reconstructing the index of refraction of the media, for which reconstructing the scattered field is an important first step.

6. REFERENCES

[1] Colton, D. and Kress, R.: *Inverse Acoustic and Electromagnetic Scattering Theory.* 2nd Ed., Springer-Verlag 1998.

[2] D. Colton, J. Coyle, and P. Monk : ”Recent Developments in Inverse Scattering Theory,” *SIAM Rev.* **42**(3), 369-414 (2000).

[3] Potthast, R.: *Point sources and multipoles in inverse scattering theory.* Chapman & Hall, 2001.

[4] Potthast, R.: A fast new method to solve inverse scattering problems. *Inverse Problems* **12**, 731-742 (1996).

[5] Potthast, R.: A point-source method method for inverse acoustic and electromagnetic obstacle scattering problems. *IMA Jour. Appl. Math.* **61**, 119-140 (1998).

[6] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems*, Kluwer Academic Publisher, Dordrecht 1996.

[7] Kreß, R.: *Linear Integral Equations.* 2nd Ed. Springer-Verlag 1999.