Integral Equation Methods in Inverse Obstacle Scattering with a Generalized Impedance Boundary Condition

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Abstract The inverse problem under consideration is to reconstruct the shape of an impenetrable two-dimensional obstacle with a generalized impedance boundary condition from the far field pattern for scattering of time-harmonic acoustic or E-polarized electromagnetic plane waves. We propose an inverse algorithm that extends the approach suggested by Johansson and Sleeman [10] for the case of the inverse problem for a sound-soft or perfectly conducting scatterer. It is based on a system of nonlinear boundary integral equations associated with a single-layer potential approach to solve the forward scattering problem which extends the integral equation method proposed by Cakoni and Kress [5] for a related boundary value problem for the Laplace equation. In addition, we also present an algorithm for reconstructing the impedance function when the shape of the scatterer is known. We present the mathematical foundations of the methods and exhibit their feasibility by numerical examples.

1 Introduction

The use of generalized impedance boundary conditions (GIBC) in the mathematical modeling of wave propagation has gained considerable attention in the literature over the last decades. This type of boundary conditions is applied to scattering problems for penetrable obstacles to model them approximately by scattering problems for impenetrable obstacles in order to reduce the cost of numerical computations. In this paper, we will consider boundary conditions that generalize the classical impedance boundary condition, which is also known as Leontovich boundary condition, by adding a term with a second order differential operator. As compared with

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the Leontovich condition, this wider class of impedance conditions provides more accurate models, for example, for imperfectly conducting obstacles (see [7, 8, 16]).

To formulate the generalized impedance condition and the corresponding scattering problem, let D be a simply connected bounded domain in $\mathbb{R}^2$ with boundary ∂D of Hölder class $C^{4,\alpha}$ and denote by ν the unit normal vector to ∂D oriented towards the complement $\mathbb{R}^2 \setminus \overline{D}$. We consider the scattering problem to find the total wave $u = u^i + u^s \in H^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{D})$ satisfying the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D} \tag{1}$$

with positive wave number k and the generalized impedance boundary condition

$$\frac{\partial u}{\partial \nu} + \mathrm{i}k \left(\lambda u - \frac{\mathrm{d}}{\mathrm{d}s}\, \mu\, \frac{\mathrm{d}u}{\mathrm{d}s} \right) = 0 \quad \text{on } \partial D \tag{2}$$

where $\mathrm{d}/\mathrm{d}s$ is the tangential derivative and $\mu \in C^1(\partial D)$ and $\lambda \in C^1(\partial D)$ are complex valued functions. We note that the classical Leontovich condition is contained in (2) as the special case where $\mu = 0$. The incident wave u^i is assumed to be a plane wave $u^i(x) = e^{\mathrm{i}k\, x\cdot d}$ with a unit vector d describing the direction of propagation, but we also can allow other incident waves such as point sources. The scattered wave u^s has to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}k u^s \right) = 0, \qquad r = |r|, \tag{3}$$

uniformly with respect to all directions. The derivative for $u|_{\partial D} \in H^{\frac{3}{2}}(\partial D)$ with respect to arc length s in (2) has to be understood in the weak sense, that is, u has to satisfy

$$\int_{\partial D} \left(\eta\, \frac{\partial u}{\partial \nu} + \mathrm{i}k\lambda\, \eta\, u + \mathrm{i}k\mu\, \frac{\mathrm{d}\eta}{\mathrm{d}s}\, \frac{\mathrm{d}u}{\mathrm{d}s} \right) \mathrm{d}s = 0 \tag{4}$$

for all $\eta \in H^{\frac{3}{2}}(\partial D)$.

The Sommerfeld radiation condition is equivalent to the asymptotic behavior of an outgoing cylindrical wave of the form

$$u^s(x) = \frac{e^{\mathrm{i}k|x|}}{\sqrt{|x|}} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|} \right) \right\}, \quad |x| \to \infty, \tag{5}$$

uniformly for all directions $\hat{x} = x/|x|$ where the function u_∞ defined on the unit circle $\mathbb{S}^1$ is known as the far field pattern of u^s. Besides the direct scattering problem to determine the scattered wave u^s for a given incident wave u^i the two inverse scattering problems that we will consider are to determine the boundary ∂D, for given impedance functions, or the impedance coefficients μ and λ, for a given boundary, from a knowledge of the far field pattern u_∞ on $\mathbb{S}^1$ for one or several incident plane waves. The first problem we will call the inverse shape problem and the second the inverse impedance problem.

For further interpretation of the generalized impedance boundary condition we refer to [1, 2, 3] where the direct and the inverse scattering problem are analyzed by variational methods. For the solution of a related boundary value problem for the Laplace equation with the generalized impedance boundary condition of the form (2), Cakoni and Kress [5] have proposed a single-layer potential approach that leads to a boundary integral equation or more precisely a boundary integro-differential equation governed by a pseudo-differential operator of order one. In Section 2 we will extend this approach to the direct scattering problem (1)–(3). As to be expected, the single-layer approach fails when k^2 is an interior Dirichlet eigenvalue for the negative Laplacian in D and to remedy this deficiency we describe a modified approach by a combined single- and double-layer approach that leads to a pseudo-differential operator of order two. For simplicity confining ourselves to the single-layer potential approach, we then proceed in Section 3 with describing the numerical solution of the integro-differential equation via trigonometric interpolation quadratures and differentiation that lead to spectral convergence.

Our analysis of the two inverse problems is based on a nonlinear boundary integral equation method in the spirit of Johansson and Sleeman [10] (see also [6, Section 5.4]) and follows the approach for the Laplace equation as developed by Cakoni and Kress [5]. We begin in Section 4 with a review on uniqueness and then proceed in Section 5 with the solution of the inverse shape problem followed by the solution of the inverse impedance problem in Section 6. In both cases we present the theoretical basis for the inverse algorithms and illustrate them by a couple of numerical examples.

2 The Boundary Integral Equation

In this section we describe a boundary integral equation method for solving the direct obstacle scattering problem and begin by establishing uniqueness of the solution. Throughout our analysis we will assume that

$$\operatorname{Re}\lambda \geq 0, \quad \operatorname{Re}\mu \geq 0, \quad |\mu| > 0, \tag{6}$$

where the first two conditions ensure uniqueness and the third condition is required for our existence analysis.

Theorem 1. *Any solution $u \in H^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{D})$ to (1)–(2) satisfying the Sommerfeld radiation condition vanishes identically.*

Proof. Inserting $\eta = \bar{u}|_{\partial D}$ in the weak form (4) of the boundary condition we obtain that

$$\int_{\partial D} \bar{u}\, \frac{\partial u}{\partial \nu}\, \mathrm{d}s = -\mathrm{i}k \int_{\partial D} \left\{ \lambda |u|^2 + \mu \left| \frac{\mathrm{d}u}{\mathrm{d}s} \right|^2 \right\} \mathrm{d}s.$$

Hence in view of our assumption (6) we can conclude that

$$\operatorname{Im} \int_{\partial D} \bar{u}\, \frac{\partial u}{\partial \nu}\, \mathrm{d}s \leq 0$$

and from this and the radiation condition the statement of the theorem follows from Rellich's lemma, see Theorem 2.13 in [6]. □

Corollary 1. *The scattering problem (1)–(3) has at most one solution.*

We recall the fundamental solution of the Helmholtz equation

$$\Phi(x,y) = \frac{\mathrm{i}}{4}\, H_0^{(1)}(k|x-y|), \quad x \neq y,$$

in $\mathbb{R}^2$ in terms of the Hankel function $H_0^{(1)}$ of the first kind of order zero. Further, following [6] we introduce the classical boundary integral operators in scattering theory given by the single- and double-layer operators

$$(S\varphi)(x) := 2\int_{\partial D} \Phi(x,y)\varphi(y)\,\mathrm{d}s(y), \quad x \in \partial D, \tag{7}$$

$$(K\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\,\mathrm{d}s(y), \quad x \in \partial D, \tag{8}$$

and the corresponding normal derivative operators

$$(K'\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\, \varphi(y)\,\mathrm{d}s(y), \quad x \in \partial D, \tag{9}$$

$$(T\varphi)(x) := 2\, \frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\,\mathrm{d}s(y), \quad x \in \partial D. \tag{10}$$

For the subsequent analysis in contemporary Sobolev spaces, we note that for $\partial D \in C^{4,\alpha}$ the operators $S : H^{\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$, $S, K : H^{\frac{3}{2}}(\partial D) \to H^{\frac{5}{2}}(\partial D)$, $T : H^{\frac{3}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ and $K' : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ are all bounded (see [11, 15]).

In a first attempt, extending the approach proposed in [5] for the Laplace equation, we try to find the solution of (1)–(3) in the form of a single-layer potential for the scattered wave

$$u^s(x) = \int_{\partial D} \Phi(x,y)\varphi(y)\,\mathrm{d}s(y), \quad x \in \mathbb{R}^2 \setminus \overline{D}, \tag{11}$$

with density $\varphi \in H^{\frac{1}{2}}(\partial D)$ and note that the regularity $\varphi \in H^{\frac{1}{2}}(\partial D)$ guarantees that $u \in H^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{D})$ (see [15]). From the asymptotics for the Hankel function $H_0^{(1)}(t)$ as $t \to \infty$, it can be deduced that the far field pattern of u^s is given by

$$u_\infty(\hat{x}) = \gamma \int_{\partial D} e^{-\mathrm{i}k\hat{x}\cdot y}\varphi(y)\,\mathrm{d}s(y), \quad \hat{x} \in \mathbb{S}^1, \tag{12}$$

where

$$\gamma = \frac{e^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8\pi k}}\,. \tag{13}$$

Letting x approach the boundary ∂D from inside $\mathbb{R}^2 \setminus \overline{D}$, from the jump relations for single-layer potentials (see [6]) we observe that the boundary condition (2) is satisfied provided φ solves the integro-differential equation

$$\varphi - K'\varphi - \mathrm{i}k\left(\lambda - \frac{\mathrm{d}}{\mathrm{d}s}\,\mu\,\frac{\mathrm{d}}{\mathrm{d}s}\right)S\varphi = g \tag{14}$$

where we set

$$g := 2\,\frac{\partial u^i}{\partial \nu}\bigg|_{\partial D} + 2\mathrm{i}k\left(\lambda - \frac{\mathrm{d}}{\mathrm{d}s}\,\mu\,\frac{\mathrm{d}}{\mathrm{d}s}\right)u^i|_{\partial D} \tag{15}$$

in terms of the incident wave u^i. After defining a bounded linear operator $A : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ by

$$A\varphi := \varphi - K'\varphi - \mathrm{i}k\left(\lambda - \frac{\mathrm{d}}{\mathrm{d}s}\,\mu\,\frac{\mathrm{d}}{\mathrm{d}s}\right)S\varphi \tag{16}$$

we can summarize the above into the following theorem.

Theorem 2. *The single-layer potential (11) solves the scattering problem (1)–(3) provided the density φ satisfies the equation*

$$A\varphi = g. \tag{17}$$

Lemma 1. *The operator $M : H^{\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ given by*

$$M\varphi := \frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} + \int_{\partial D} \varphi\,\mathrm{d}s \tag{18}$$

is bounded and has a bounded inverse.

Proof. We parametrize the boundary ∂D with the arc length s as parameter and identify $H^p(\partial D)$ with $H^p_{\mathrm{per}}[0,L]$ where L is the length of ∂D and $H^p_{\mathrm{per}}[0,L] \subset H^p[0,L]$ is the subspace of L periodic functions (or more precisely bounded linear functionals if $p < 0$) (see [12, Section 8.5]). Using the Fourier series representation of $H^r_{\mathrm{per}}[0,L]$ it can be seen that indeed $M : H^{\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is an isomorphism. □

Lemma 2. *The operator $A - \mathrm{i}k\mu MS : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is compact.*

Proof. The boundedness of the operators $S, K' : H^{\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ and $K' : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ mentioned above implies that all terms in the sum (16) defining the operator A are bounded from $H^{\frac{1}{2}}(\partial D)$ into $H^{\frac{1}{2}}(\partial D)$ except the term

$$\varphi \mapsto \mathrm{i}k\,\frac{\mathrm{d}}{\mathrm{d}s}\,\mu\,\frac{\mathrm{d}}{\mathrm{d}s}\,S\varphi.$$

Therefore, after splitting

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mu\,\frac{\mathrm{d}S\varphi}{\mathrm{d}s} = \frac{\mathrm{d}^2 S\varphi}{\mathrm{d}s^2} + \frac{\mathrm{d}\mu}{\mathrm{d}s}\,\frac{\mathrm{d}S\varphi}{\mathrm{d}s}$$

we observe that the operator $A - \mathrm{i}k\mu MS : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is bounded since both $1/\mu$ and λ/μ are in $C^1(\partial D)$ by our assumptions on μ and λ. Hence the statement of the theorem follows from the compact embedding of $H^{\frac{1}{2}}(\partial D)$ into $H^{-\frac{1}{2}}(\partial D)$. $\square$

Theorem 3. *Provided k^2 is not a Dirichlet eigenvalue for the negative Laplacian in D, for each $g \in H^{-\frac{1}{2}}(\partial D)$ the equation (17) has a unique solution $\varphi \in H^{\frac{1}{2}}(\partial D)$ and this solution depends continuously on g.*

Proof. Since under our assumption on k the operator $S : H^{\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ is an isomorphism, by Lemma 1 and our assumptions on μ the operator $\mathrm{i}k\mu MS : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ also is an isomorphism. Therefore, in view of Lemma 2, by the Riesz theory it suffices to show that the operator A is injective. Assume that $\varphi \in H^{\frac{1}{2}}(\partial D)$ satisfies $A\varphi = 0$. Then, by Theorem 2 the single-layer potential u defined by (11) solves the scattering problem for the incident wave $u^i = 0$. Hence, by the uniqueness Theorem 1 we have $u = 0$ in $\mathbb{R}^2 \setminus \overline{D}$. Taking the boundary trace of u it follows that $S\varphi = 0$ and consequently $\varphi = 0$. $\square$

To remedy the failure of the single-layer potential approach at the interior Dirichlet eigenvalues, as in the case of the classical impedance condition, we modify it into the form of a combined single- and double-layer potential for the scattered wave

$$u^s(x) = \int_{\partial D} \left\{ \Phi(x,y) + \mathrm{i}\,\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} \varphi(y)\,\mathrm{d}s(y), \quad x \in \mathbb{R}^2 \setminus \overline{D}, \tag{19}$$

with density $\varphi \in H^{\frac{3}{2}}(\partial D)$. The regularity $\varphi \in H^{\frac{3}{2}}(\partial D)$ implies $u \in H^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{D})$. Letting x approach the boundary ∂D from inside $\mathbb{R}^2 \setminus \overline{D}$, we observe that the boundary condition (2) is satisfied provided φ solves the integro-differential equation

$$\varphi - K'\varphi - \mathrm{i}T\varphi - \mathrm{i}k\left(\lambda - \frac{\mathrm{d}}{\mathrm{d}s}\,\mu\,\frac{\mathrm{d}}{\mathrm{d}s}\right)(S\varphi + \mathrm{i}\varphi + \mathrm{i}K\varphi) = g \tag{20}$$

with g given by (15). We define a bounded linear operator $B : H^{\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ by

$$B\varphi := \varphi - K'\varphi - \mathrm{i}T\varphi - \mathrm{i}k\left(\lambda - \frac{\mathrm{d}}{\mathrm{d}s}\,\mu\,\frac{\mathrm{d}}{\mathrm{d}s}\right)(S\varphi + \mathrm{i}\varphi + \mathrm{i}K\varphi) \tag{21}$$

and then have the following theorem.

Theorem 4. *The combined single- and double-layer potential (19) solves the scattering problem (1)–(3) provided the density φ satisfies the equation*

$$B\varphi = g. \tag{22}$$

Lemma 3. *The operator $B - k\mu M : H^{\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is compact.*

Proof. The boundedness of $S, K : H^{\frac{3}{2}}(\partial D) \to H^{\frac{5}{2}}(\partial D)$, $K' : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ and $T : H^{\frac{3}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ mentioned above implies that all terms in the sum (21) defining the operator A are bounded from $H^{\frac{3}{2}}(\partial D)$ into $H^{\frac{1}{2}}(\partial D)$ except the term

$$\varphi \mapsto k\, \frac{\mathrm{d}}{\mathrm{d}s}\, \mu\, \frac{d\varphi}{ds}\,.$$

Therefore, as in the proof of Lemma 2 we can deduce that the operator $B - k\mu M : H^{\frac{3}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is bounded and the statement follows from the compact embedding of $H^{\frac{1}{2}}(\partial D)$ into $H^{-\frac{1}{2}}(\partial D)$. $\square$

Theorem 5. *For each $g \in H^{-\frac{1}{2}}(\partial D)$ the integral equation (22) has a unique solution $\varphi \in H^{\frac{3}{2}}(\partial D)$ and this solution depends continuously on g.*

Proof. By our assumption on μ we have that $k\mu M : H^{\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is an isomorphism. Therefore, in view of Theorem 4 and Lemma 3 by the Riesz theory it suffices to show that the operator B is injective. Assume that $\varphi \in H^{\frac{3}{2}}(\partial D)$ satisfies $B\varphi = 0$. Then, by Theorem 4 the combined single- and double-layer potential u defined by (19) solves the scattering problem for the incident wave $u^i = 0$. Hence, by the uniqueness Theorem 1 we have $u = 0$ in $\mathbb{R}^2 \setminus \overline{D}$. Taking the boundary trace of u it follows that $S\varphi + \mathrm{i}\varphi + \mathrm{i}K\varphi = 0$. From this proceeding as in the corresponding existence proof for the scattering problem with Dirichlet boundary condition (see Theorem 3.11 in [6]) we can conclude that $\varphi = 0$. $\square$

Summarizing, we finally have our main result of this section.

Theorem 6. *The direct scattering problem (1)–(3) has a unique solution.*

In addition to the potential approach for setting up the boundary integral equations, of course, following the so-called direct approach one can also derive integral equations based on Green's representation formula. Passing to the boundary ∂D in Huygens' principle (see Theorem 3.14 in [6]) and incorporating the boundary condition (2) we obtain the equation

$$\eta - K\eta - \mathrm{i}kS\left(\lambda - \frac{\mathrm{d}}{\mathrm{d}s}\,\mu\,\frac{\mathrm{d}}{\mathrm{d}s}\right)\eta = 2u^i|_{\partial D} \tag{23}$$

for the boundary trace $\eta := u|_{\partial D}$ of the total field. Obviously, the operator on the left-hand side of (23) is the adjoint of A with respect to the L^2 bilinear form and therefore, by the Fredholm alternative, the equation (23) also is uniquely solvable, provided k^2 is not a Dirichlet eigenvalue for the negative Laplacian in D.

3 Numerical Solution

For the numerical solution, for simplicity we confine ourselves to the equation (14). We employ a collocation method based on numerical quadratures using trigonometric polynomial approximations as the most efficient method for solving boundary integral equations for scattering problems in planar domains with smooth boundaries (see [6]). Here, additionally we need to be concerned with presenting an approximation for the operator $\varphi \mapsto \frac{\mathrm{d}}{\mathrm{d}s}\,\mu\,\frac{d\varphi}{ds}$ as the new feature in the integro-differential equations for the generalized impedance boundary condition. For this, we apply trigonometric differentiation.

Both for the numerical solution and later on for the presentation of our inverse algorithm we assume that the boundary curve ∂D is given by a regular 2π periodic parameterization

$$\partial D = \{z(t) : 0 \le t \le 2\pi\}. \tag{24}$$

Then, via $\psi = \varphi \circ z$ we introduce the parameterized single-layer operator by

$$(\widetilde{S}\psi)(t) := \frac{\mathrm{i}}{2}\int_0^{2\pi} H_0^{(1)}(k|z(t)-z(\tau)|)\,|z'(\tau)|\,\psi(\tau)\,\mathrm{d}\tau$$

and the parameterized normal derivative operator by

$$(\widetilde{K}'\psi)(t) := \frac{\mathrm{i}k}{2}\int_0^{2\pi} \frac{[z'(t)]^\perp \cdot [z(\tau)-z(t)]}{|z'(t)|\,|z(t)-z(\tau)|}\,H_1^{(1)}(k|z(t)-z(\tau)|)\,|z'(\tau)|\,\psi(\tau)\,\mathrm{d}\tau$$

for $t \in [0,2\pi]$. Here we made use of $H_0^{(1)\prime} = -H_1^{(1)}$ with the Hankel function $H_1^{(1)}$ of order zero and of the first kind. Furthermore, we denote $a^\perp := (a_2,-a_1)$ for any vector $a = (a_1,a_2)$, that is, $a^\perp$ is obtained by rotating a clockwise by 90 degrees. Then the parameterized form of (14) is given by

$$\psi - \widetilde{K}'\psi - \mathrm{i}k\lambda\circ z\widetilde{S}\psi + \frac{1}{|z'|}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mu\circ z}{|z'|}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{S}\psi = g\circ z \tag{25}$$

We construct approximations via trigonometric interpolation quadratures and trigonometric differentiation based on equidistant interpolation points $t_j = j\pi/n$ for $j = 1,\dots,2n$ with $n \in \mathbb{N}$. For the operators $\widetilde{S}$ and $\widetilde{K}'$ we make use of approximation $\widetilde{S}_n$ and $\widetilde{K}_n'$ via trigonometric interpolation quadratures that take care of the logarithmic singularities of the Hankel functions as described in Section 3.5 of [6] or in [14]. We refrain from repeating the details.

To approximate the operator $\varphi \mapsto \frac{\mathrm{d}}{\mathrm{d}s}\,\mu\,\frac{\mathrm{d}\varphi}{\mathrm{d}s}$ we simply use numerical differentiation via trigonometric interpolation, i.e., we approximate the derivative ψ' of a 2π periodic function ψ by the derivative $(P_n\psi)'$ of the unique trigonometric polynomial $P_n\psi$ of degree n (without the term $\sin nt$) that interpolates $(P_n\psi)(t_j) = \psi(t_j)$ for $j = 1,\dots,2n$. For the resulting weights we refer to [12, Section 13.5]. We set

$P_n'\psi := (P_n\psi)'$ and approximate

$$\frac{1}{|z'|}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mu\circ z}{|z'|}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{S}\psi \approx \frac{1}{|z'|}P_n'\frac{\mu\circ z}{|z'|}P_n'\widetilde{S}_n\psi.$$

Summarizing, our numerical solution method approximates the integro-differential equation (25) by

$$\psi_n - \widetilde{K}_n'\psi_n - \mathrm{i}k\lambda\circ z\widetilde{S}_n\psi_n + \frac{1}{|z'|}P_n'\frac{\mu\circ z}{|z'|}P_n'\widetilde{S}_n\psi_n = g\circ z \tag{26}$$

which is solved for the trigonometric polynomial ψ_n by collocation at the nodal points t_j for $j = 1,\dots,2n$.

Since the operators

$$\varphi \mapsto \frac{\mathrm{d}^2}{\mathrm{d}s^2}S\varphi \quad \text{and} \quad \varphi \mapsto \frac{\mathrm{d}}{\mathrm{d}s}S\frac{\mathrm{d}\varphi}{\mathrm{d}s}$$

have the same principal part, the error and convergence analysis for numerically solving the hypersingular equation of the first kind with the operator T, defined in (10), via Maue's formula and trigonometric differentiation as carried out in [13] and based on Theorem 13.12 and Corollary 13.13 in [12], can be transferred to the approximation (26) with only minor modifications. In particular, such an analysis would predict spectral convergence in the case of analytic μ,λ and z. However, since our main emphasis is on the inverse scattering problem we refrain from carrying out the details. Instead of this we will conclude with a numerical example exhibiting the spectral convergence. Before doing so we note, that an approximate solution of (20) including an error analysis can be obtained analogously using the approximations for S,K,K' from [6] and the approximation of T via Maue's formula that we just mentioned (see [13]).

For numerical examples we consider scattering by an apple-shaped obstacle with parametric representation

$$z(t) = \frac{0.5+0.4\cos t+0.1\sin 2t}{1+0.7\cos t}\,(\cos t,\sin t), \quad 0\le t\le 2\pi, \tag{27}$$

(see Fig. 1) and by a peanut-shaped obstacle with parametric representation

$$z(t) = \sqrt{\cos^2 t+0.25\sin^2 t}\,(\cos t,\sin t), \quad 0\le t\le 2\pi, \tag{28}$$

(see Fig. 2). As impedance functions we choose

$$\lambda(z(t)) = \frac{1}{1-0.1\sin 2t} \quad \text{and} \quad \mu(z(t)) = \frac{1}{1+0.3\cos t} \tag{29}$$

for $t\in[0,2\pi]$ and note that for both examples we can interpret the impedance functions as given in a neighborhood of ∂D depending only on the polar angle.

After approximately solving the integro-differential equation for the density φ the far field pattern is obtained from (12) by the composite trapezoidal rule. Rather than presenting tables with the far field pattern for plane wave incidence we find it more convenient to just illustrate the spectral convergence by Table 1 which shows the maximum norm (over the collocation points) of the error $E_n := \|u_\infty - u_{\infty,n}\|_\infty$ between the exact and the approximate far field pattern for a point source $u_s = \frac{i}{4} H_0^{(1)}(k|x-x_0|)$ located at some $x_0 \in D$ which has far field pattern $u_\infty(\hat{x}) = \gamma e^{-ik\hat{x}\cdot x_0}$. In the examples we chose $x_0 = (0.1, 0.2)$.

	$2n$	$E_{n,\text{apple}}$	$E_{n,\text{peanut}}$
$k=2$	16	5.02e-04	5.32e-05
	32	3.55e-05	1.33e-07
	64	5.52e-08	8.19e-14
	128	1.16e-13	1.45e-14
$k=8$	16	1.00e-02	1.00e-01
	32	2.43e-05	1.95e-05
	64	1.38e-08	3.71e-14
	128	5.86e-14	8.94e-15

Table 1 Error decay for apple-shaped and peanut-shaped scatterer.

4 Inverse Scattering: Uniqueness

We now turn our attention to the inverse scattering problems. The most general inverse scattering problem is the *inverse shape and impedance problem* to determine ∂D, μ and λ from a knowledge of one (or finitely many) far field patterns u_∞ of solutions u to (1)–(3). In this paper we will be only concerned with two less general cases, namely the *inverse shape problem* and the *inverse impedance problem*. The inverse shape problem consists in determining ∂D from one (or finitely many) far field patterns knowing the impedance coefficients μ and λ. With the roles reversed, the inverse impedance problem requires to determine the impedance functions μ and λ from one (or finitely many) far field patterns for a known shape ∂D.

The first question to ask is what is the minimum amount of data, i.e., the minimal number of far field patterns, to guaranty the uniqueness of the solution for the inverse impedance problem or the inverse shape problem. The following theorem shows that three far field patterns uniquely determine both impedance functions λ and μ provided that ∂D is known.

Theorem 7. *For a given shape ∂D, three far field patterns corresponding to the scattering of three plane waves with different incident directions uniquely determine the impedance functions μ and λ.*

Proof. Plane waves with different directions clearly are linearly independent. Consequently the corresponding total waves u_1, u_2, u_3 are also linearly independent. Therefore, the proof of Theorem in [5] for the case of the Laplace equation can be carried over without any changes to the Helmholtz equation since it only uses the differential equation on the boundary as given by the generalized impedance boundary condition. □

Extending the counter example given in [5] for the Laplace case, the following example illustrates non-uniqueness issues for the inverse impedance problem using two far field patterns. Let D be a disc of radius R centered at the origin, let μ and λ be constants satisfying (6), and consider the two incident waves given in polar coordinates by $u^i(r,\theta) = J_n(kr)\, e^{\pm \mathrm{i} n\theta}$ in terms of the Bessel function J_n of order $n \in \mathbb{N}$. Then the corresponding total wave is given by

$$u(r,\theta) = \left(J_n(kr) - a_n H_n^{(1)}(kr)\right) e^{\pm \mathrm{i} n\theta}$$

with the Hankel function $H_n^{(1)}$ of the first kind of order n and

$$a_n = \frac{kR^2 J_n'(kR) + \mathrm{i}k(n^2\mu + \lambda R^2) J_n(kR)}{kR^2 H_n^{(1)\prime}(kR) + \mathrm{i}k(n^2\mu + \lambda R^2) H_n^{(1)}(kR)}\,. \tag{30}$$

We note that the uniqueness Theorem 1 ensures that the denominator in (30) is different from zero. Clearly, there are infinitely many combinations of positive real numbers μ and λ giving the same value for a_n, that is, the same two linearly independent total fields.

The following uniqueness result for the full inverse shape and impedance problem was obtained by Bourgeois, Chaulet and Haddar [2].

Theorem 8. *Both the shape and the impedance functions of a scattering obstacle with generalized impedance condition are uniquely determined by the far field patterns for an infinite number of incident waves with different incident directions and one fixed wave number.*

The main idea of the proof in [6, Theorem 5.5] for the case $\mu = 0$ remains valid. We only need to convince ourselves that the mixed reciprocity relation for scattering of point sources and plane waves, see [6, Theorem 3.16] extends from the case $\mu = 0$ to the general case as consequence of the weak form (4) of the generalized impedance condition.

We conclude this short section on uniqueness for the inverse problem with outlining the proof for the identifiability of a disc and its constant impedance coefficients from the far field pattern for one incident plane wave.

Theorem 9. *A disc with constant impedance coefficients is uniquely determined by the far field pattern for one incident plane wave.*

Proof. Using polar coordinates the Jacobi–Anger expansion (see [6]) reads

$$e^{\mathrm{i}k\,x\cdot d} = \sum_{n=-\infty}^{\infty} \mathrm{i}^n J_n(kr)\, e^{\mathrm{i}n\theta}, \quad x \in \mathbb{R}^2, \tag{31}$$

where θ is the angle between x and d. From this it can be seen that the scattered wave u^s for scattering from a disc of radius R centered at the origin has the form

$$u^s(x) = \sum_{n=-\infty}^{\infty} a_n\, \mathrm{i}^n H_n^{(1)}(kr)\, e^{\mathrm{i}n\theta}, \quad r > R, \tag{32}$$

with the coefficients a_n from (30). Using the asymptotics of the Bessel and Hankel functions for large n (see [6]) uniform convergence can be established for the series (32) in compact subsets of $\mathbb{R}^2 \setminus 0$. In particular, this implies that the scattered wave u^s has an extension as solution to the Helmholtz equation across the boundary into the interior of the disc with the exception of the center.

Now assume that two discs D_1 and D_2 with centers z_1 and z_2 have the same far field pattern $u_{\infty,1} = u_{\infty,2}$ for scattering of one incident plane wave. Then by Rellich's lemma (see [6]) the scattered waves coincide $u_1^s = u_2^s$ in $\mathbb{R}^2 \setminus (D_1 \cup D_2)$ and we can identify $u^s = u_1^s = u_2^s$ in $\mathbb{R}^2 \setminus (D_1 \cup D_2)$. Now assume that $z_1 \neq z_2$. Then u_1^s has an extension into $\mathbb{R}^2 \setminus \{z_1\}$ and u_2^s an extension into $\mathbb{R}^2 \setminus \{z_2\}$. Therefore, u^s can be extended from $\mathbb{R}^2 \setminus (D_1 \cup D_2)$ into all of $\mathbb{R}^2$, that is, u^s is an entire solution to the Helmholtz equation. Consequently, since u^s also satisfies the radiation condition it must vanish identically $u^s = 0$ in all of $\mathbb{R}^2$. Therefore the incident field $u^i(x) = e^{\mathrm{i}k\,x\cdot d}$ must satisfy the generalized impedance condition on D_1 with radius R_1. Elementary differentiations show that this implies

$$R_1^2 \cos\theta + R_1^2 \lambda + k^2 \mu \sin^2\theta + \mathrm{i}k\mu\cos\theta = 0$$

for all $\theta \in [0, 2\pi]$. However this is a contradiction and therefore $z_1 = z_2$.

In order to show that D_1 and D_2 have the same radius and the same impedance coefficients, we observe that by symmetry, or by inspection of the explicit solution given above, the far field pattern for scattering of plane waves from a disc with constant impedance coefficients depends only on the angle between the observation direction and the incident direction. Hence, knowledge of the far field pattern for one incident direction implies knowledge of the far field pattern for all incident directions. Now the statement follows from the above Theorem 8. □

5 Solution of the Inverse Shape Problem

We now proceed describing an iterative algorithm for approximately solving the inverse shape problem by extending the method proposed by Johansson and Sleeman [10] for sound-soft or perfectly conducting obstacles. After introducing the far field operator

$$S_\infty : H^{\frac{1}{2}}(\partial D) \to L^2(\mathbb{S}^1)$$

by

$$(S_\infty \varphi)(\hat{x}) := \gamma \int_{\partial D} e^{-\mathrm{i}k\hat{x}\cdot y} \varphi(y)\,\mathrm{d}s(y), \quad \hat{x} \in \mathbb{S}^1, \tag{33}$$

from (11) and (12) we observe that the far field pattern for the solution to the scattering problem (1)–(3) is given by

$$u_\infty = S_\infty \varphi \tag{34}$$

in terms of the solution to (14). We note that S_∞ is compact and state the following theorem as theoretical basis of our inverse algorithm. For this we note that the operators and the right-hand side g depend on the boundary curve ∂D.

Theorem 10. *For a given incident field u^i and a given far field pattern u_∞, assume that ∂D and the density φ satisfy the system*

$$\varphi - K'\varphi - \mathrm{i}k\left(\lambda - \frac{\mathrm{d}}{\mathrm{d}s}\,\mu\,\frac{\mathrm{d}}{\mathrm{d}s}\right) S\varphi = g \tag{35}$$

and

$$S_\infty \varphi = u_\infty \tag{36}$$

where g is given in terms of the incident field by (15). Then ∂D solves the inverse shape problem.

The ill-posedness of the inverse shape problem is reflected through the ill-posedness of the second equation (36), the far field equation that we denote as *data equation*). Note that the system (35)–(36) is linear with respect to the density φ and nonlinear with respect to the boundary ∂D. This opens up a variety of approaches to solve (35)–(36) by linearization and iteration. In this paper, we are going to proceed as follows. Given an approximation for the unknown ∂D we solve the equation (35) that we denote as *field equation* for the unknown density φ. Then, keeping φ fixed we linearize the data equation (36) with respect to the boundary ∂D to update the approximation.

To describe this in more detail, we also need the parameterized version

$$\widetilde{S}_\infty : H^{\frac{1}{2}}[0,2\pi] \to L^2(\mathbb{S}^1)$$

of the far field operator given by

$$(\widetilde{S}_\infty \psi)(\hat{x}) := \gamma \int_0^{2\pi} e^{-\mathrm{i}k\hat{x}\cdot z(\tau)}\,|z'(\tau)|\,\psi(\tau)\,\mathrm{d}\tau, \quad \hat{x} \in \mathbb{S}^1. \tag{37}$$

Then the parameterized form of (35)–(36) is given by

$$\psi - \widetilde{K}'\psi - \mathrm{i}k\lambda\circ z\widetilde{S}\psi + \frac{1}{|z'|}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\mu\circ z}{|z'|}\,\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{S}\psi = g\circ z \tag{38}$$

and

$$\widetilde{S}_\infty(\psi, z) = u_\infty \tag{39}$$

where $\psi = \varphi \circ z$.

The Fréchet derivative $\widetilde{S}'_\infty$ of the operator $\widetilde{S}_\infty$ with respect to the boundary curve z in the direction ζ is given by

$$\widetilde{S}'_\infty(\psi;\zeta)(\hat{x}) := \gamma \int_0^{2\pi} e^{-\mathrm{i}k\hat{x}\cdot z(\tau)} \left[-\mathrm{i}k\hat{x}\cdot\zeta(\tau)\,|z'(\tau)| + \frac{z'(\tau)\cdot\zeta'(\tau)}{|z'(\tau)|} \right] \psi(\tau)\,\mathrm{d}\tau$$

for $\hat{x} \in \mathbb{S}^1$. Then the linearization of (39) at z with respect to the direction ζ becomes

$$\widetilde{S}_\infty \psi + \widetilde{S}'_\infty(\psi;\zeta) = u_\infty \tag{40}$$

and is a linear equation for the update ζ.

Now, given an approximation for the boundary curve ∂D with parameterization z, each iteration step of the proposed inverse algorithm consists of two parts.

1. We solve the well-posed field equation (38) for ψ. This can be done through the numerical method described in Section 3.
2. Then we solve the ill-posed linearized equation (40) for ζ and obtain an updated approximation for ∂D with the parameterization $z + \zeta$. Since the kernels of the integral operators in (40) are smooth, for its numerical approximation the composite trapezoidal rule can be employed. Because of the ill-posedness the solution of (40) requires stabilization, for example, by Tikhonov regularization.

These two steps are now iterated until some stopping criterion is satisfied.

In principle, the parameterization of the update is not unique. To cope with this ambiguity, one possibility that we pursued in our numerical examples is to allow only parameterizations of the form

$$z(t) = r(t)(\cos t, \sin t), \quad 0 \le t \le 2\pi, \tag{41}$$

with a non-negative function r representing the radial distance of ∂D from the origin. Consequently, the perturbations are of the form

$$\zeta(t) = q(t)(\cos t, \sin t), \quad 0 \le t \le 2\pi, \tag{42}$$

with a real function q. In the approximations we assume r and its update q to have the form of a trigonometric polynomial of degree J, in particular,

$$q(t) = \sum_{j=0}^{J} a_j \cos jt + \sum_{j=1}^{J} b_j \sin jt. \tag{43}$$

Then the update equation (40) is solved in the least squares sense, penalized via Tikhonov regularization, for the unknown coefficients $a_0, \ldots, a_J$ and $b_1, \ldots, b_J$ of the trigonometric polynomial representing the update q. As experienced in the application of the above approach for related problems, it is advantageous to use an

H^p Sobolev penalty term rather than an L^2 penalty in the Tikhonov regularization, i.e, to interpret $\widetilde{S}'_\infty$ as an ill-posed linear operator $\widetilde{S}'_\infty : H^p[0,2\pi] \to L^2(\mathbb{S}^1)$ for some small $p \in \mathbb{N}$.

As a theoretical basis for the application of Tikhonov regularization from [9] we cite that, after the restriction to star-like boundaries of the form (42, the operator $\widetilde{S}'_\infty$ is injective if k_0^2 is not a Neumann eigenvalue for the negative Laplacian in D.

The above algorithm has a straightforward extension for the case of more than one incident wave. Assume that $u_1^i, \dots, u_M^i$ are M incident waves with different incident directions and $u_{\infty,1}, \dots, u_{\infty,M}$ the corresponding far field patterns for scattering from ∂D. Given an approximation z for the boundary we first solve the field equations (38) for the M different incident fields to obtain M densities $\psi_1, \dots, \psi_M$. Then we solve the linearized equations

$$\widetilde{S}_\infty \psi_m + \widetilde{S}'_\infty(\psi_m; \zeta) = u_{\infty,m}, \quad m = 1, \dots, M, \tag{44}$$

for the update ζ by interpreting them as one ill-posed equation with an operator from $H^p[0,2\pi]$ into $(L^2(\mathbb{S}^1))^M$ and applying Tikhonov regularization.

The numerical examples are intended as proof of concept and not as indications of an already fully developed method. In particular, the regularization parameters and the number of iterations are chosen by trial and error instead of, for example, a discrepancy principle. In all examples, to avoid committing an inverse crime the synthetic far field data are obtained by solving the integral equation for the combined single- and double-layer approach whereas the inverse solver is based on the single-layer approach.

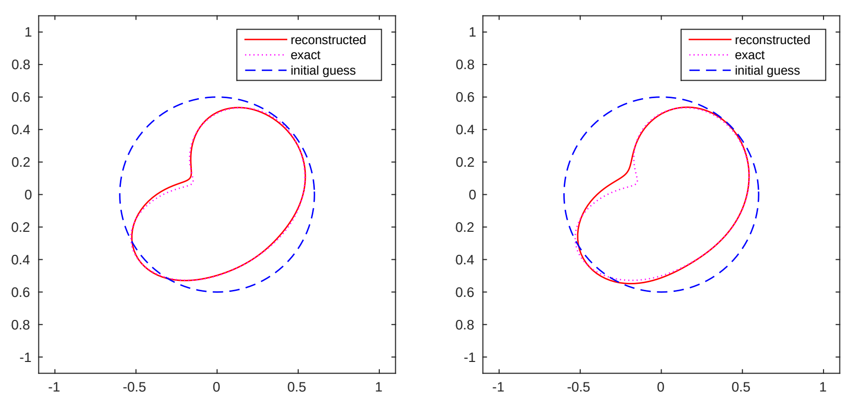

Fig. 1 Reconstruction of the apple (27) for exact data after 30 iterations (left) and for 5% noise after 10 iterations (right).

For both examples the impedance functions are given by (29). The number of quadrature points is $2n = 64$ both on the boundary curve and on the circle for the far field pattern. The wave number is $k = 2$. The degree of the polynomials (43) is chosen as $J = 4$ and the regularization parameter for an H^2 regularization of the linearized data equation (40) is $\alpha = 0.05 \times 0.9^m$ for the m-th iteration step. For the perturbed data, random noise is added point wise with relative error in the L^2 norm. The iterations are started with an initial guess given by a circle of radius 0.6 centered at the origin. In both examples we used two incident waves, for the apple shape the incident directions are $d = (\pm 1, 0)$ and for the peanut shape $d = (0, \pm 1)$. In the figures the exact ∂D is given as dotted (magenta), the reconstruction as full (red) and the initial guess as dashed (blue) curve.

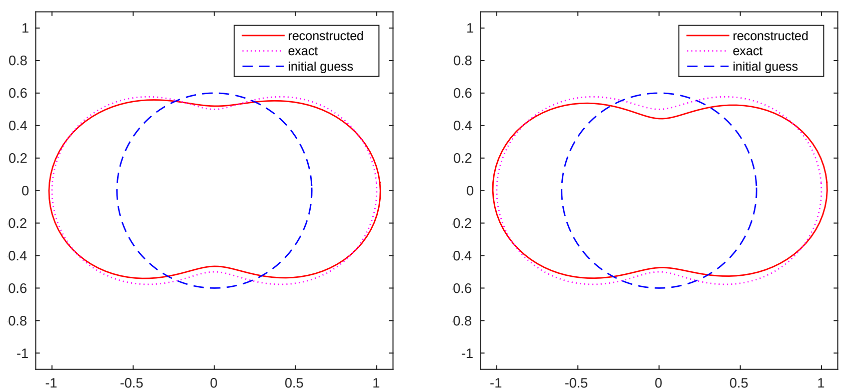

Fig. 2 Reconstruction of the peanut (28) for exact data after 30 iterations (left) and for 5% noise after 10 iterations (right).

6 Solution of the Inverse Impedance Problem

Turning to the solution of the inverse impedance problem, we note that we can understand the data equation (36) as its main basis. Knowing the boundary ∂D, assuming again that k^2 is not a Dirichlet eigenvalue for the negative Laplacian in D we can represent u^s from u_∞ as a single-layer potential with density φ on ∂D. In order to attain the given far field pattern the density has to satisfy

$$S_\infty \varphi = u_\infty. \tag{45}$$

Once the density φ is known, the values of u and $\partial_\nu u$, i.e., the Cauchy data of u on the boundary can be obtained through the jump relations

$$u|_{\partial D} = u^i|_{\partial D} + \frac{1}{2} S\varphi \tag{46}$$

and
$$\frac{\partial u}{\partial \nu}\bigg|_{\partial D} = \frac{\partial u^i}{\partial \nu}\bigg|_{\partial D} + \frac{1}{2}\,K'\varphi - \frac{1}{2}\,\varphi. \tag{47}$$

For the numerical solution of (45) and the evaluation of (46) and (47) the approximations of the integral operators described in Section 3 are available. The derivative of $u|_{\partial D}$ with respect to s can be obtained by trigonometric differentiation. Knowing the Cauchy data on ∂D we now can recover the impedance functions μ and λ from the boundary condition (2).

The uniqueness result of Theorem 7 suggests that we need three incident plane waves with different directions leading to three far field patterns $u_{\infty,1}, u_{\infty,2}, u_{\infty,3}$ to reconstruct λ and μ. Solving the corresponding data equations (45) by Tikhonov regularization and using (46) and (47), we obtain three Cauchy pairs $u_1, \partial_\nu u_1$ and $u_2, \partial_\nu u_2$ and $u_3, \partial_\nu u_3$ for which we can exploit the boundary condition to construct μ and λ. For this we proceed somewhat differently than in [5] and mimic the idea of the proof of Theorem 7.

Multiplying the impedance condition (2) for u_1 by u_2 and the impedance condition for u_2 by u_1 and subtract we obtain

$$\mathrm{i}k\,\frac{\mathrm{d}}{\mathrm{d}s}\,\mu\left(u_1\,\frac{\mathrm{d}u_2}{\mathrm{d}s} - u_2\,\frac{\mathrm{d}u_1}{\mathrm{d}s}\right) = u_1\,\frac{\partial u_2}{\partial \nu} - u_2\,\frac{\partial u_1}{\partial \nu} \quad \text{on } \partial D.$$

From this it follows that

$$\mathrm{i}k\mu\left(u_1\,\frac{\mathrm{d}u_2}{\mathrm{d}s} - u_2\,\frac{\mathrm{d}u_1}{\mathrm{d}s}\right) = \mathscr{I}\left\{u_1\,\frac{\partial u_2}{\partial \nu} - u_2\,\frac{\partial u_1}{\partial \nu}\right\} + C_{12} \quad \text{on } \partial D \tag{48}$$

where C_{12} is a complex constant and $\mathscr{I}$ denotes integration over ∂D from a fixed $x_0 \in \partial D$ to $x \in \partial D$. Proceeding the same way with the two other possible combinations of u_2 and u_3 and of u_3 and u_1 we obtain two analogous equations with two more constants C_{23} and C_{31}. We approximate the unknown (parameterized) impedance function μ by a trigonometric polynomial of degree J and collocate the parametrized three equations of the form (48) at the $2n$ collocation points $t_j = j\pi/n$, $j = 1,\dots,2n$. The resulting linear system of $6n$ equations for the $(2J+1)$ Fourier coefficients of μ_{approx} and the three integration constants C_{12}, C_{23}, C_{31} then is solved in the least squares sense.

Having reconstructed μ, the remaining coefficient λ can be obtained from the impedance condition for any of the three functions u_1, u_2, or u_3. For symmetry, approximating the unknown function λ also by a trigonometric polynomial of degree J we collocate the boundary condition (2) for all three solutions u_1, u_2, and u_3 and solve the resulting linear system of $6n$ equations for the $(2J+1)$ Fourier coefficients of λ_{approx} in the least squares sense.

For both our numerical examples the impedance functions are given by (29). The wave number is $k = 1$ and the three incident directions are $d = (1,0)$ and $d = (\cos 2\pi/3, \pm\sin 2\pi/3)$. As in the examples of Section 4 the number of quadrature points is $2n = 64$ on each curve. The integration $\mathscr{I}$ is approximated by trigonometric interpolation quadrature. The degree of the polynomials for the approximation of the impedance functions is chosen as $J = 2$. We approximate the density φ via H^2 Tikhonov regularization of (45) by a trigonometric polynomial of degree $J_\varphi = 12$. The regularization parameter α is chosen by trial and error as $\alpha_{\text{exact}} = 10^{-10}$ and $\alpha_{\text{noise}} = 10^{-5}$.

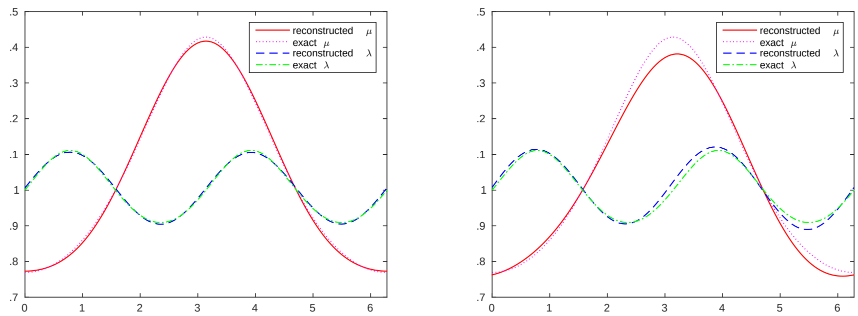

Fig. 3 Reconstruction of the impedance functions for the ellipse (49) for exact data (left) and 1% noise (right).

Fig. 3 and Fig. 4 show the reconstruction of the impedances for an ellipse with parametrization

$$z(t) = (\cos t, 0.7 \sin t), \quad 0 \leq t \leq 2\pi, \tag{49}$$

and for the peanut (28). The exact μ is given as dotted (magenta) curve and the reconstruction as full (red) curve, the exact λ is dashed-dotted (green) and the reconstruction dashed (blue). In general, the examples and our further numerical experiments indicate that the simultaneous reconstruction of both impedance functions is very sensitive to noise.

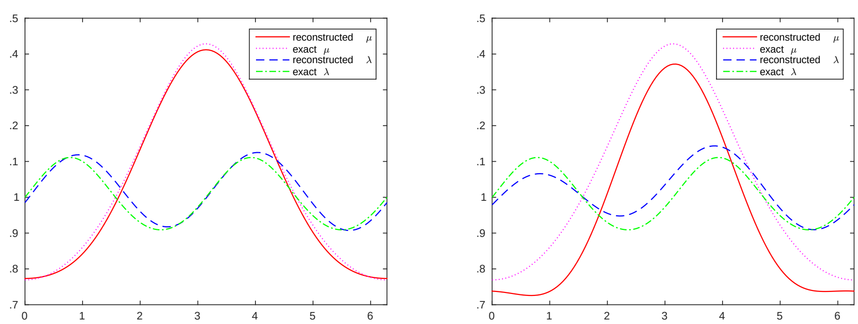

Fig. 4 Reconstruction of the impedance functions for the peanut (28) for exact data (left) and 1% noise (right).

In conclusion, we note that we have presented a method for the reconstruction of the shape (for known impedance functions) and a method for the reconstruction of the impedance functions (for known shape). Further research is required for the solution of the full inverse problem by simultaneous linearization of the system (35) and (36) with respect to both the shape and the impedance analogous to [4].

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