The DB Boundary Condition in Electromagnetic Scattering Revisited

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Abstract

For the solution of the scattering problem for time-harmonic electromagnetic waves with boundary conditions for the normal components of both the electric and the magnetic field an integral equation method proposed by Gülzow in 1988 is reconsidered. It is made more concise, made symmetric with respect to the electric and magnetic field, and also extended from the classical Hölder spaces to a contemporary Sobolev space setting. For this regime, new reciprocity principles for scattering of plane waves and dipole fields are established and a related far field operator is discussed. Finally, the corresponding inverse scattering problem to recover the shape of the scatterer from far field data is discussed with the main emphasis on uniqueness results.

1 Introduction

The propagation of time-harmonic electromagnetic waves in a homogeneous isotropic medium in $\mathbb{R}^3$ is described by the reduced Maxwell equations

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0. \tag{1.1}$$

for the electric field E and the magnetic field H and a positive wave number k. For the scattering of a given electromagnetic wave E^i, H^i by a scatterer described by a bounded domain $D \subset \mathbb{R}^3$ with boundary ∂D and exterior unit normal vector ν in addition to the classical boundary conditions such

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as the perfect conductor and the impedance conditions, in 1956 Rumsey [22] suggested a boundary condition of the form

$$\nu \cdot E = \nu \cdot H = 0 \quad \text{on } \partial D \tag{1.2}$$

for the total wave $E = E^i + E^s$, $H = H^i + H^s$ with the scattered wave E^s, H^s required to satisfy the Silver–Müller radiation condition

$$\lim_{r\to\infty} [H^s(x) \times x - rE^s(x)] = 0 \tag{1.3}$$

where $r = |x|$ and where the limit is assumed to hold uniformly in all directions $x/|x|$. Uniqueness of a solution for a simply connected scatterer D was settled in 1970 by Yee [24]. Existence of a solution by a boundary integral equation method was established in 1986 by the author [12] based on well-posed equations of the second kind and by Gülzow [8], one of the author's PhD students, in his thesis by a well-posed hypersingular integral equation of the first kind. The analysis in [8, 12] also covers the case of multiply connected scatterers.

The work in [8, 12] has been cited a number of times within the last decade both in the engineering and the mathematics literature. For brevity the term DB boundary conditions for (1.2) was introduced by Lindell and Sihvola [14, 15, 16, 17] who also investigated its relations to metamaterials. The DB boundary conditions also occurred in the context of electromagnetic cloaking (see [23, 25]). Weighted more on the analysis side, in 2010 Epstein and Greengard [4] presented an alternative well-posed integral equation of the second kind to the DB boundary conditions in the frame work of Debye sources. In 2011, Markkanen, Ylä-Oijala, and Sihvola [19] provided a further integral equation approach based on the Stratton–Chu representation formulas for the Maxwell equations.

These recent investigations motivated the author, after almost 35 years, to pick up on this topic again and, in particular, extend some of the analysis related to inverse scattering problems with the perfect conductor boundary condition to the case of the DB boundary conditions.

The plan of the paper is as follows. In Section 2 we begin with a careful description of Gülzows boundary integral equation approach in a classical Hölder space setting and include some improvements and additions. Then we proceed in Section 3 with the extension of the analysis into a Sobolev space setting with the appropriate energy spaces for the Maxwell equations. This then is followed in Section 4 by the investigation of the scattering of plane electromagnetic waves and of electromagnetic dipole fields. The main result of this section will be two reciprocity principles. One of them will be used for the analysis of the far field operator which is defined as an integral operator on the L^2 space of tangential fields on the unit sphere with the kernel given by the electric far field pattern of the scattered wave with the incident direction and the observation direction as its two variables. The other reciprocity principle plays a role in Section 5 which has as its topic uniqueness results for the inverse problem to determine the shape of the scatterer from far field patterns for plane wave scattering. Reflecting the previous analysis, in the final Section 6 proposals are made for two types of solution methods based either on the integral equations of Sections 2 and 3 or on the far field operator as introduced in Section 4.

2 The exterior DB boundary value problem

We begin with a review of uniqueness and existence of the solution to the electromagnetic scattering problem with DB boundary conditions in the classical Hölder space setting following the approach by Gülzow and simultaneously making it more concise. After renaming the unknowns, this scattering problem is a special case of the following exterior boundary value problem. We assume that $D \subset \mathbb{R}^3$ is a bounded domain with a connected $C^{2,\alpha}$, $0 < \alpha < 1$, boundary ∂D and exterior unit normal vector ν and confine ourselves to the case where D is simply connected. Then the exterior DB boundary value problem is finding a solution E, H of the Maxwell equations (1.1) belonging to $C^{0,\alpha}(\mathbb{R}^3 \setminus D) \cap C^1(\mathbb{R}^3 \setminus \bar{D})$ satisfying the Silver–Müller radiation condition (1.3) and the boundary conditions

$$\nu \cdot E = f \quad \text{and} \quad \nu \cdot H = g \quad \text{on } \partial D \tag{2.1}$$

for given functions f, g in $C^{0,\alpha}(\partial D)$. Clearly, by Stokes' integral theorem f, g have to fulfill

$$\int_{\partial D} f\, ds = \int_{\partial D} g\, ds = 0 \tag{2.2}$$

as necessary solvability conditions.

The following consequence of the homogeneous DB boundary conditions will play a central role in the following analysis including also the uniqueness of a solution. Therefore we promote it to the rank of a lemma.

Lemma 2.1 *Assume that E, H is a solution to the Maxwell equations in $C^{0,\alpha}(\bar{D}) \cap C^1(D)$ or in $C^{0,\alpha}(\mathbb{R}^3 \setminus D) \cap C^1(\mathbb{R}^3 \setminus \bar{D})$ satisfying $\nu \cdot E = \nu \cdot H = 0$ on ∂D. Then there exist $\varphi, \psi \in C^{1,\alpha}(\partial D)$ such that*

$$\nu \times E = -\nu \times \operatorname{Grad} \varphi \quad \text{and} \quad \nu \times H = \nu \times \operatorname{Grad} \psi \tag{2.3}$$

and we have that

$$\int_{\partial D} \nu \cdot (E \times H)\, ds = \int_{\partial D} \nu \cdot (E \times \bar{H})\, ds = 0. \tag{2.4}$$

Proof. We begin by noting that for the surface divergence Div of tangential fields on ∂D by the continuity of $\operatorname{curl} E = ikH$ up to the boundary we may use the vector identity

$$\operatorname{Div}(\nu \times E) = -\nu \cdot \operatorname{curl} E \tag{2.5}$$

(see [3, Section 6.3]). On the simply connected surface ∂D, by the Hodge decomposition (see [9]) we can express

$$\nu \times E = \operatorname{Grad} \chi - \nu \times \operatorname{Grad} \varphi \quad \text{on } \partial D$$

with two scalar functions $\varphi, \chi \in C^{1,\alpha}(\partial D)$ and the surface gradient Grad. From this, noting that

$$\operatorname{Div}(\nu \times \operatorname{Grad} \varphi) = 0 \tag{2.6}$$

and using (2.5), the first Maxwell equation and the homogeneous boundary condition for H we obtain that

$$\operatorname{Div} \operatorname{Grad} \chi = \operatorname{Div}(\nu \times E) = -ik\, \nu \cdot H = 0 \quad \text{on } \partial D.$$

By the the Gauß surface divergence theorem we have that

$$\int_{\partial D} \left\{ \bar{\chi} \operatorname{Div} \operatorname{Grad} \chi + |\operatorname{Grad} \chi|^2 \right\} ds = \int_{\partial D} \operatorname{Div}(\bar{\chi} \operatorname{Grad} \chi)\, ds = 0,$$

and from this we obtain that $\chi = \text{const}$. Thus we have $\nu \times E = -\nu \times \operatorname{Grad} \varphi$ and analogously $\nu \times H = \nu \times \operatorname{Grad} \psi$ with $\psi \in C^{1,\alpha}(\partial D)$. As consequence of (2.6) we have

$$\operatorname{Div}[\psi(\nu \times \operatorname{Grad} \varphi)] = \operatorname{Grad} \psi \cdot (\nu \times \operatorname{Grad} \varphi) = \nu \cdot (\operatorname{Grad} \varphi \times \operatorname{Grad} \psi)$$

and the integrals in (2.4) vanish by the Gauß surface divergence theorem. □

We note that instead of appealing to the Hodge decomposition we could validate the representations in (2.3) also with the help of Stokes' integral theorem as done in [12] (which, of course, is connected to the Hodge theory).

Theorem 2.2 *The exterior DB boundary value problem has at most one solution.*

Proof. Assume that E, H solve the homogeneous exterior DB boundary value problem. Then by (2.4) we have $\int_{\partial D} \nu \cdot (E \times \bar{H})\, ds = 0$ and this implies $E = H = 0$ by Theorem 6.11 in [3] as a consequence of the radiation condition (1.3). $\square$

At this point we mention that in the case of a multiply connected domain D with topological genus p in order to ensure uniqueness circulations for E and H have to be prescribed with respect to p closed curves that form a basis of the first homology group of $\mathbb{R}^3 \setminus D$ (see [12]).

For the existence analysis, in terms of the fundamental solution

$$\Phi(x,y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}\,, \quad x \neq y,$$

to the Helmholtz equation in $\mathbb{R}^3$ following [3, Section 3.1] we introduce the classical boundary integral operators in scattering theory given by the single- and double-layer operators

$$(S\varphi)(x) := 2 \int_{\partial D} \Phi(x,y)\varphi(y)\, ds(y) \tag{2.7}$$

and

$$(K\varphi)(x) := 2 \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\, ds(y) \tag{2.8}$$

and the normal derivative of the double-layer potential

$$(T\varphi)(x) := 2\, \frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\, ds(y) \tag{2.9}$$

for $x \in \partial D$. We note that S and K are bounded operators from $C^{0,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$ and consequently compact from $C^{1,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$ by the compact embedding of $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$. The operator T is bounded from $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$. The single-layer potential defines a bounded linear operator from $C^{0,\alpha}(\partial D)$ into $C^{1,\alpha}(\bar{D})$ and into $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$ and the double-layer potential is bounded from $C^{1,\alpha}(\partial D)$ into $C^{1,\alpha}(\bar{D})$ and into $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$ (see Theorems 3.3 and 3.4 in [3]).

We consider solutions to the Maxwell equations of the form

$$
\begin{aligned}
E(x) &= \operatorname{curl}\operatorname{curl}\int_{\partial D}\Phi(x,y)\,\nu(y)\,\varphi(y)\,ds(y)\\
&\qquad -ik\operatorname{curl}\int_{\partial D}\Phi(x,y)\,\nu(y)\,\psi(y)\,ds(y), \qquad (2.10)\\
H(x) &= \frac{1}{ik}\operatorname{curl}E(x), \quad x\in\mathbb{R}^3\setminus\partial D,
\end{aligned}
$$

with scalar functions $\varphi,\psi\in C^{1,\alpha}(\partial D)$. (In (2.10) we deviate from the work of Gülzow [8] by the factor of $-ik$ in front of the curl-term in the representation of E in order to achieve symmetry in the resulting integral equation system.) In view of $\operatorname{curl}\operatorname{curl} = -\Delta + \operatorname{grad}\operatorname{div}$ we note that (2.10) implies

$$
\begin{aligned}
H(x) = -ik\ \operatorname{curl}&\int_{\partial D}\Phi(x,y)\,\nu(y)\,\varphi(y)\,ds(y)\\
&-\operatorname{curl}\operatorname{curl}\int_{\partial D}\Phi(x,y)\,\nu(y)\,\psi(y)\,ds(y), \quad x\in\mathbb{R}^3\setminus\partial D.
\end{aligned}
\tag{2.11}
$$

As linear combinations of derivatives of single- and double-layer potentials the Cartesian components of E and H satisfy the Sommerfeld radiation condition and consequently the Silver–Müller radiation condition by Theorem 6.8 in [3]. With the aid of

$$\operatorname{curl}_x[\Phi(x,y)\nu(y)\varphi(y)] = -\varphi(y)\operatorname{Grad}_y\Phi(x,y)\times\nu(y)$$

by Stokes' integral theorem (see Theorem 2.1 in [2]) we conclude that

$$\operatorname{curl}\int_{\partial D}\Phi(x,\cdot)\,\nu\varphi\,ds = -\int_{\partial D}\Phi(x,\cdot)\,\nu\times\operatorname{Grad}\varphi\,ds. \tag{2.12}$$

Using $\operatorname{div}_x\{\Phi(x,y)\nu(y)\} = -\operatorname{grad}_y\Phi(x,y)\cdot\nu(x)$ and $\operatorname{curl}\operatorname{curl} = -\Delta + \operatorname{grad}\operatorname{div}$ yields

$$\operatorname{curl}\operatorname{curl}\int_{\partial D}\Phi(x,\cdot)\,\nu\varphi\,ds = k^2\int_{\partial D}\Phi(x,\cdot)\,\nu\varphi\,ds - \operatorname{grad}\int_{\partial D}\frac{\partial\Phi(x,\cdot)}{\partial\nu}\,\varphi\,ds. \tag{2.13}$$

From (2.12) and (2.13) by the mapping properties of the scalar single-and double layer potentials mentioned above we observe that the linear mapping from φ, ψ to E, H defined by (2.10) is bounded from $C^{1,\alpha}(\partial D) \times C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\bar{D}) \times C^{0,\alpha}(\bar{D})$ and into $C^{0,\alpha}(\mathbb{R}^3 \setminus D) \times C^{0,\alpha}(\mathbb{R}^3 \setminus D)$. By the jump relations for the single- and double-layer potentials together with (2.12) and (2.13) we obtain the jump relations

$$\begin{aligned} \nu \cdot E_\pm &= -\frac{1}{2}\, T\varphi + \frac{k^2}{2}\, Q\varphi + \frac{ik}{2}\, P\psi, \\ \nu \times (E_+ - E_-) &= -\nu \times \operatorname{Grad} \varphi, \end{aligned} \tag{2.14}$$

where

$$E_\pm(x) := \lim_{h \to +0} E(x \pm h\nu(x)), \quad x \in \partial D,$$

and where we have set

$$P\varphi := \nu \cdot S(\nu \times \operatorname{Grad} \varphi) \quad \text{and} \quad Q\varphi := \nu \cdot S(\nu\varphi). \tag{2.15}$$

Note that the jump for the tangential component in (2.14) we use the tangential derivative of the jump relation for the double-layer potential. (Alternatively, here we can also apply the jump relation for the curl of vector potentials from [3, Theorem 6.9].) Analogous, we have

$$\begin{aligned} \nu \cdot H_\pm &= \frac{ik}{2}\, P\varphi + \frac{1}{2}\, T\psi - \frac{k^2}{2}\, Q\psi, \\ \nu \times (H_+ - H_-) &= \nu \times \operatorname{Grad} \psi. \end{aligned} \tag{2.16}$$

From (2.14) and (2.16) we obtain that E, H as given by (2.10) satisfies the DB boundary conditions for the exterior problem provided the densities φ and ψ solve the equation

$$A \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 2 \begin{pmatrix} f \\ g \end{pmatrix} \tag{2.17}$$

with the operator $A : C^{1,\alpha}(\partial D) \times C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D) \times C^{0,\alpha}(\partial D)$ given by

$$A := \begin{pmatrix} -T + k^2 Q & ikP \\ ikP & T - k^2 Q \end{pmatrix}. \tag{2.18}$$

For $j = 0, 1$, we define

$$C_0^{j,\alpha}(\partial D) := \left\{ \varphi \in C^{j,\alpha}(\partial D) : \int_{\partial D} \varphi \, ds = 0 \right\}.$$

By construction the operator A is given by the normal components of curls on ∂D and consequently, by Stokes' integral theorem its range is contained in the subspace $Y := C_0^{0,\alpha}(\partial D) \times C_0^{0,\alpha}(\partial D)$. Therefore we may consider the operator A as an operator from $X := C_0^{1,\alpha}(\partial D) \times C_0^{1,\alpha}(\partial D)$ into Y. We note that the right-hand side of (2.17) belongs to Y by the necessary solvability conditions (2.2).

For the operators entering into A we have that T is bounded from $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$ and that P and Q are bounded from $C^{1,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$ and consequently compact from $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$ due to the compact imbedding of $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$. The single-layer operator S_0 for the limiting case $k = 0$ has a bounded inverse $S_0^{-1} : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ and we have the relation

$$S_0 T_0 = K_0^2 - I \tag{2.19}$$

with the identity operator I and the limits T_0 and K_0 of T and K as k tends to zero (see [13, Section 7.6]). If we define

$$B := \begin{pmatrix} S_0 & 0 \\ 0 & S_0 \end{pmatrix}$$

then the equation (2.18) and

$$BA \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 2B \begin{pmatrix} f \\ g \end{pmatrix} \tag{2.20}$$

are equivalent. Making use of (2.19) we can write

$$-S_0 T = I - K_0^2 - S_0(T - T_0).$$

The operator $T - T_0$ has the same leading singularity as the operator S_0 and can be seen to satisfy the assumptions of Theorem 2.7 in [2]. Therefore $T - T_0$ is bounded from $C(\partial D)$ into $C^{0,\alpha}(\partial D)$ and consequently compact from $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$. Hence $S_0(T - T_0)$ is compact from $C^{1,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$. Thus the operator $BA : X \to B(X) \subset C^{1,\alpha}(\partial D) \times C^{1,\alpha}(\partial D)$ is of the form identity plus compact and we can apply the Riesz theory, i.e., uniqueness for a solution of (2.18) in X applies existence of a solution.

Assume that φ, ψ solve the homogeneous from of equation (2.18). Then the corresponding field (2.10) solves the homogeneous exterior DP boundary value problem and the uniqueness Theorem 2.2 implies that $E = H = 0$ in

$\mathbb{R}^3 \setminus \bar{D}$. By (2.14) and (2.16) we observe that E, H also solve the homogeneous DP boundary value problem in D. From [12, Theorem 3.3] we know that there exists a countable set of positive wave numbers k, called interior eigenvalues, accumulating only at infinity for which the homogeneous DP boundary value problem in D has a finite number of nontrivial solutions. If k is not such an eigenvalue, then we have $E = H = 0$ in D and (2.14) and (2.16) imply that $\operatorname{Grad}\varphi = \operatorname{Grad}\psi = 0$ on ∂D. From this, in view of the definition of $C_0^{1,\alpha}(\partial D)$, we obtain injectivity of A on $C_0^{1,\alpha}(\partial D) \times C_0^{1,\alpha}(\partial D)$. Consequently, when k is not an interior eigenvalue the operator A is an isomorphism from $C_0^{1,\alpha}(\partial D) \times C_0^{1,\alpha}(\partial D)$ onto $C_0^{0,\alpha}(\partial D) \times C_0^{0,\alpha}(\partial D)$ with the inverse operator A^{-1} bounded by the open mapping theorem. From this we can conclude that the exterior DB boundary value problem is uniquely solvable and that the linear mapping taking the boundary data f, g onto the unique solution E, H given by (2.10) is bounded from $C^{0,\alpha}(\partial D) \times C^{0,\alpha}(\partial D)$ into $C^{0,\alpha}(\mathbb{R}^3 \setminus D) \times C^{0,\alpha}(\mathbb{R}^3 \setminus D)$.

When k is an interior eigenvalue, for a solution φ, ψ to the homogeneous form of equation (2.17) the field (2.10) is an eigenelement in D and from the jump relations (2.14) and (2.16) we have

$$\nu \times E = \nu \times \operatorname{Grad}\varphi \quad \text{and} \quad -\nu \times H = \nu \times \operatorname{Grad}\psi \quad \text{on } \partial D. \tag{2.21}$$

Conversely, let (E, H) be a nontrivial solution to the homogeneous DP boundary value problem in D. Then by Lemma 2.1 there exist $\varphi, \psi \in C^{1,\alpha}(\partial D)$ such that (2.21) holds, and by the Stratton-Chu formulas (see [3, Theorem 6.1]) we can represent

$$-E(x) = \operatorname{curl}\int_{\partial D} \Phi(x,\cdot)\, \nu \times \operatorname{Grad}\varphi \, ds - ik \int_{\partial D} \Phi(x,\cdot)\, \nu \times \operatorname{Grad}\psi \, ds$$

and

$$-H(x) = -ik \int_{\partial D} \Phi(x,\cdot)\, \nu \times \operatorname{Grad}\varphi \, ds - \operatorname{curl}\int_{\partial D} \Phi(x,\cdot)\, \nu \times \operatorname{Grad}\psi \, ds$$

for $x \in D$. Letting x approach ∂D and making use of (2.12) and (2.13) together with $\nu \cdot E = \nu \cdot H = 0$ on ∂D it can be seen that φ, ψ is in the null space of A. Thus we have a one-to-one correspondence between the eigenelements of the interior DB boundary value problem and the elements of the null space of the operator A.

We now modify the approach (2.10) for the existence analysis in the case when k is an interior eigenvalue. The operators S and T are known to be self-adjoint with respect to the bilinear form

$$\langle \varphi, \psi \rangle := \int_{\partial D} \varphi \psi \, ds$$

(see [3, p. 48]). This implies that P and Q also are self-adjoint with respect to this bilinear form. For Q this is obvious and for P, with the aid of (2.5) and the Gauß surface divergence theorem, from (2.12) and (2.15) we indeed obtain

$$\int_{\partial D} \psi P \varphi ds = \int_{\partial D} \psi \operatorname{Div}[\nu \times S(\nu\varphi)] \, ds = -\int_{\partial D} \operatorname{Grad} \psi \cdot [\nu \times S(\nu\varphi)] \, ds$$

$$= \int_{\partial D} S[\nu \times \operatorname{Grad} \psi] \cdot \nu\varphi \, ds = \int_{\partial D} \varphi P \psi \, ds.$$

This all implies that A is self-adjoint with respect to the bilinear form

$$\left\langle \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix} \right\rangle := \int_{\partial D} (\varphi_1 \psi_1 + \varphi_2 \psi_2) \, ds \tag{2.22}$$

and that the operators operators $BA : X \to Y$ and $AB : Y \to Y$ are adjoint in the dual systems $\langle X, B(X) \rangle$ and $\langle Y, Y \rangle$ generated by this bilinear form.

By the Fredholm alternative for compact operators in dual systems (see [13, Chapter 4] the equation $BA\chi = B\xi$ is solvable if and only if the right hand side $B\xi$ satisfies $\langle B\xi, \eta \rangle = 0$ for all η in the null space of AB. This implies that the equation $A\chi = \xi$ is solvable if and only if its right hand side ξ satisfies $\langle \xi, \gamma \rangle = 0$ for all γ in the null space of A.

In order to be able to satisfy this solvability condition we modify the approach (2.10) as follows. Let $E_m, H_m,\ m = 1, \dots, p$, be a basis of interior eigenelements with corresponding elements φ_m, ψ_m in the null space of A satisfying $\nu \times E_m = \nu \times \operatorname{Grad} \varphi_m$ and $\nu \times H_m = -\nu \times \operatorname{Grad} \psi_m$ on ∂D according to (2.21). Since $E_m \in C^{0,\alpha}(\bar{D})$ the volume potentials

$$U_m(x) := \int_D \overline{E_m(y)} \Phi(x, y) \, dy, \quad x \in \mathbb{R}^3$$

are in $C^1(\mathbb{R}^3)$ and belong to $C^2(D)$ with $\Delta U_m + k^2 U_m = -\overline{E}_m$ in D and to $C^2(\mathbb{R}^3 \setminus \bar{D})$ with $\Delta U_m + k^2 U_m = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ (see [6, Section 4.2]). In particular, it follows that $\nu \cdot U_m \in C^{0,\alpha}(\partial D)$. By Gauß' integral theorem using $\operatorname{div} E_m = 0$ in D and $\nu \cdot E_m = 0$ on ∂D we find

$$\operatorname{div} U_m(x) = -\int_D \overline{E_m(y)} \cdot \operatorname{grad}_y \Phi(x,y)\,dy - \int_D \operatorname{div}_y\{\Phi(x,y)\overline{E_m(y)}\}\,dy = 0$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$. Therefore U_m and

$$V_m := \frac{1}{ik} \operatorname{curl} U_m$$

solve the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$. By Gauß' integral theorem again and $\operatorname{curl} E_m = ikH_m$ in D we have that

$$\operatorname{curl} U_m(x) = -ik \int_D \overline{H_m(y)} \Phi(x,y)\,dy - \int_{\partial D} \nu(y) \times \overline{E_m(y)}\, \Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3,$$

and from this we observe that also $\nu \cdot V_m \in C^{0,\alpha}(\partial D)$.

We introduce the $p \times p$ matrix

$$w_{\ell m} := \left\langle \begin{pmatrix} \nu \cdot U_m \\ \nu \cdot V_m \end{pmatrix}, \begin{pmatrix} \varphi_\ell \\ \psi_\ell \end{pmatrix} \right\rangle, \quad \ell, m = 1, \dots, p, \tag{2.23}$$

and show that it is invertible. Let $\beta_m, , m = 1, \dots, p$, be a solution to the homogeneous equation

$$\sum_{m=1}^{p} w_{\ell m} \beta_m = 0, \quad \ell = 1, \dots, p,$$

and consider

$$\varphi := \sum_{m=1}^{p} \beta_m \varphi_m, \quad \psi := \sum_{m=1}^{p} \beta_m \psi_m,$$

and the corresponding fields E, H, U and V. Then

$$\int_{\partial D} \{\varphi\, \nu \cdot U + \psi\, \nu \cdot V\}\,ds = \sum_{\ell,m=1}^{p} w_{\ell m} \beta_m \beta_\ell = 0.$$

With the aid of Stokes' theorem, the rules for the spat product, Green's vector integral theorem and $\Delta U + k^2 U = -\bar{E}$ in D we compute

$$\begin{aligned}
&k^2 \int_{\partial D} \{\varphi\, \nu \cdot U + \psi\, \nu \cdot V\}\, ds \\
&= \int_{\partial D} \{\varphi\, \nu \cdot \operatorname{curl}\operatorname{curl} U - ik\psi\, \nu \cdot \operatorname{curl} U\}\, ds \\
&= \int_{\partial D} \{\nu \cdot (\operatorname{curl} U \times \operatorname{Grad} \varphi) - ik\nu \cdot (U \times \operatorname{Grad} \psi)\}\, ds \\
&= \int_{\partial D} \{(\nu \times E) \cdot \operatorname{curl} U - (\nu \times U) \cdot \operatorname{curl} E\}\, ds \\
&= \int_D \{E \Delta U - U \Delta E\}\, dx = -k^2 \int_D |E|^2 dx.
\end{aligned}$$

(Here, for the application of Stokes' theorem we first integrate over a parallel surface $\partial D_h = \{x + \nu(x)h : x \in \partial D\}$ for sufficiently small $h > 0$ and then pass to the limit $h \to 0$.) From the last two equations we conclude that

$$\sum_{m=1}^{p} \beta_m E_m = E = 0$$

in D and this implies $\beta_m = 0$, $m = 1, \ldots, p$, since the E_m are linearly independent. Thus the matrix $w_{\ell m}$ is invertible.

Therefore, by solving the linear system

$$\sum_{m=1}^{p} w_{\ell m} \beta_m = \left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} \varphi_\ell \\ \psi_\ell \end{pmatrix} \right\rangle, \quad \ell = 1, \ldots, p,$$

we now can determine a unique linear combination

$$U := \sum_{m=1}^{p} \beta_m U_m, \quad V := \sum_{m=1}^{p} \beta_m V_m \tag{2.24}$$

such that the perturbed right-hand side

$$2 \begin{pmatrix} f - \nu \cdot U \\ g - \nu \cdot V \end{pmatrix}$$

of (2.17) satisfies the solvability condition for the operator A when k is an interior eigenvalue. Clearly for U, V given by (2.24) we can estimate

$$\|\nu \cdot U\|_{0,\alpha} + \|\nu \cdot V\|_{0,\alpha} \leq C\{\|f\|_{0,\alpha} + \|g\|_{0,\alpha}\} \tag{2.25}$$

for some constant $C > 0$ given in terms of the inverse of the matrix $w_{\ell m}$.

Consequently with a solution φ, ψ of the modified equation (2.17) we obtain a solution to the exterior DB problem by adding the field U, V to the field given by (2.10). We can make the solution to the modified equation (2.17) unique by requiring that it belongs to the L^2 orthogonal complement $N(A)^\perp$ in X of the null space of A. By the open mapping theorem the inverse of this bijective mapping from $\{\chi \in X : \chi \perp N(A)\}$ onto

$$\{F \in Y : \langle F, \Psi \rangle = 0,\ \Psi \in N(A)\}$$

is bounded.

This way, in view of (2.25) it can be seen that also in the case when k is an interior eigenvalue, the linear mapping of the boundary data f, g onto the unique solution E, H of the exterior DB boundary value problem is bounded from $C^{0,\alpha}(\partial D) \times C^{0,\alpha}(\partial D)$ into $C^{0,\alpha}(\mathbb{R}^3 \setminus D) \times C^{0,\alpha}(\mathbb{R}^3 \setminus D)$.

So finally we can summarize the results of this section into the following theorem.

Theorem 2.3 *The exterior DB boundary value problem has a unique solution. The linear mapping from the boundary data onto the solution is bounded from $C^{0,\alpha}(\partial D) \times C^{0,\alpha}(\partial D)$ into $C^{0,\alpha}(\mathbb{R}^3 \setminus D) \times C^{0,\alpha}(\mathbb{R}^3 \setminus D)$.*

For numerical implementations, unfortunately our approach suffers from the non-uniqueness issue at the interior eigenvalues. However, it has the advantage that it only uses the scalar boundary integral operators S and T. For these, for example, the spectral methods based on spherical harmonics due to Ganesh, Graham and Sloan [5, 7] are easier to implement than for the corresponding vector boundary integral operators that occur in the solution of the perfect conductor scattering problem. To validate this approach for a particular domain D and wave number k one can utilize a new algorithm for solving nonlinear eigenvalue problems for integral operators that are analytic with respect to the eigenvalue parameter, such as the homogeneous form of the system (2.17) with eigenvalues k, as proposed by Beyn [1]. For the finite number n of eigenvalues lying in a fixed interval, this method reduces the large nonlinear eigenvalue problem to a linear eigenvalue problem of small size n (see also [3, Section 10.2]).

For an integral equation approach that does not suffer from the non-uniqueness issue at the interior eigenvalue we refer to [12]. This method requires the knowledge of a tangential vector $c \in C^{1,\alpha}(\partial D)$ with

$$\operatorname{Div} c = -ikg \tag{2.26}$$

which is immediately available in the scattering problem for an incident wave E^i, H^i via

$$ikg = -ik\nu \cdot H^i = \nu \cdot \operatorname{curl} E^i = -\operatorname{Div}(\nu \times E^i)$$

and also can be constructed in a straightforward manner in the general case (see [12]). Then an auxiliary problem to find a vector field W and a scalar function u both satisfying the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ and the boundary condition

$$\nu \times W = c,$$

$$\nu \cdot W + \frac{\partial u}{\partial \nu} + \int_{\partial D} u \, ds = f,$$

$$\operatorname{div} W - k^2 u = 0$$

on ∂D together with the radiation conditions

$$\lim_{r \to \infty} \operatorname{curl} W(x) \times x + x \operatorname{div} W(x) - ikrW(x)] = 0,$$

$$\lim_{r \to \infty} \operatorname{grad} u(x) \cdot x - ikru(x)] = 0,$$

where $r = |x|$ and where the limits are assumed to hold uniformly in all directions $x/|x|$ as in the Silver-Müller radiation condition (1.3). Then

$$E := W + \operatorname{grad} u, \quad H := \frac{1}{ik} \operatorname{curl} E$$

solve the exterior DB boundary value problem. For the auxiliary problem in [12] an integral equation approach in a Hölder space setting is presented with uniquely solvable equations for all wave numbers $k > 0$. However, there is a price to pay for that since a system of now three integral equations for a tangential vector field and two scalar functions on ∂D need to be solved.

3 The DB problem in Sobolev spaces

In this section we briefly indicate how Theorems 2.2 and 2.3 can be extended to the case of weak solutions in a Sobolev space setting just by tools from functional analysis and look for solutions in the appropriate energy space

$$H^1_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D}) := \left\{ E \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D}) : \mathrm{curl}\, E \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D}) \right\}$$

for the Maxwell equations. For detailed descriptions of this Sobolev space we refer to [10, 20, 21]. By the corresponding trace theorem the tangential trace $\nu \times E|_{\partial D}$ of a vector field $E \in H^1_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$ is in the trace space $H^{-1/2}(\mathrm{Div}, \partial D)$ defined by the tangential fields on ∂D whose surface divergence in the weak sense belongs to $H^{-1/2}(\partial D)$. In view of the identity (2.5) for smooth fields, this implies that for solutions $E, H \in H^1(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$ to the Maxwell equations the normal traces $\nu \cdot E|_{\partial D}$ and $\nu \cdot H|_{\partial D}$ are well defined and belong to $H^{-1/2}(\partial D)$. Therefore, the DB boundary condition is well defined in $H^1_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$ and, accordingly, for the given boundary values we assume $f, g \in H^{-1/2}(\partial D)$.

The proof of the uniqueness Theorem 2.2 carries over to the weak case. For the existence analysis we seek a solution in the form (2.10) with density functions $\varphi, \psi \in H^{1/2}(\partial D)$. The scalar single-layer potential defines a bounded linear operator from $H^{-1/2}(\partial D)$ into $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ and the double-layer potential is a bounded operator from $H^{1/2}(\partial D)$ into $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$. In view of (2.12) and (2.13) this implies that the fields in (2.10) define a bounded operator from $H^{1/2}(\partial D) \times H^{1/2}(\partial D)$ into $H^1_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D}) \times H^1_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$. For the operators entering into A in the Sobolev space setting we have that T is bounded from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$ and that P and Q are bounded from $H^{1/2}(\partial D)$ into $H^{1/2}(\partial D)$ and consequently compact from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$. We note that these mapping properties of the single- and double-layer potentials in the Sobolev spaces can be obtained by purely functional analytic tools from the corresponding mapping properties in Hölder spaces as stated in the previous section (see Corollaries 3.7 and 3.8 in [3]). This all implies that E, H defined by (2.10) with densities $\varphi, \psi \in H^{1/2}(\partial D)$ satisfy the DB boundary condition with the normal derivatives understood as traces in $H^{-1/2}(\partial D)$ provided the equation (2.17) is satisfied.

Noting that the single-layer operator S_0 also has a bounded inverse as operator from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$ the same procedure as in the case of the Hölder spaces equivalently transfers the equation into the form identity plus compact. We now apply the Riesz-Fredholm theory in the dual system

$$\langle C^{1,\alpha}(\partial D) \times C^{1,\alpha}(\partial D), H^{1/2}(\partial D) \times H^{1/2}(\partial D)\rangle$$

with the bilinear form (2.22) for the two operators BA on $C^{1,\alpha}(\partial D) \times C^{1,\alpha}(\partial D)$ and AB on $H^{1/2}(\partial D) \times H^{1/2}(\partial D)$. Since the two operators are adjoint with respect to this dual system, by the Fredholm alternative the dimensions of their null spaces must coincide. Since B is bijective this implies that the null space of A in $C^{1,\alpha}(\partial D) \times C^{1,\alpha}(\partial D)$ and the null space of A in $H^{1/2}(\partial D) \times H^{1/2}(\partial D)$ must have the same finite dimension and consequently both null spaces must coincide. With this the existence analysis for the equation (2.17) can be carried over from the Hölder space setting to the Sobolev space setting.

Analogously to the previous section we define

$$H_0^j(\partial D) := \left\{\varphi \in H_0^j(\partial D) : \int_{\partial D} \varphi ds = 0\right\}$$

for $j = \pm 1/2$. Then, when k is not an interior DB eigenvalue A is an isomorphism from $H_0^{1/2}(\partial D) \times H_0^{1/2}(\partial D)$ onto $H_0^{-1/2}(\partial D) \times H_0^{-1/2}(\partial D)$ with the inverse operator A^{-1} being bounded. We omit the analysis for the case when k is an interior eigenvalue and finally summarize our considerations into the following theorem.

Theorem 3.1 *In the Sobolev space $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ the exterior DB boundary value problem possesses a unique solution. The mapping taking the boundary values onto the solution is bounded from $H^{-1/2}(\partial D) \times H^{-1/2}(\partial D)$ into $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D}) \times H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$.*

4 Direct DB scattering

We now turn to the scattering of an electromagnetic wave given by a solution E^i, H^i of the Maxwell equations that is C^2 smooth in some domain containing $\bar{D}$. Combining the Stratton–Chu formulas for the scattered wave E^s, H^s and for the incident wave E^i, H^i (and noting that for the incident wave the Stratton–Chu formula yields zero for points $x \in \mathbb{R}^3 \setminus \bar{D}$) for the total wave

$E = E^i + E^s, H = H^i + H^s$ we obtain

$$
\begin{aligned}
E^s(x) = \operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y) \\
-\frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y)
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
H^s(x) = \frac{1}{ik} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times E(y)\, \Phi(x,y)\, ds(y) \\
+ \operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x,y)\, ds(y)
\end{aligned}
\tag{4.2}
$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$. From Lemma 2.1 we know that the boundary condition $\nu \cdot E = \nu \cdot H = 0$ on ∂D implies that $\nu \times E = -\nu \times \operatorname{Grad} \varphi$ and $\nu \times H = \nu \times \operatorname{Grad} \psi$ with some $\varphi, \psi \in C^{1,\alpha}(\partial D)$. Inserting this into (4.1) and (4.2) yields

$$
\begin{aligned}
E(x) - E^i(x) = \operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi(x,y)\, \nu(y)\, \varphi(y)\, ds(y) \\
-ik \operatorname{curl} \int_{\partial D} \Phi(x,y)\, \nu(y)\, \psi(y)\, ds(y)
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
H(x) - H^i(x) = -ik \operatorname{curl} \int_{\partial D} \Phi(x,y)\, \nu(y)\, \varphi(y)\, ds(y) \\
- \operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi(x,y)\, \nu(y)\, \psi(y)\, ds(y)
\end{aligned}
\tag{4.4}
$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$. Taking the normal component on the boundary ∂D, this so-called direct approach via the representation formulas leads to the system

$$
A \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = -2 \begin{pmatrix} \nu \cdot E^i \\ \nu \cdot H^i \end{pmatrix}. \tag{4.5}
$$

As typical for the approach via integral equations of the first kind (4.5) coincides with (2.17) in the particular case of the scattering problem.

We recall that every solution E, H to the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the Silver–Müller radiation condition has the asymptotic form

$$\begin{aligned} E(x) &= \frac{e^{ik|x|}}{|x|}\left\{E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\} \\ H(x) &= \frac{e^{ik|x|}}{|x|}\left\{H_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\} \end{aligned} \tag{4.6}$$

as $|x| \to \infty$ uniformly in all directions $\hat{x} = x/|x|$ where the vector fields E_∞ and H_∞ defined on the unit sphere $\mathbb{S}^2$ are known as the electric far field pattern and magnetic far field pattern, respectively. They satisfy

$$H_\infty = \nu \times E_\infty \quad \text{and} \quad \nu \cdot E_\infty = \nu \cdot H_\infty = 0 \tag{4.7}$$

with the unit outward normal ν on $\mathbb{S}^2$ (see [3, Theorem 6.9]). There is a one-to-one correspondence between solutions to the Maxwell equations and their far field patterns in the sense that a vanishing far field pattern $E_\infty = 0$ on $\mathbb{S}^2$ by Rellich's lemma implies that $E = H = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ (see [3, Theorem 6.10]). For the far field of (2.10) we note that

$$E_\infty(\hat{x}) = \frac{k^2}{4\pi}\,\hat{x} \times \int_{\partial D} e^{-ik\,\hat{x}\cdot y}\left\{(\nu(y) \times \hat{x})\,\varphi(y) + \nu(y)\,\psi(y)\right\} ds(y) \tag{4.8}$$

and

$$H_\infty(\hat{x}) = \frac{k^2}{4\pi}\,\hat{x} \times \int_{\partial D} e^{-ik\,\hat{x}\cdot y}\left\{\nu(y)\,\varphi(y) - (\nu(y) \times \hat{x})\,\psi(y)\right\} ds(y) \tag{4.9}$$

for $\hat{x} \in \mathbb{S}^2$. Observing that $\nu(\hat{x}) = \hat{x}$ for $\hat{x} \in \mathbb{S}^2$ we see that (4.7) is satisfied.

We now consider the scattering of electromagnetic plane waves with incident direction $d \in \mathbb{S}^2$ and polarization vector p. they are described by the matrices $E^i(x,d)$ and $H^i(x,d)$ that are defined by their multiplication with the polarization vector as

$$\begin{aligned} E^i(x,d)p &:= \frac{i}{k}\,\operatorname{curl}\operatorname{curl} p\, e^{ik\,x\cdot d} = ik\,(d \times p) \times d\, e^{ik\,x\cdot d}, \\ H^i(x,d)p &:= \operatorname{curl} p\, e^{ik\,x\cdot d} = ik\, d \times p\, e^{ik\,x\cdot d}. \end{aligned} \tag{4.10}$$

Because of the linearity of the direct scattering problem with respect to the incident field, we can express the scattered waves by matrices $E^s(x,d)$

and $H^s(x, d)$, the total waves by matrices $E(x, d)$ and $H(x, d)$, and the far field patterns by $E_\infty(\hat{x}, d)$ and $H_\infty(\hat{x}, d)$, respectively. The latter map the polarization vector p onto the far field patterns $E_\infty(\hat{x}, d)p$ and $H_\infty(\hat{x}, d)p$, respectively.

Analogously to Theorem 6.30 in [3] we can establish the following reciprocity result for DB scattering.

Theorem 4.1 *The electric far field pattern for the scattering of plane electromagnetic waves by a DB scatterer satisfies the reciprocity relation*

$$E_\infty(\hat{x}, d) = [E_\infty(-d, -\hat{x})]^\top, \quad \hat{x}, d \in \mathbb{S}^2. \tag{4.11}$$

Proof. As in the proof of Theorem 6.30 in [3] from the Gauß integral theorem, the Maxwell equations for the incident and the scattered fields, the radiation condition for the scattered field and the far field representation of Theorem 6.9 in [3] we find

$$\begin{aligned} &4\pi\big\{q \cdot E_\infty(\hat{x}, d)p - p \cdot E_\infty(-d, -\hat{x})q\big\} \\ &= \int_{\partial D} \big\{\nu \times E(\,\cdot\,, d)p \cdot H(\,\cdot\,, -\hat{x})q + \nu \times H(\,\cdot\,, d)p \cdot E(\,\cdot\,, -\hat{x})q\big\}\, ds \end{aligned} \tag{4.12}$$

in terms of the total fields E and H. From this the reciprocity relation (4.11) follows with the aid of Lemma 2.1. □

For the scattering of an electric dipole of the form

$$\begin{aligned} E_e^i(x, z)p &:= \frac{i}{k}\, \mathrm{curl}_x\, \mathrm{curl}_x\, p\Phi(x, z), \\ H_e^i(x, z)p &:= \mathrm{curl}_x\, p\Phi(x, z) \end{aligned} \tag{4.13}$$

with a polarization vector $p \in \mathbb{R}^3$ we denote the scattered fields by $E_e^s(x, z)$ and $H_e^s(x, z)$, the total fields by $E_e(x, z)$ and $H_e(x, z)$ and the far field patterns of the scattered wave by $E_{e,\infty}^s(\hat{x}, z)$ and $H_{e,\infty}^s(\hat{x}, z)$.

Analogous to the previous proof, following the proof of Theorem 6.31 in [3] with the aid Lemma 2.1 we can establish the following result connecting scattering of plane waves and dipole fields.

Theorem 4.2 *For scattering of electric dipoles and plane waves we have the mixed reciprocity relation*

$$4\pi E_{e,\infty}^s(-d, z) = [E^s(z, d)]^\top, \quad z \in \mathbb{R}^3 \setminus \bar{D},\ d \in \mathbb{S}^2. \tag{4.14}$$

These reciprocity principles can play a role in a future investigation of the inverse scattering problem for the DB boundary condition as they did for the perfect conductor case. We illustrate this by the following analysis on the far field operator where we define

$$L_t^2(\mathbb{S}^2) := \{g \in L^2(\mathbb{S}^2) : \nu \cdot g = 0\}$$

as the space of all tangential L^2 vector fields on $\mathbb{S}^2$. We note that fields of the form (4.16) occurring in the following theorem are called electromagnetic Herglotz pairs.

Theorem 4.3 *The far field operator* $F : L_t^2(\mathbb{S}^2) \to L_t^2(\mathbb{S}^2)$ *defined by*

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} E_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2, \tag{4.15}$$

is injective and has dense range if and only if there exists a solution E, H *of the Maxwell equations in* D *which satisfies the homogeneous DB boundary condition on* ∂D *and is of the form*

$$E(x) = \int_{\mathbb{S}^2} e^{ik\, x\cdot d} a(d)\, ds(d), \quad H(x) = \frac{1}{ik} \operatorname{curl} E(x), \quad x \in \mathbb{R}^3, \tag{4.16}$$

for some $a \in L_t^2(\mathbb{S}^2)$.

Proof. We begin by noting the following consequence of the linearity and well-posedness of the DB scattering problem. For $g \in L_t^2(\mathbb{S}^2)$ the solution to the DB scattering problem for the incident wave

$$\tilde{E}^i(x) = \int_{\mathbb{S}^2} E^i(x, d) g(d)\, ds(d), \quad \tilde{H}^i(x) = \int_{\mathbb{S}^2} H^i(x, d) g(d)\, ds(d)$$

is given by

$$\tilde{E}^s(x) = \int_{\mathbb{S}^2} E^s(x, d) g(d)\, ds(d), \quad \tilde{H}^s(x) = \int_{\mathbb{S}^2} H^s(x, d) g(d)\, ds(d)$$

for $x \in \mathbb{R}^3 \setminus \bar{D}$ and has the far field pattern

$$\tilde{E}_\infty(\hat{x}) = \int_{\mathbb{S}^2} E_\infty(\hat{x}, d) g(d)\, ds(d), \quad \tilde{H}_\infty(\hat{x}) = \int_{\mathbb{S}^2} H_\infty(\hat{x}, d) g(d)\, ds(d)$$

for $\hat{x} \in \mathbb{S}^2$. From the reciprocity relation (4.11) it can seen be that the L^2 adjoint $F^* : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ of F is given by

$$F^* g = \overline{RFR\bar{g}}, \tag{4.17}$$

where $R : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ is defined by

$$(Rg)(d) := g(-d).$$

Hence the operator F is injective if and only if its adjoint F^* is injective. Observing the property $N(F^*)^\perp = \overline{F(L^2_t(\mathbb{S}^2))}$ for bounded operators F in a Hilbert space it suffices to characterize the null space of F.

From the one-to-one correspondence of Herglotz pairs and their kernel functions (see [3, Theorem 3.27] and the remark at the beginning of the proof we have that the existence of a nontrivial $g \in L^2_t(\mathbb{S}^2)$ is equivalent to the existence of a nontrivial Herglotz pair $\tilde{E}^i, \tilde{H}^i$ (with kernel $a = ikg$) for which the electric far field pattern of the corresponding scattered field $\tilde{E}^s$ has vanishing far field pattern $\tilde{E}_\infty = 0$ on $\mathbb{S}^2$. The latter is equivalent to $\tilde{E}^s = \tilde{H}^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. This in turn, by the boundary condition $\nu \cdot \tilde{E}^i + \nu \cdot \tilde{E}^s = \nu \cdot \tilde{H}^i + \nu \cdot \tilde{H}^s = 0$ on ∂D and the uniqueness of the solution to the exterior DB boundary value problem, is equivalent to $\nu \cdot \tilde{E}^i = \nu \cdot \tilde{H}^i = 0$ on ∂D and the proof is finished. $\square$

Let

$$u_n(x) := j_n(k|x|)\, Y_n\left(\frac{x}{|x|}\right)$$

where j_n is a spherical Bessel function and Y_n a spherical harmonic of order n. In [3, p. 264] it is shown that the electromagnetic spherical wave functions

$$E_n := \operatorname{curl}\{x u_n(x)\} \quad \text{and} \quad H_n := \frac{1}{ik} \operatorname{curl} E_n$$

provide an electromagnetic Herglotz pair (and the same is true for $E = H_n$ and $H = -E_n$). We have

$$E_n(x) = \operatorname{grad} u_n(x) \times x$$

and consequently

$$\nu \cdot E_n = 0$$

and

$$ik\,\nu \cdot H_n = (\nu \cdot \operatorname{curl} E_n) = -\operatorname{Div} \nu \times E_n = R \operatorname{Div} \operatorname{Grad} u_n = -Rn(n+1)u_n$$

on spheres of radius R centered at the origin. Therefore, if kR is equal to a zero of the spherical Bessel function j_n then k is an interior eigenvalue and the corresponding eigenfields are Herglotz pairs.

We conclude the analysis of the far field operator with the following result for which the proof is completely analogous to that of Theorem 6.39 in [3] for the perfect conductor case with the application of the perfect conductor boundary condition replaced by Lemma 2.1 similar as in the proof of Theorem 4.1.

Theorem 4.4 *The far field operator F is compact and normal, i.e.,*

$$FF^* = F^*F,$$

and hence has an infinite number of eigenvalues. Further, the operator

$$I + \frac{1}{2\pi}F$$

is unitary.

5 Inverse DB scattering: uniqueness

In the direct scattering problem as discussed in the previous sections, given the boundary of the scatterer and the incident wave, we want to find the scattered wave and in particular its behavior at large distances of the scatterer which is characterized by the far field pattern. The inverse scattering problem we want to consider now is to determine the boundary of the scatterer from the knowledge of the far field pattern for the scattering of plane electromagnetic waves of the form (4.10) as incident fields. Our main result of this section will be the following uniqueness result as analogue of the corresponding theorem for scattering from a perfect conductor (see [3, Theorem 7.1]).

Theorem 5.1 *Assume that D_1 and D_2 are two scatterers with DB boundary condition such that for a fixed wave number k for all plane waves of the form* (4.10) *the electric far field patterns for both scatterers coincide for all incident directions d and all polarizations p. Then $D_1 = D_2$.*

Our proof will consist of two parts. In the first step, from the coincidence of the scattered waves for two scatterers D_1 and D_2 for all incident plane waves of the form (4.10) we will deduce that the scattered waves also coincide for all dipole fields of the form (4.13) with sources located in a point z in $\mathbb{R}^3 \setminus (\bar{D}_1 \cup \bar{D}_2)$ as incident waves. Then in the second step, assuming that $D_1 \neq D_2$ (and without loss of generality that $D_1 \setminus (\bar{D}_1 \cap \bar{D}_2)$ is nonempty), we will arrive at a contradiction by letting the dipole location z tend to a boundary point of ∂D_1 which does not belong to $\bar{D}_2$. For the first part of the proof we will provide two alternatives: one proof via approximation of dipole fields by plane waves in the spirit of the original idea of the uniqueness proof for acoustic waves by Kirsch and Kress [11] and another and simpler proof via the mixed reciprocity principle of Theorem 4.2. We will start with the first possibility where we will need the following lemma.

Lemma 5.2 *Let B be a bounded domain with a connected C^2 boundary ∂B, and for $z \notin B$ let E_e^1, H_e^1 be the electric dipole field as given by* (4.13) *with polarization $p \in \mathbb{R}^3$. Then there exists a sequence E_n, H_n in the span of electromagnetic plane waves*

$$V := \operatorname{span}\{E^i(\cdot, d)p, H^i(\cdot, d), p) : d \in \mathbb{S}^2, p \in \mathbb{R}^3\}$$

such that $E_n \to E$, $H_n \to H$, $n \to \infty$, uniformly on compact subsets of B.

Proof. We make use of the result of Lemma 3.3 in [11] which states that for every solution $u \in C^2(B)$ of the Helmholtz equation there exists a sequence u_n in the span of acoustic plane waves

$$U := \operatorname{span}\{e^{ik\,x\cdot d} : d \in \mathbb{S}^2\}$$

such that $u_n \to u$, $n \to \infty$, with uniform convergence on compact subsets of B including the derivatives up to second order. (The lemma in [11] actually only states convergence up to the first derivatives, but inspection of the proof shows that we also have convergence for the second and third derivatives.) Choosing $u = \Phi(\cdot, z)$ we obtain

$$E_n := \frac{i}{k} \operatorname{curl}\operatorname{curl} p u_n \to E, \quad H_n := \frac{1}{ik} \operatorname{curl} E_n \to H, \quad n \to \infty,$$

which ends the proof. $\square$

For the proof of Theorem 5.1 we assume that $D_1 \neq D_2$. Then by Rellich's lemma we conclude that the scattered waves $E^s(\cdot\,; d, p)$ coincide for all directions d and all polarizations p in the unbounded component G of the complement of $\bar{D}_1 \cup \bar{D}_2$. Choose $z \in G$ and consider as incident fields the dipole fields $E_e^i(x, z), H_e^i(x, z)$, with scattered fields $E_e^s(x, z), H_e^s(x, z)$ and far field patterns $E_{e,\infty}^s(\hat{x}, z), H_{e,\infty}^s(\hat{x}, z)$. To show that

$$\tilde{E}_1^s = \tilde{E}_2^s \quad \tilde{H}_1^s = \tilde{H}_2^s \quad \text{in } G \tag{5.1}$$

we choose a bounded simply connected C^2 domain B with connected boundary ∂B such that $\bar{D}_1 \cup \bar{D}_2 \subset B$ and that $z \notin \bar{B}$. Then, by Lemma 5.2, there exists a sequence E_n, H_n in V such that

$$E_n \to \tilde{E}^i, \quad H_n \to \tilde{H}^i, \quad n \to \infty, \tag{5.2}$$

uniformly on $\bar{D}_1 \cup \bar{D}_2$. Since the E_n, H_n are linear combinations of plane waves, the corresponding scattered waves $E_{n,1}^s, H_{n,1}^s$ and $E_{n,2}^s, H_{n,2}^s$ for the obstacles D_1 and D_2 satisfying the boundary conditions

$$\nu \cdot [E_{n,j}^s + E_n] = \nu \cdot [H_{n,j}^s + H_n] = 0 \quad \text{on } \partial D_j, \quad j = 1, 2, \tag{5.3}$$

coincide in G, that is,

$$E_{n,1}^s = E_{n,2}^s \quad \text{and} \quad H_{n,1}^s = H_{n,2}^s \quad \text{in } G. \tag{5.4}$$

The well-posedness of the exterior DB problem from Theorem 2.3, the boundary conditions (5.3) and the convergence (5.2) imply that

$$E_{n,j}^s \to \tilde{E}_j^s, \quad H_{n,j}^s \to \tilde{H}_j^s, \quad n \to \infty,$$

uniformly on compact subsets of G for $j = 1, 2$. From this and (5.4) the statement (5.1) follows.

The alternative for proving (5.1) via the mixed reciprocity principle is straightforward. Coincidence of the far field patterns for plane wave incidence by Rellich's lemma implies coincidence of the corresponding scattered fields. By (4.14) this implies coincidence of the far field patterns for dipole field incidence and from this in turn again be Rellich's lemma (5.1) follows.

For the second part of the proof, without loss of generality we assume there exists $x^* \in \partial G$ such that $x^* \in \partial D_1$ and $x^* \notin \bar{D}_2$. Then we can choose $h > 0$ such that the sequence

$$x_m := x^* + \frac{h}{m}\, \nu(x^*), \quad m = 1, 2, \ldots, \tag{5.5}$$

is contained in G. Consider the fields in (5.1) with z replaced by x_m and polarization $p = \nu(x^*)$. Since $x^* \notin \bar{D}_2$, for scattering from D_2 we have

$$\|\nu \cdot [\tilde{H}_e^i(\cdot; x_m)\nu(x^*)] - \nu \cdot [\tilde{H}_e^i(\cdot; x^*)\nu(x^*)]\|_{C^{1,\alpha}(\partial D_2} \to 0, \quad m \to \infty,$$

and the same convergence for $\tilde{E}_e^i$. Therefore, by the well-posedness from Theorem 2.3 we have that

$$\lim_{m\to\infty} \nu(x^*) \cdot [\tilde{H}_{n,2}^s(x^*, x_m)\nu(x^*)] = \nu(x^*) \cdot [\tilde{H}_2^s(x^*, x^*)\nu(x^*)]. \tag{5.6}$$

On the other hand, from the boundary condition corresponding to the obstacle D_1 we find that

$$|\nu(x^*)\cdot[\tilde{H}_2^s(x^*, x_m)\nu(x^*)]| = |\operatorname{curl}\{\Phi(x^*, x_m)\nu(x^*)\}| = \frac{|1 - ik|x^* - x_m|\,|}{4\pi|x^* - x_m|^2} \to \infty$$

as $m \to \infty$. This is a contradiction to (5.1) and (5.6). Therefore $D_1 = D_2$ and the proof is complete. □

In addition this uniqueness result for one wave number it is also possible to prove a uniqueness theorem for fixed incident direction and polarization.

Theorem 5.3 *Assume that D_1 and D_2 are two scatterers with DB boundary condition such that for plane waves with one fixed incident direction and polarization the electric far field patterns of both scatterers coincide for all wave numbers contained in some open interval in $(0, \infty)$. Then $D_1 = D_2$.*

The proof is the same as that for the corresponding result on the perfect conductor boundary condition in [3, Theorem 7.2]. In the application of Green's vector theorem at the end of that proof one has to apply again the Hodge decomposition and use Lemma 2.1.

For uniqueness with only a few waves we have the following result.

Theorem 5.4 *A convex polyhedron with DB boundary condition is uniquely determined by the electric far field patterns for two incident plane wave of the same wave number with two pairs of incident directions d_1, d_2 and polarizations p_1, p_2 such that the planes spanned by $d_1 \times p_1$ and $d_2 \times p_2$ are not parallel.*

The proof is the same as that for the corresponding result on sound soft acoustic scattering in [3, Theorem 5.5]. The condition of the theorem ensures that for each plane P in $\mathbb{R}^3$ at least one of the two plane waves $E^i(\cdot; d_1)p_1$ and $E^i(\cdot; d_2)p_2$ has nonzero normal component on P. It is to be expected that the theorem remains valid without the simplifying assumption of convexity as in the perfect conductor case (see [18]).

6 Outlook on solving the inverse problem

Despite the lack of a general uniqueness result for one or a few incident fields analogous to inverse scattering for the perfect conductor boundary condition one approach for solving the inverse problem for DB scatterers can be based on the two equations (4.5) and (4.8) for the unknowns φ, ψ and ∂D, given the incident field E_i, H_i and the (measured) far field pattern E_∞. We call the first equation the field equation and the second the data equation. The equations are linear with respect to φ, ψ and nonlinear with respect to the main unknown ∂D. There are two immediate options for an iterative solution. In a first method, given an approximation of the unknown boundary ∂D one can solve the field equation (4.5) for φ and ψ. Then keeping φ and ψ fixed, the ill-posed data equation (4.8) is linearized with respect to ∂D in order to update the boundary approximation. These two steps are then iterated. In a second approach, the two equations are solved by Newton iterations, that is, by linearizing both equations with respect to all three unknowns. For details on the corresponding methods for the inverse scattering problem in acoustics and for a perfect conductor we refer to [3].

A different approach using many incident fields are sampling methods such as the linear sampling method and the factorization method. These are based on the far field operator F of Section 4 and described, for example, in [3] for the perfect conductor case.

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