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ON A SCHWARZ METHOD FOR A SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEM WITH DISCONTINUOUS BOUNDARY DATA *

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1. Introduction

We are concerned with a two dimensional steady state convection-diffusion problem with discontinuous outflow boundary conditions. It is well known that, where the boundary conditions are sufficiently smooth and compatible, such a problem can be solved with uniform accuracy with respect to the small parameter ε using a standard finite difference operator on special piecewise uniform meshes [3], [1] and [7]. Where the outflow boundary data are only weakly regular and compatible, parameter-uniform solutions may also be obtained by this method [1]. However, for large values of ε (and by extension for small ε with a sufficiently large number of mesh intervals), orders of convergence are small and pointwise errors are large.

Numerical methods for singularly perturbed problems comprising domain decomposition and Schwarz iterative techniques have been examined by a number of authors, for example, in [3], [7], [6], [4] and [5]. In [3], Miller et al. examine a continuous overlapping Schwarz method for a singularly perturbed convection-diffusion equation with arbitrary fixed interface positions and find it to be uniformly convergent with respect to the perturbation parameter. In [5] MacMullen et al. consider the corresponding discrete overlapping Schwarz method for the same problem and find that, in the discrete case the numerical solution obtained converges to the solution of upwinding on a quasi-uniform mesh. Furthermore, they show that if the interface positions for the overlapping discretised domains are based on layer-resolving piecewise uniform fitted meshes then the numerical solutions obtained fail to converge to the analytical solution. As an alternative they construct a discrete non-overlapping Schwarz method on uniform meshes with artificial Dirichlet interface conditions for singularly perturbed linear convection-diffusion problems in two dimensions with sufficiently smooth and compatible boundary data and show it to be first order convergent for $\varepsilon \leq N^{-1}$. We examine experimentally the performance of such methods extended to the class of singularly perturbed convection-diffusion problems with more general boundary conditions described below.

We consider the following model problem in a domain Ω, the unit square.

$$Lu \equiv \varepsilon \Delta u_\varepsilon + \frac{\partial u_\varepsilon}{\partial x} + \frac{\partial u_\varepsilon}{\partial y} = 0 \; in \; \Omega, \qquad (1)$$

$$u_\varepsilon(x,0) = 5 - 4x^{1/6},\; u_\varepsilon(x,1) = 1,\; x \in (0,1)$$
$$u_\varepsilon(0,y) = y^{1/2},\; u_\varepsilon(1,y) = 1 \; y \in (0,1)$$

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where $0 < \varepsilon \leq 1$. The problem exhibits regular layers along the outflow boundaries, a corner boundary layer at the outflow boundary corner as well as a singularity of the solution discontinuity in a neighbourhood of this corner. We implement domain decomposition methods to isolate the neighbourhood of the singularity, along with a discrete Schwarz iterative technique with the aim of producing parameter-uniformly accurate solutions for all values of epsilon on the whole domain in the presence of such a singularity.

2. Numerical Methods and Results

2..1. Non-iterative discrete method on piecewise uniform fitted meshes

Hereafter referred to as the Direct Method. A tensor product of two piecewise-uniform fitted meshes Ω^N is used on Ω, where the transition parameter σ is chosen as

$$\sigma = \min\{1/2, \varepsilon \ln N\} \tag{2}$$

along with the upwind finite difference operator

$$L_\varepsilon^N Z_i^j = \left[\varepsilon(\delta_x^2 + \delta_y^2) + D_x^+ + D_y^+\right] Z_i^j. \tag{3}$$

The differences between the numerical solutions for various values of N and the numerical solution for $N = 256$, which are indicative of nodal errors are shown in Table 1, where

$$E_\varepsilon^N = \max_{x_i, y_j \in \Omega_\varepsilon^N} |U_\varepsilon^N - \bar{U}_\varepsilon^{256}|,\ E^N = \max_\varepsilon E_\varepsilon^N.$$

The computed orders of convergence for various values of N and ε, defined by

$$D_\varepsilon^N = \max_{x_i, y_j \in \Omega_\varepsilon^N} |U_\varepsilon^N - \bar{U}_\varepsilon^{2N}|,$$
$$p_\varepsilon^N = \log_2 D_\varepsilon^N / D_\varepsilon^{2N}.$$

are shown in Table 2. It is clear from Tables (1) and (2) that the method fails for problem (1) for large values of ε. Furthermore, while at first glance these results appear to be satisfactory for small ε, as N increases the results worsen for smaller values of ε. It can therefore be inferred that the bad behaviour for large ε will be replicated for small ε where N is large enough. We therefore conclude that the direct method fails when applied to problem (1).

2..2. Discrete overlapping Schwarz methods

Domain decomposition with restricted overlap region

Given the location of the incompatibility it is natural to consider partitioning the solution domain Ω into the following two overlapping subregions

$$\Omega_S = (0,1)^2 \backslash (0, \frac{\sigma}{a}) \times (0, \frac{\sigma}{b}),\ a, b > 1$$
$$\Omega_P = \{(x,y) \in (0,R)^2 : x^2 + y^2 < R^2\},\ R = C_R\sigma,\ C_R < 1$$

where C_R is a constant, pictured in Figure (1).

It is important to note that, in order for the discrete overlapping Schwarz method to converge to the correct solution, the overlap region between subdomains must be sufficiently large. From the definition of Ω_S and Ω_P it follows that the width of the overlap region at its minimum point is

$$\left(C_R - \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}\right)\sigma,$$

Table 1: Maximum pointwise errors E_ε^N and E^N for the Direct Method applied to problem (1).

ε	Number of intervals N			
	16	32	64	128
1	0.0218	0.0243	0.0244	0.0191
2^{-1}	0.0184	0.0227	0.0237	0.0188
2^{-2}	0.0122	0.0190	0.0223	0.0183
2^{-3}	0.0242	0.0150	0.0186	0.0172
2^{-4}	0.0390	0.0223	0.0176	0.0140
2^{-6}	0.0835	0.0471	0.0227	0.0145
2^{-8}	0.1202	0.0739	0.0376	0.0149
2^{-10}	0.1412	0.0907	0.0489	0.0192
2^{-12}	0.1535	0.0992	0.0543	0.0220
2^{-14}	0.1620	0.1045	0.0569	0.0230
2^{-16}	0.1687	0.1084	0.0587	0.0234
2^{-18}	0.1740	0.1116	0.0601	0.0238
2^{-20}	0.1782	0.1141	0.0612	0.0240
2^{-22}	0.1816	0.1161	0.0621	0.0243
2^{-24}	0.1843	0.1177	0.0628	0.0245
2^{-26}	0.1865	0.1189	0.0634	0.0246
2^{-28}	0.1882	0.1199	0.0638	0.0248
E^N	0.1882	0.1199	0.0638	0.0248

Table 2: Computed orders of convergence p^N for the Direct Method applied to problem (1).

ε	Number of intervals N		
	16	32	64
1	-0.18	-0.11	-0.08
2^{-1}	-0.30	-0.16	-0.10
2^{-2}	-0.70	-0.30	-0.15
2^{-3}	0.50	-0.17	-0.30
2^{-4}	0.40	0.00	0.00
2^{-6}	0.63	0.74	0.00
2^{-8}	0.43	0.66	0.69
2^{-10}	0.36	0.55	0.68
2^{-12}	0.35	0.53	0.62
2^{-14}	0.35	0.53	0.62
2^{-16}	0.34	0.54	0.63
2^{-18}	0.34	0.55	0.64
2^{-20}	0.34	0.56	0.65
2^{-22}	0.34	0.56	0.65
2^{-24}	0.34	0.57	0.66
2^{-26}	0.34	0.57	0.66
2^{-28}	0.34	0.57	0.66

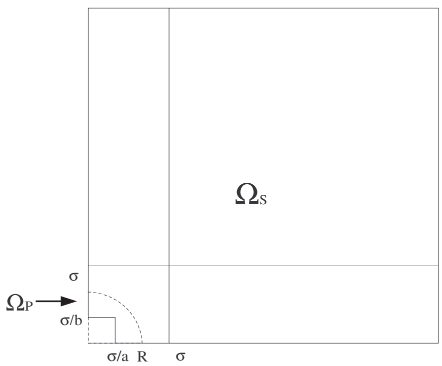


Figure 1: Domain decomposition with restricted overlap region

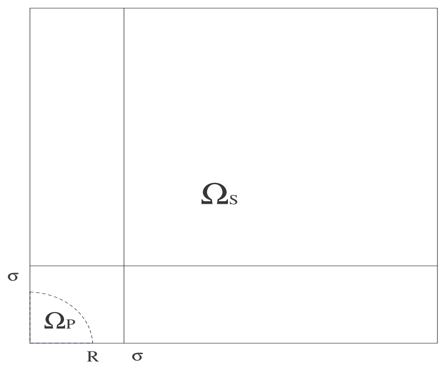


Figure 2: Domain decomposition with maximal overlap region

thus placing the restriction on our choice of C_R that

$$C_R > \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}.$$

However, in addition to considering the necessary overlap width required by the discrete Schwarz method for convergence, we must also bear in mind the choice of R, the radius of the quarter-disk subregion. In the presence of the incompatibility we require R to be small in order to cluster more grid points in the neighbourhood of the singularity. Since the width of the overlap is proportional to σ, for large values of ε the above restriction imposed by this decomposition prevents us from choosing R sufficiently small to resolve the singularity.

Domain decomposition with maximal overlap region

We overcome this obstacle by letting

$$\Omega_S = \Omega$$

thereby removing the restriction on the choice of R, which we define as

$$R = \sqrt{2}\sigma/8$$

ensuring that the overlap region is sufficiently large for the convergence of the Schwarz method, pictured in Figure (2).

A tensor product of two piecewise-uniform fitted meshes Ω_S^N is used on Ω, with σ defined as in (2). Uniform meshes Ω_P^N are used on Ω_P.

Schwarz Method 1

On Ω_P we use the following translation

$$v = e^{(x+y)/2\varepsilon} u_\varepsilon,$$

yielding the translated equation

$$2\varepsilon^2 \Delta v - v = 0.$$

In polar coordinates the equation becomes

$$2\varepsilon^2 (v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta}) - v = 0.$$

Our discrete iterative method is then as follows:
For each $k \geq 1$*,*

$$U_\varepsilon^{[k]}(x,y) = \begin{cases} U_S^{[k]}(x,y), & (x,y) \in \Omega_S^N \\ \bar{U}_P^{[k]}(x,y), & (x,y) \in \Omega_P \cap \Omega_S^N \end{cases}$$

where $\bar{U}_i^{[k]}$ *is the bilinear interpolant of* $U_i^{[k]}$*. Then for* $k = 1$*,*

$$L_\varepsilon^N U_S^{[1]} = 0, \ (x_i, y_j) \in \Omega_S^N,$$

and

$$L_P^N U_P^{[1]} = 0, \ (r_i, \theta_j) \in \Omega_P^N, \ U_P^{[1]}(R, \theta_j) = \bar{U}_S^{[1]}(R, \theta_j),$$

where $U_P^{[1]}(0, \theta_j)$ *is a linear interpolant of* $g_B(0)$ *and* $g_L(0)$*.*
For $k > 1$*,*

$$L_\varepsilon^N U_S^{[k]} = 0, \ (x_i, y_j) \in \Omega_S^N,$$
$$U_S^{[k]}(x_i, y_j) = \bar{U}_P^{[k-1]}(x_i, y_j), \ x_i \in (0, \sigma/8), \ y_j \in (0, \sigma/8),$$

and

$$L_P^N U_P^{[k]} = 0, \ (r_i, \theta_j) \in \Omega_P^N, \ U_P^{[k]}(R, \theta_j) = \bar{U}_S^{[k-1]}(R, \theta_j),$$

where $U_P^{[1]}(0,\theta_j)$ *is a linear interpolant of* $g_B(0)$ *and* $g_L(0)$*.*

We define the finite difference operators as

$$L_\varepsilon^N Z_i^j = \left[\varepsilon(\delta_x^2+\delta_y^2)+D_x^+ + D_y^+\right] Z_i^j$$
$$L_P^N Z_i^j = 2\varepsilon^2\left[\delta_r^2+\frac{1}{r_i}D_r^0+\frac{1}{r_i^2}\delta_\theta^2\right] Z_i^j - Z_i^j \tag{4}$$

where

$$\delta_x^2 Z(x_i,y_i) = \frac{2}{x_{i+1}-x_{i-1}}(D_x^+ - D_x^-)Z(x_i,y_j)$$

with

$$D_x^+ Z(x_i,y_j) = \frac{Z(x_{i+1},y_j)-Z(x_i,y_j)}{x_{i+1}-x_i},\ D_x^- Z(x_i,y_j) = \frac{Z(x_i,y_j)-Z(x_{i-1},y_j)}{x_i-x_{i-1}},$$
$$D_x^0 Z(x_i,y_j) = \frac{Z(x_{i+1},y_j)-Z(x_{i-1},y_j)}{x_{i+1}-x_{i-1}}.$$

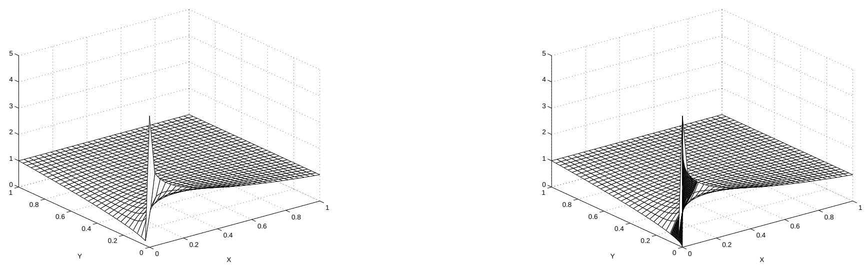
The numerical solutions are depicted in Figures 3 and 4 for $N = 32$ and $\varepsilon = 2^{-1}$ and 2^{-6} respectively, alongside plots of the numerical solutions obtained by the Direct Method for the same parameters for the purpose of comparison. In Table 3 the required iteration counts are given for a tolerance level of

$$\max_{x_i,y_j\in\bar\Omega_\varepsilon^N} |U_S^{[k]}(x_i,y_j) - U_S^{[k-1]}(x_i,y_j)| \le 10^{-4}$$

The differences between the numerical solutions for various values of N and the numerical solution for $N = 256$, which are indicative of nodal errors are shown in Table 4 and the computed orders of convergence for various values of N and ε are shown in Table 5. Both tables show a notable improvement in the magnitude of the error and the orders of convergence for large values of ε on the results obtained by the Direct Method, seen in Tables 1 and 2. However, there is a noticeable drop in the convergence rates for intermediate values of ε which requires further consideration.

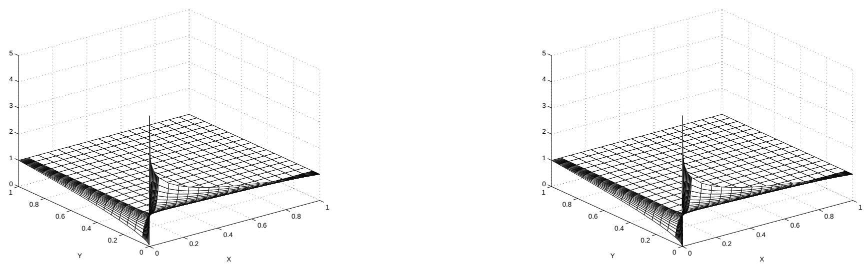
Table 3: Iteration counts for Schwarz Method 1 applied to problem (1) .

	Number of intervals N				
ε	16	32	64	128	256
1	3	6	5	4	3
2^{-1}	3	6	5	4	3
2^{-2}	3	6	5	4	3
2^{-3}	3	6	5	3	3
2^{-4}	3	6	5	3	4
2^{-6}	3	6	5	4	4
2^{-8}	3	6	5	4	4
2^{-10}	3	6	5	4	4
2^{-12}	2	6	5	4	4
2^{-14}	2	6	5	4	4
2^{-16}	2	6	5	4	4
2^{-18}	2	6	5	4	4
2^{-20}	2	6	5	4	4
2^{-22}	2	6	5	4	4
2^{-24}	2	6	5	4	4
2^{-26}	2	6	5	4	4
2^{-28}	2	6	5	4	4



Direct Method   Schwarz Method

Figure 3: Numerical solutions of problem (1) for $\varepsilon = 2^{-1},\ N = 32$



Direct Method   Schwarz Method

Figure 4: Numerical solutions of problem (1) for $\varepsilon = 2^{-6},\ N = 32$

Table 4: Maximum pointwise errors E_ε^N *and* E^N for Schwarz Method 1 applied to problem (1).

	Number of intervals N			
ε	16	32	64	128
1	0.0218	0.0076	0.0019	0.0004
2^{-1}	0.0184	0.0065	0.0015	0.0003
2^{-2}	0.0122	0.0054	0.0025	0.0008
2^{-3}	0.0242	0.0150	0.0080	0.0028
2^{-4}	0.0390	0.0223	0.0114	0.0087
2^{-6}	0.0835	0.0471	0.0227	0.0082
2^{-8}	0.1202	0.0739	0.0376	0.0138
2^{-10}	0.1412	0.0907	0.0489	0.0192
2^{-12}	0.1535	0.0992	0.0543	0.0220
2^{-14}	0.1620	0.1045	0.0569	0.0230
2^{-16}	0.1687	0.1084	0.0587	0.0234
2^{-18}	0.1740	0.1116	0.0601	0.0238
2^{-20}	0.1783	0.1141	0.0612	0.0240
2^{-22}	0.1816	0.1161	0.0621	0.0243
2^{-24}	0.1843	0.1177	0.0628	0.0245
2^{-26}	0.1865	0.1189	0.0634	0.0246
2^{-28}	0.1882	0.1199	0.0638	0.0248
E^N	0.1882	0.1199	0.0638	0.0248

Table 5: Computed orders of convergence p_ε^N for Schwarz Method 1 applied to problem (1).

	Number of intervals N		
ε	16	32	64
1	1.49	1.91	1.92
2^{-1}	1.48	2.02	2.05
2^{-2}	1.13	1.34	0.93
2^{-3}	0.50	1.23	0.91
2^{-4}	0.65	0.28	0.28
2^{-6}	0.63	0.74	0.82
2^{-8}	0.43	0.66	0.81
2^{-10}	0.36	0.55	0.68
2^{-12}	0.35	0.53	0.62
2^{-14}	0.35	0.53	0.62
2^{-16}	0.34	0.54	0.63
2^{-18}	0.34	0.55	0.64
2^{-20}	0.34	0.56	0.65
2^{-22}	0.34	0.56	0.65
2^{-24}	0.34	0.57	0.66
2^{-26}	0.34	0.57	0.66
2^{-28}	0.34	0.57	0.66

Schwarz Method 2

On Ω_P we use the alternative translation

$$w = e^{(x+y)/2\varepsilon} u_\varepsilon - \phi(x, y)$$

where ϕ is defined as follows

$$\phi(x, y) = 10/\pi(\pi/2 - \arctan(y/x))$$

yielding the translated equation

$$2\varepsilon^2 \Delta w - w = \phi.$$

This has the effect of resolving the incompatibilty at the outflow corner. Solving the translated equation using the domain decomposition, finite difference operators (4) and iterative method as before, the numerical results obtained do not differ appreciably from those for Schwarz Method 1.

Schwarz Method 3

It is interesting to consider an alternative Schwarz method with a square subregion, $\Omega_C = (0,R)^2,\ R = \sigma/8$ at the outflow corner replacing the quarter-disk region used in the two Schwarz methods above. The finite difference operator in this region is then the same as that used on the entire domain, (3). The iterative method is essentially similar to that used in the previous two Schwarz methods. Tables (6) and (7) show that the performance of this method is no better than that of the direct method for large values of epsilon when applied to problem (1).

Table 6: Maximum pointwise errors E_ε^N *and* E^N for Schwarz Method 3 applied to problem (1).

ε	Number of intervals N			
	16	32	64	128
1	0.0221	0.0246	0.0244	0.0191
2^{-1}	0.0190	0.0235	0.0238	0.0188
2^{-2}	0.0110	0.0204	0.0223	0.0183
2^{-3}	0.0242	0.0150	0.0187	0.0172
2^{-4}	0.0390	0.0223	0.0171	0.0139
2^{-6}	0.0835	0.0471	0.0227	0.0145
2^{-8}	0.1202	0.0739	0.0376	0.0149
2^{-10}	0.1412	0.0907	0.0489	0.0192
2^{-12}	0.1535	0.0992	0.0543	0.0220
2^{-14}	0.1620	0.1045	0.0569	0.0230
2^{-16}	0.1687	0.1084	0.0587	0.0234
2^{-18}	0.1740	0.1116	0.0601	0.0238
2^{-20}	0.1782	0.1141	0.0612	0.0240
2^{-22}	0.1816	0.1161	0.0621	0.0243
2^{-24}	0.1843	0.1177	0.0628	0.0245
2^{-26}	0.1865	0.1189	0.0634	0.0246
2^{-28}	0.1882	0.1199	0.0638	0.0248
E^N	0.1882	0.1199	0.0638	0.0248

Table 7: Computed orders of convergence p_ε^N for Schwarz Method 3 applied to problem (1).

ε	Number of intervals N		
	16	32	64
1	-0.17	-0.10	-0.07
2^{-1}	-0.30	-0.13	-0.09
2^{-2}	-0.74	-0.23	-0.14
2^{-3}	0.50	-0.18	-0.29
2^{-4}	0.42	0.01	-0.01
2^{-6}	0.63	0.74	0.00
2^{-8}	0.43	0.66	0.70
2^{-10}	0.36	0.55	0.68
2^{-12}	0.35	0.53	0.62
2^{-14}	0.35	0.53	0.62
2^{-16}	0.34	0.54	0.63
2^{-18}	0.34	0.55	0.64
2^{-20}	0.34	0.56	0.65
2^{-22}	0.34	0.56	0.65
2^{-24}	0.34	0.57	0.66
2^{-26}	0.34	0.57	0.66
2^{-28}	0.34	0.57	0.66

In conclusion, while the results obtained by Schwarz Methods 1 and 2 applied to problem (1) are an improvement on those obtained by the Direct Method for this problem, further development of the method is required in order to solve problem (1) ε-uniformly.

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