Int. Conference on Boundary and Interior Layers
BAIL 2006
G. Lube, G. Rapin (Eds)

A second order uniform convergent method for a singularly perturbed parabolic system of reaction–diffusion type*

C. Clavero, J.L. Gracia and F. Lisbona

Department of Applied Mathematics
University of Zaragoza. Spain
clavero@unizar.es, jlgracia@unizar.es, lisbona@unizar.es

1. Introduction

In this work we are interested in solving singularly perturbed parabolic boundary value problems given by

$$\begin{cases} L_{\vec{\varepsilon}}\vec{u} \equiv \dfrac{\partial \vec{u}}{\partial t} + L_{x,\vec{\varepsilon}}\vec{u} = \vec{f}, \qquad (x,t) \in Q = \Omega \times (0,T] = (0,1) \times (0,T], \\ \vec{u}(0,t) = \vec{0},\ \vec{u}(1,t) = \vec{0}, \quad \forall t \in [0,T], \\ \vec{u}(x,0) = \vec{0}, \quad \forall x \in \Omega, \end{cases} \tag{1}$$

where

$$L_{x,\vec{\varepsilon}} \equiv \begin{pmatrix} -\varepsilon_1 \frac{\partial^2}{\partial x^2} & \\ & -\varepsilon_2 \frac{\partial^2}{\partial x^2} \end{pmatrix} + A, \qquad A = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix},$$

with $0 < \varepsilon_1 \le \varepsilon_2 \le 1$. The components of the right hand side $\vec{f}(x,t) = (f_1(x,t), f_2(x,t))^T$ and the matrix A are supposed smooth enough and also that the following positivity condition is satisfied

$$a_{i,1} + a_{i,2} \ge \alpha > 0,\ a_{ii} > 0, \quad i = 1,2, \quad a_{ij} \le 0 \ \text{ if } \ i \ne j. \tag{2}$$

In addition, we suppose that sufficient compatibility conditions among the data of the differential equation hold, in order that the exact solution $\vec{u} \in C^{4,3}(\bar{Q})$, i.e, continuity up to fourth order in space and up to third order in time.

This problem is a simple model of the classical linear double–diffusion model for saturated flow in fractured porous media (Barenblatt system) developed in [1]. Also this problem can be used to model diffusion process in bones (see [4]). It is well–known that the exact solution of these problems has a multiscale character, i.e., there are boundary layers. Therefore, it is necessary to dispose of efficient numerical methods (uniformly convergent methods) to approximate the solution independently of the values of the diffusion parameters ε_1 and ε_2.

Recently some papers (see [7], [8], [9] [10] and [11]) study uniform convergent numerical methods to solve singularly perturbed elliptic linear systems on a special piecewise uniform Shishkin mesh, for different relations between the diffusion parameters: $i)$ $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = 1$; $ii)$ $\varepsilon_1 = \varepsilon_2 = \varepsilon$; $iii)$ $\varepsilon_1, \varepsilon_2$ arbitrary. Here, we are interested in increasing the uniform convergence order of the numerical method given in [6], which was used to solve a parabolic coupled system of type (1). With this aim we will combine the Crank-Nicolson method to discretize the time variable joint to the central finite differences discretization in space. Previously this method has been used in the framework of singularly perturbed problems; for instance, in [2] it was considered to solve a class of 1D parabolic problems of convection-diffusion type.

We denote by $\Gamma_0 = \{(x,0) \,|\, x \in \Omega\}$, $\Gamma_1 = \{(x,t) \,|\, x = 0,1,\ t \in [0,T]\}$, $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)^T$, with $0 < \varepsilon_1 \le \varepsilon_2 \le 1$, the vectorial singular perturbation parameter. We write

*This research will be partially supported by the project MEC/FEDER MTM 2004-01905 and the Diputación General de Aragón.

$\vec{v} \leq \vec{w}$ if $v_i \leq w_i$, $i = 1, 2$, $|\vec{v}| = (|v_1|, |v_2|)^T$, $\vec{C} = (C, C)^T$, where C is any positive constant independent of the diffusion parameters ε_1, ε_2 and the discretization parameters N and Δt, $\|f\|_H$ is the maximum norm of f on the closed set H and $\|\vec{f}\|_H = \max\{\|f_1\|_H, \|f_2\|_H\}$.

2. The numerical method: uniform convergence

Before defining the numerical method we give some results showing the asymptotic behaviour of the exact solution of the problem (1); the proofs of these results can be found in [3] and [6]. First, it is easy to prove that the differential operator satisfies a maximum principle: If $\vec{\psi} \geq \vec{0}$ on Γ and $L_{\vec{\varepsilon}}\vec{\psi} \geq \vec{0}$ in Q, then $\vec{\psi} \geq \vec{0}$ for all $(x, t) \in \bar{Q}$.

Now we consider the decomposition $\vec{u} = \vec{v} + \vec{w}$, where the regular component $\vec{v}$ is the solution of the problem

$$\begin{cases} L_{\vec{\varepsilon}}\vec{v} = \vec{f}, \quad \text{in } Q, \\ \vec{v}(x,0) = \vec{0}, \quad \text{on } \Gamma_0, \\ \vec{v} = \vec{z}, \quad \text{on } \Gamma_1, \end{cases} \tag{3}$$

where $\vec{z}$ satisfies the following initial value problem

$$\begin{cases} \vec{z}_t + A\vec{z} = \vec{f}, \quad (x,t) \in \{0,1\} \times (0,T], \\ \vec{z}(x,0) = \vec{0} \quad x \in \{0,1\}, \end{cases} \tag{4}$$

and the singular component $\vec{w}$ is the solution of the problem

$$\begin{cases} L_{\vec{\varepsilon}}\vec{w} = \vec{0}, \quad \text{in } Q, \\ \vec{w} = \vec{u} - \vec{v}, \quad \text{on } \Gamma. \end{cases} \tag{5}$$

Lemma 1 *The regular component $\vec{v} = (v_1, v_2)^T$ satisfies*

$$\begin{gathered} \left\| \frac{\partial^k \vec{v}}{\partial t^k} \right\|_{\bar{Q}} \leq C, \ 0 \leq k \leq 3, \quad \left\| \frac{\partial^k \vec{v}}{\partial x^k} \right\|_{\bar{Q}} \leq C, \ k = 1, 2, \\ \left\| \frac{\partial^k v_1}{\partial x^k} \right\|_{\bar{Q}} \leq C\varepsilon_1^{1-k/2}, \quad \left\| \frac{\partial^k v_2}{\partial x^k} \right\|_{\bar{Q}} \leq C\varepsilon_2^{1-k/2}, \ k = 3, 4, \\ \left\| \frac{\partial^2 \vec{v}}{\partial t \partial x} \right\|_{\bar{Q}} \leq C, \quad \left\| \frac{\partial^3 \vec{v}}{\partial t \partial x^2} \right\|_{\bar{Q}} \leq C, \quad \left\| \frac{\partial^3 \vec{v}}{\partial t^2 \partial x} \right\|_{\bar{Q}} \leq C, \quad \left\| \frac{\partial^4 \vec{v}}{\partial t^2 \partial x^2} \right\|_{\bar{Q}} \leq C. \end{gathered} \tag{6}$$

To establish the behaviour of the singular component we introduce the auxiliary function $B_\gamma(x) = e^{-x\sqrt{\alpha/\gamma}} + e^{-(1-x)\sqrt{\alpha/\gamma}}$, where γ is an arbitrary positive constant and α was defined in (2).

Lemma 2 *The singular component $\vec{w} = (w_1, w_2)^T$ satisfies*

$$\left| \frac{\partial^k \vec{w}}{\partial t^k} \right| \leq B_{\varepsilon_2}(x)\vec{C}, \quad \forall (x,t) \in \bar{Q}, \ 0 \leq k \leq 3, \tag{7}$$

and also

$$|w_1(x)| \le CB_{\varepsilon_2}(x), \quad |w_2(x)| \le CB_{\varepsilon_2}(x), \tag{8}$$

$$\left|\frac{\partial w_1}{\partial x}\right| \le C(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + {\varepsilon_2}^{-1/2}B_{\varepsilon_2}(x)), \quad \left|\frac{\partial w_2}{\partial x}\right| \le C{\varepsilon_2}^{-1/2}B_{\varepsilon_2}(x), \tag{9}$$

$$\left|\frac{\partial^2 w_1}{\partial x^2}\right| \le C(\varepsilon_1^{-1}B_{\varepsilon_1}(x) + {\varepsilon_2}^{-1}B_{\varepsilon_2}(x)), \quad \left|\frac{\partial^2 w_2}{\partial x^2}\right| \le C{\varepsilon_2}^{-1}B_{\varepsilon_2}(x), \tag{10}$$

$$\left|\frac{\partial^3 w_1}{\partial x^3}\right| \le C(\varepsilon_1^{-3/2}B_{\varepsilon_1}(x) + {\varepsilon_2}^{-3/2}B_{\varepsilon_2}(x)), \tag{11}$$

$$\left|\frac{\partial^3 w_2}{\partial x^3}\right| \le C{\varepsilon_2}^{-1}(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + {\varepsilon_2}^{-1/2}B_{\varepsilon_2}(x)), \tag{12}$$

$$\left|\frac{\partial^4 w_1}{\partial x^4}\right| \le C(\varepsilon_1^{-2}B_{\varepsilon_1}(x) + {\varepsilon_2}^{-2}B_{\varepsilon_2}(x)), \tag{13}$$

$$\left|\frac{\partial^4 w_2}{\partial x^4}\right| \le C{\varepsilon_2}^{-1}(\varepsilon_1^{-1}B_{\varepsilon_1}(x) + {\varepsilon_2}^{-1}B_{\varepsilon_2}(x)). \tag{14}$$

To approximate the solution of (1), firstly we use the Crank–Nicolson method to discretize in time. This scheme, on a uniform mesh $\bar{\omega}^M = \{k\Delta t,\ 0 \le k \le M,\ \Delta t = T/M\}$, can be written as

$$\begin{cases} \vec{u}^0 = \vec{u}(x,0) = \vec{0}, \\ \begin{cases} \left(I + \dfrac{\Delta t}{2}L_{x,\vec{\varepsilon}}\right)\vec{u}^{n+1} = \dfrac{\Delta t}{2}(\vec{f}^n + \vec{f}^{n+1}) + \left(I - \dfrac{\Delta t}{2}L_{x,\vec{\varepsilon}}\right)\vec{u}^n, \quad n = 0,1,\cdots,M-1, \\ \vec{u}^{n+1}(0) = \vec{0}, \quad \vec{u}^{n+1}(1) = \vec{0}, \end{cases} \end{cases} \tag{15}$$

where $\vec{f}^n = \vec{f}(x,t_n)$. In order to study the convergence of this method, we consider the following auxiliary problem

$$\begin{cases} \widehat{\vec{u}}^n = \vec{u}(x,t_n), \\ \begin{cases} \left(I + \dfrac{\Delta t}{2}L_{x,\vec{\varepsilon}}\right)\widehat{\vec{u}}^{n+1} = \dfrac{\Delta t}{2}(\vec{f}^n + \vec{f}^{n+1}) + \left(I - \dfrac{\Delta t}{2}L_{x,\vec{\varepsilon}}\right)\vec{u}(x,t_n), \\ \widehat{\vec{u}}^{n+1}(0) = \vec{0}, \quad \widehat{\vec{u}}^{n+1}(1) = \vec{0}. \end{cases} \end{cases} \tag{16}$$

To study the local error and also the uniform-stability of this method, we follow similar ideas to these ones developed in [2], and therefore we obtain the following result.

Theorem 3 *The local and global errors associated to the Crank-Nicolson method satisfy*

$$\|\vec{u}(x,t_{n+1}) - \widehat{\vec{u}}^{n+1}(x)\|_{\bar{\Omega}} = \mathcal{O}((\Delta t)^3), \quad \|\vec{u}(x,t_{n+1}) - \vec{u}^{n+1}(x)\|_{\bar{\Omega}} = \mathcal{O}((\Delta t)^2).$$

Then, the Crank-Nicolson method is uniformly convergent of second order.

Nevertheless, for the posterior analysis we need a more precise information about the asymptotic behaviour of the components and their derivatives of the exact solution of the problem (16). Then, we write this solution as a sum of the regular and the singular components, i.e., $\widehat{\vec{u}}^{n+1} = \widehat{\vec{v}}^{n+1} + \widehat{\vec{w}}^{n+1}$, which are respectively the solution of the following problems:

$$\left(I + \frac{\Delta t}{2}L_{x,\vec{\varepsilon}}\right)\widehat{\vec{v}}^{n+1}(x) = \frac{\Delta t}{2}(\vec{f}^n + \vec{f}^{n+1}) + \left(I - \frac{\Delta t}{2}L_{x,\vec{\varepsilon}}\right)\vec{v}(x,t_n), \quad x \in (0,1), \tag{17}$$

$$\left(I + \frac{\Delta t}{2}A\right)\widehat{\vec{v}}^{n+1}(x) = \frac{\Delta t}{2}(\vec{f}^n + \vec{f}^{n+1}) + \left(I - \frac{\Delta t}{2}A\right)\vec{v}(x,t_n), \quad x = 0,1, \tag{18}$$

and

$$\left(I + \frac{\Delta t}{2} L_{x,\vec{\varepsilon}}\right) \widehat{\vec{w}}^{n+1} = \left(I - \frac{\Delta t}{2} L_{x,\vec{\varepsilon}}\right) \vec{w}(x, t_n), \tag{19}$$

$$\widehat{\vec{w}}^{n+1}(0) = \widehat{\vec{u}}^{n+1}(0) - \widehat{\vec{v}}^{n+1}(0), \quad \widehat{\vec{w}}^{n+1}(1) = \widehat{\vec{u}}^{n+1}(1) - \widehat{\vec{v}}^{n+1}(1), \tag{20}$$

where $\vec{v}$ is the regular component, solution of (3)-(4), and $\vec{w}$ is the singular component, solution of (5).

Lemma 4 *The regular component* $\widehat{\vec{v}}^{n+1} = (\widehat{v}_1^{n+1}, \widehat{v}_2^{n+1})^T$ *satisfies*

$$\left\| \frac{d^k \widehat{v}_i^{n+1}}{dx^k} \right\|_{\bar{\Omega}} \le C, \ 0 \le k \le 2, \quad \left\| \frac{d^k \widehat{v}_i^{n+1}}{dx^k} \right\|_{\bar{\Omega}} \le C \varepsilon_i^{1-k/2}, \ 3 \le k \le 4, \ i = 1, 2. \tag{21}$$

The singular component $\widehat{\vec{w}}^{n+1} = (\widehat{w}_1^{n+1}, \widehat{w}_2^{n+1})^T$ *satisfies*

$$|\widehat{w}_1^{n+1}(x)| \le C B_{\varepsilon_2}(x), \quad |\widehat{w}_2^{n+1}(x)| \le C B_{\varepsilon_2}(x), \tag{22}$$

$$\left| \frac{d\widehat{w}_1^{n+1}}{dx} \right| \le C(\varepsilon_1^{-1/2} B_{\varepsilon_1}(x) + \varepsilon_2{}^{-1/2} B_{\varepsilon_2}(x)), \quad \left| \frac{d\widehat{w}_2^{n+1}}{dx} \right| \le C \varepsilon_2{}^{-1/2} B_{\varepsilon_2}(x), \tag{23}$$

$$\left| \frac{d^2\widehat{w}_1^{n+1}}{dx^2} \right| \le C(\varepsilon_1^{-1} B_{\varepsilon_1}(x) + \varepsilon_2{}^{-1} B_{\varepsilon_2}(x)), \quad \left| \frac{d^2\widehat{w}_2^{n+1}}{dx^2} \right| \le C \varepsilon_2{}^{-1} B_{\varepsilon_2}(x), \tag{24}$$

$$\left| \frac{d^3\widehat{w}_1^{n+1}}{dx^3} \right| \le C \varepsilon_1^{-1}(\varepsilon_1^{-1/2} B_{\varepsilon_1}(x) + \varepsilon_2{}^{-1/2} B_{\varepsilon_2}(x)), \tag{25}$$

$$\left| \frac{d^3\widehat{w}_2^{n+1}}{dx^3} \right| \le C \varepsilon_2{}^{-1}(\varepsilon_1^{-1/2} B_{\varepsilon_1}(x) + \varepsilon_2{}^{-1/2} B_{\varepsilon_2}(x)), \tag{26}$$

$$\left| \frac{d^4\widehat{w}_1^{n+1}}{dx^4} \right| \le C \varepsilon_1^{-1}(\varepsilon_1^{-1} B_{\varepsilon_1}(x) + \varepsilon_2{}^{-1} B_{\varepsilon_2}(x)), \tag{27}$$

$$\left| \frac{d^4\widehat{w}_2^{n+1}}{dx^4} \right| \le C \varepsilon_2{}^{-1}(\varepsilon_1^{-1} B_{\varepsilon_1}(x) + \varepsilon_2{}^{-1} B_{\varepsilon_2}(x)). \tag{28}$$

Moreover, if $\varepsilon_1 < \varepsilon_2$*, then the singular component can be decomposed as*

$$\widehat{w}_1^{n+1} = \widehat{w}_{1,\varepsilon_1}^{n+1} + \widehat{w}_{1,\varepsilon_2}^{n+1}, \quad \widehat{w}_2^{n+1} = \widehat{w}_{2,\varepsilon_1}^{n+1} + \widehat{w}_{2,\varepsilon_2}^{n+1},$$

where

$$\left| \frac{d^2\widehat{w}_{1,\varepsilon_1}^{n+1}}{dx^2} \right| \le C \varepsilon_1^{-1} B_{\varepsilon_1}(x), \quad \left| \frac{d^3\widehat{w}_{1,\varepsilon_2}^{n+1}}{dx^3} \right| \le C \varepsilon_1{}^{-1} \varepsilon_2{}^{-1/2} B_{\varepsilon_2}(x), \tag{29}$$

$$\left| \frac{d^2\widehat{w}_{2,\varepsilon_1}^{n+1}}{dx^2} \right| \le C \varepsilon_2{}^{-1} B_{\varepsilon_1}(x), \quad \left| \frac{d^3\widehat{w}_{2,\varepsilon_2}^{n+1}}{dx^3} \right| \le C \varepsilon_2{}^{-3/2} B_{\varepsilon_2}(x). \tag{30}$$

Also, if $\varepsilon_1 < \varepsilon_2$ *the singular component can be decomposed as*

$$\widehat{w}_1^{n+1} = \widehat{z}_{1,\varepsilon_1}^{n+1} + \widehat{z}_{1,\varepsilon_2}^{n+1}, \quad \widehat{w}_2^{n+1} = \widehat{z}_{2,\varepsilon_1}^{n+1} + \widehat{z}_{2,\varepsilon_2}^{n+1},$$

where

$$\left| \frac{d^2\widehat{z}_{1,\varepsilon_1}^{n+1}}{dx^2} \right| \le C \varepsilon_1^{-1} B_{\varepsilon_1}(x), \quad \left| \frac{d^4\widehat{z}_{1,\varepsilon_2}^{n+1}}{dx^4} \right| \le C \varepsilon_1{}^{-1} \varepsilon_2{}^{-1} B_{\varepsilon_2}(x), \tag{31}$$

$$\left| \frac{d^2\widehat{z}_{2,\varepsilon_1}^{n+1}}{dx^2} \right| \le C \varepsilon_2{}^{-1} B_{\varepsilon_1}(x), \quad \left| \frac{d^4\widehat{z}_{2,\varepsilon_2}^{n+1}}{dx^4} \right| \le C \varepsilon_2{}^{-2} B_{\varepsilon_2}(x). \tag{32}$$

Remark 5 *We note that the bounds given in Lemma 4 for the third and fourth derivatives of* $\widehat{w}_1$*, are worse (with respect to the diffusion parameters) than the corresponding ones for the singular component* $\dfrac{\partial^3 w_1}{\partial x^3}$ *and* $\dfrac{\partial^4 w_1}{\partial x^4}$ *of the continuous problem. Fortunately, these different bounds do not introduce any additional difficulty in the posterior analysis of the uniform convergence of the spatial discretization of our numerical scheme, because of these derivatives are scaled with the corresponding diffusion parameter* ε_1*.*

Now we use the classical central difference operator defined on a piecewise uniform mesh $\bar{\Omega}^N$ of Shishkin type (see [5]), to discretize the problem (15). To construct this mesh, we must have into account that the solution of the continuous problem, in the spatial direction, has two overlapping boundary layers at both end points $x = 0$ and $x = 1$. Then, the mesh is defined by using two transition parameters (see [8] for a theoretical justification), which are defined by

$$\tau_{\varepsilon_2} = \min\left\{1/4, 2\sqrt{\varepsilon_2/\alpha}\ln N\right\}, \quad \tau_{\varepsilon_1} = \min\left\{\tau_{\varepsilon_2}/2, 2\sqrt{\varepsilon_1/\alpha}\ln N\right\}.$$

In the five subintervals $[0, \tau_{\varepsilon_1}]$, $[\tau_{\varepsilon_1}, \tau_{\varepsilon_2}]$, $[\tau_{\varepsilon_2}, 1-\tau_{\varepsilon_2}]$, $[1-\tau_{\varepsilon_2}, 1-\tau_{\varepsilon_1}]$ and $[1-\tau_{\varepsilon_1}, 1]$ we distribute uniformly $N/8+1$, $N/8+1$, $N/2+1$, $N/8+1$ and $N/8+1$ mesh points respectively. So, the mesh points are given by

$$x_j = \begin{cases} jh_{\varepsilon_1}, & j = 0, \ldots, N/8, \\ x_{N/8} + (j - N/8)h_{\varepsilon_2}, & j = N/8+1, \ldots, N/4, \\ x_{N/4} + (j - N/4)H, & j = N/4+1, \ldots, 3N/4, \\ x_{3N/4} + (j - 3N/4)h_{\varepsilon_2}, & j = 3N/4+1, \ldots, 7N/8, \\ x_{7N/8} + (j - 7N/8)h_{\varepsilon_1}, & j = 7N/8+1, \ldots, N, \end{cases}$$

where $h_{\varepsilon_1} = 8\tau_{\varepsilon_1}/N$, $\quad h_{\varepsilon_2} = 8(\tau_{\varepsilon_2} - \tau_{\varepsilon_1})/N$, $\quad H = 2(1 - 2\tau_{\varepsilon_2})/N$. If $\tau_{\varepsilon_1} \neq 1/8$ and $\tau_{\varepsilon_2} = 1/4$, we modify slightly the mesh points that lie in $[\tau_{\varepsilon_2}, 1 - \tau_{\varepsilon_2}]$ in order that the mesh be uniform outside the boundary layers. Then, in this case, the mesh points are defined by

$$x_j = \begin{cases} jh_{\varepsilon_1}, & j = 0, \ldots, N/8, \\ x_{N/8} + (j - N/8)\hat{H}, & j = N/8+1, \ldots, 7N/8, \\ x_{7N/8} + (j - 7N/8)h_{\varepsilon_1}, & j = 7N/8+1, \ldots, N, \end{cases}$$

where now $\hat{H} = 4(1 - 2\tau_{\varepsilon_1})/(3N)$.

We denote the local step sizes by $h_j = x_j - x_{j-1},\ j = 1, \ldots, N$. On this piecewise uniform mesh, the finite difference scheme is defined by

$$\begin{cases} \vec{U}_j^0 = \vec{0}, \qquad 0 \le j \le N, \\ \begin{cases} \text{For } n = 0, \ldots, M-1, \\ L_{\vec{\varepsilon}}^N \vec{U}_j^{n+1} \equiv \left(I + \dfrac{\Delta t}{2} L_{x,\vec{\varepsilon}}^N\right) \vec{U}_j^{n+1} = \left(I - \dfrac{\Delta t}{2} L_{x,\vec{\varepsilon}}^N\right) \vec{U}_j^n + \dfrac{\Delta t}{2}(\vec{f}_j^{n+1} + \vec{f}_j^n), \\ \vec{U}_0^{n+1} = \vec{U}_N^{n+1} = \vec{0}, \end{cases} \end{cases} \tag{33}$$

where

$$L_{x,\vec{\varepsilon}}^N \equiv \begin{pmatrix} -\varepsilon_1 & \\ & -\varepsilon_2 \end{pmatrix} \delta^2 + AI, \qquad \delta^2 Z_j = \frac{2}{h_j + h_{j+1}} \left(\frac{Z_{j+1} - Z_j}{h_{j+1}} - \frac{Z_j - Z_{j-1}}{h_j}\right).$$

In order to prove the uniform convergence of the totally discrete scheme (33), first we must study the convergence of the central finite difference scheme used to discretize the auxiliary problem (16). The, the scheme is given by

$$\begin{cases} \widehat{\vec{U}}_j^n = \widehat{\vec{u}}^n(x_j), \qquad 0 \le j \le N, \\ \begin{cases} L_{\vec{\varepsilon}}^N \widehat{\vec{U}}_j^{n+1} \equiv \left(I + \dfrac{\Delta t}{2} L_{x,\vec{\varepsilon}}^N\right) \widehat{\vec{U}}_j^{n+1} = \left(I - \dfrac{\Delta t}{2} L_{x,\vec{\varepsilon}}\right) \widehat{\vec{u}}^n(x_j) + \dfrac{\Delta t}{2}(\vec{f}_j^{n+1} + \vec{f}_j^n), \quad 0 < j < N, \\ \widehat{\vec{U}}_0^{n+1} = \widehat{\vec{U}}_N^{n+1} = \vec{0}. \end{cases} \end{cases} \tag{34}$$

Theorem 6 *Let $\widehat{\vec{u}}^{n+1}(x)$ the solution of (16) and $\{\widehat{\vec{U}}_j^{n+1}\}$ the solution of (34). Then, the error satisfies*

$$\|\widehat{\vec{u}}^{n+1}(x_j) - \widehat{\vec{U}}_j^{n+1}\|_{\bar{\Omega}^N} \le C(N^{-1}\ln N)^2. \tag{35}$$

Then, the spatial discretization is uniformly convergent with order almost two.

Proof. We only give the outlines of the proof. First, it is easy to prove that the discrete operator $L_{\vec{\varepsilon}}^N$ is uniformly stable and also satisfies a discrete maximum principle.

To obtain appropriate bounds of the local error, we consider a decomposition in the form $\widehat{\vec{U}}^{n+1} = \widehat{\vec{V}}^{n+1} + \widehat{\vec{W}}^{n+1}$, $n = 0, 1, \cdots, M-1$, where the regular part is the solution of the problem

$$\begin{cases} \widehat{\vec{V}}_j^n = \widehat{\vec{v}}^n(x_j), \qquad 0 \le j \le N, \\ \begin{cases} L_{\vec{\varepsilon}}^N \widehat{\vec{V}}_j^{n+1} \equiv \left(I + \dfrac{\Delta t}{2} L_{x,\vec{\varepsilon}}^N\right) \widehat{\vec{V}}_j^{n+1} = \left(I - \dfrac{\Delta t}{2} L_{x,\vec{\varepsilon}}\right) \widehat{\vec{v}}^n(x_j) + \dfrac{\Delta t}{2}(\vec{f}_j^{n+1} + \vec{f}_j^n), \quad 0 < j < N, \\ \widehat{\vec{V}}_0^{n+1} = \widehat{\vec{v}}^{n+1}(0), \quad \widehat{\vec{V}}_N^{n+1} = \widehat{\vec{v}}^{n+1}(1), \end{cases} \end{cases} \tag{36}$$

and the singular component is the solution of the problem

$$\begin{cases} \widehat{\vec{W}}_j^n = \widehat{\vec{w}}^n(x_j), \qquad 0 \le j \le N, \\ \begin{cases} L_{\vec{\varepsilon}}^N \widehat{\vec{W}}_j^{n+1} \equiv \left(I + \dfrac{\Delta t}{2} L_{x,\vec{\varepsilon}}^N\right) \widehat{\vec{W}}_j^{n+1} = \left(I - \dfrac{\Delta t}{2} L_{x,\vec{\varepsilon}}\right) \widehat{\vec{w}}^n(x_j), \quad 0 < j < N, \\ \widehat{\vec{W}}_0^{n+1} = \widehat{\vec{w}}^{n+1}(0), \quad \widehat{\vec{W}}_N^{n+1} = \widehat{\vec{w}}^{n+1}(1). \end{cases} \end{cases} \tag{37}$$

Then, using Taylor expansion and the barrier function technique it is possible to obtain the required result (see [3] for details). ■

Theorem 7 *Let $\vec{u}(x,t)$ be the solution of (1) and $\{\vec{U}_j^{n+1}\}$ the solution of (33). Then, the error, at the mesh points, satisfies the bound*

$$\|\vec{u}(x_j, t_{n+1}) - \vec{U}_j^{n+1}\|_{\bar{Q}^N} \le C(N^{-2+q}\ln^2 N + (\Delta t)^2), \quad 0 < q < 1, \tag{38}$$

where $N, \Delta t$ and q are such that $N^{-q} \le C\Delta t$.

Proof. We split the global error at the time t_{n+1} in the form

$$\|\vec{u}(x_j,t_{n+1}) - \vec{U}_j^{n+1}\|_{\bar{Q}} \le \|\vec{u}(x_j,t_{n+1}) - \widehat{\vec{u}}^{n+1}(x_j)\|_{\bar{Q}} + \|\widehat{\vec{u}}^{n+1}(x_j) - \widehat{\vec{U}}_j^{n+1}\|_{\bar{Q}} + \|\widehat{\vec{U}}_j^{n+1} - \vec{U}_j^{n+1}\|_{\bar{Q}}.$$

We deduce the result from Theorems 3 and 6 and using a recursive argument (see [2]). ■

Remark 8 *Theorem 7 proves the second order of uniform convergence of the method (except by the logarithmic factor), under the relation $N^{-q} \le C\Delta t$ between the discretization parameters N and Δt. Nevertheless, from the numerical point of view, this is an artificial relation that we have never needed to obtain the good results based on our numerical method. Then, we conjecture that this restriction is not necessary, but theoretically it is an open question.*

3. Numerical results

Here we show the maximum errors and the numerical orders of uniform convergence obtained using the method (33) to solve two different problems. The first one is given by

$$\left.\begin{array}{l} \dfrac{\partial u_1}{\partial t} - \varepsilon_1 \dfrac{\partial^2 u_1}{\partial x^2} + (2+x)u_1 - (1+x)u_2 = x^2(1-x)^2, \\ \dfrac{\partial u_2}{\partial t} - \varepsilon_2 \dfrac{\partial^2 u_2}{\partial x^2} + (e^x+1)u_2 - (1+x)u_1 = x^2(1-x)^2, \end{array}\right\} \quad (x,t) \in (0,1)\times(0,1], \tag{39}$$
$$u(0,t) = u(1,t) = 0, \quad t \in [0,T], \quad u(x,0) = 0, \quad x \in (0,1],$$

for which the exact solution is unknown; then to find the pointwise errors at the mesh points $\{(x_j,t_n)\}$, we use a variant of the double mesh principle (see [5]). Then, we calculate the numerical solution $\{\hat{\vec{U}}_j^n\}$ on the mesh $\{(\hat{x}_j,\hat{t}_n)\}$ that contains the mesh points of the original mesh and their midpoints, i.e.,

$$\begin{array}{ll} \hat{x}_{2j}=x_j, \quad j=0,\dots,N, & \hat{x}_{2j+1}=(x_j+x_{j+1})/2, \quad j=0,\dots,N-1, \\ \hat{t}_{2n}=t_n, \quad n=0,\dots,M, & \hat{t}_{2n+1}=(t_n+t_{n+1})/2, \quad n=0,\dots,M-1. \end{array}$$

At the mesh points of the coarse mesh we calculate the maximum errors and the uniform errors by $\vec{d}_{\vec{\varepsilon},N,\Delta t}=\max_{0\le n\le M}\max_{0\le j\le N}|\vec{U}_j^n-\hat{\vec{U}}_{2j}^{2n}|$, $\vec{d}_{N,\Delta t}=\max_S d_{\vec{\varepsilon},N,\Delta t}$, where for each fixed value de ε_2, the set $S=\{\varepsilon_1 \mid \varepsilon_1=\varepsilon_2, 2^{-2}\varepsilon_2,\dots,2^{-58},2^{-60}\}$ in order to permit that the maximum errors stabilize.

¿From these values we obtain the corresponding orders of convergence and the uniform orders of convergence in a standard way: $\vec{p}=\dfrac{\log(\vec{d}_{\vec{\varepsilon},N,\Delta t}/\vec{d}_{\vec{\varepsilon},2N,\Delta t/2})}{\log 2}$, $\quad \vec{p}_{uni}=\dfrac{\log(\vec{d}_{N,\Delta t}/\vec{d}_{2N,\Delta t/2})}{\log 2}$. Tables 1 and 2 display the results obtained in this case. From they we see that the method gives second order of uniform convergence in agreement with Theorem 7.

Table 1: Maximum errors and uniform orders of convergence for the component u_1 for the example (39)

	N=64 $\Delta t=0.5$	N=128 $\Delta t=0.5/2$	N=256 $\Delta t=0.5/2^2$	N=512 $\Delta t=0.5/2^3$	N=1024 $\Delta t=0.5/2^4$
$[\vec{d}_{N,\Delta t}]_1$	2.624E-03	1.077E-03	3.419E-04	9.385E-05	2.315E-05
$[\vec{p}_{uni}]_1$	1.284	1.656	1.865	2.019	

Table 2: Maximum errors and uniform orders of convergence for the component u_2 for the example (39)

	N=64 $\Delta t=0.5$	N=128 $\Delta t=0.5/2$	N=256 $\Delta t=0.5/2^2$	N=512 $\Delta t=0.5/2^3$	N=1024 $\Delta t=0.5/2^4$
$[\vec{d}_{N,\Delta t}]_2$	2.626E-03	1.085E-03	3.461E-04	9.650E-05	2.398E-05
$[\vec{p}_{uni}]_2$	1.275	1.648	1.842	2.009	

The second problem that we consider is the Barenblatt system

$$\left.\begin{array}{l} \dfrac{\partial u_1}{\partial t}-\varepsilon_1\dfrac{\partial^2 u_1}{\partial x^2}+u_1-u_2=1, \\ \dfrac{\partial u_2}{\partial t}-\varepsilon_2\dfrac{\partial^2 u_2}{\partial x^2}+u_2-u_1=1, \end{array}\right\} \quad (x,t)\in(0,1)\times(0,1], \tag{40}$$
$$u(0,t)=u(1,t)=0, \quad t\in[0,T], \quad u(x,0)=0, \quad x\in(0,1],$$

where again we use the same technique as before to find the maximum errors and the numerical orders of convergence. Tables 3 and 4 display the results obtained in this case. ¿From they we see that now the method does not give second order of uniform convergence and we observe numerically the influence of the compatibility conditions between data in order to achieve the required order of uniform convergence.

References

[1] G.I. Barenblatt, I.P. Zheltov, I.N. Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks, *J. Appl. Math. and Mech.* **24** (1960) 1286–1303.

Table 3: Maximum errors and uniform orders of convergence for the component u_1 for the example (40)

	N=64 $\Delta t = 0.5$	N=128 $\Delta t = 0.5/2$	N=256 $\Delta t = 0.5/2^2$	N=512 $\Delta t = 0.5/2^3$	N=1024 $\Delta t = 0.5/2^4$
$[\vec{d}_{N,\Delta t}]_1$	4.971E-02	1.768E-02	7.713E-03	3.839E-03	1.915E-03
$[\vec{p}_{uni}]_1$	1.492	1.197	1.007	1.003	

Table 4: Maximum errors and uniform orders of convergence for the component u_2 for the example (40)

	N=64 $\Delta t = 0.5$	N=128 $\Delta t = 0.5/2$	N=256 $\Delta t = 0.5/2^2$	N=512 $\Delta t = 0.5/2^3$	N=1024 $\Delta t = 0.5/2^4$
$[\vec{d}_{N,\Delta t}]_2$	4.971E-02	1.768E-02	7.449E-03	3.728E-03	1.902E-03
$[\vec{p}_{uni}]_2$	1.492	1.247	0.998	0.971	

[2] C. Clavero, J.L. Gracia, J.C. Jorge, High–order numerical methods for one–dimensional parabolic singularly perturbed problems with regular layers, *Numerical Methods for Partial Differential Equations* **21** (2005) 149–169.

[3] C. Clavero, J.L. Gracia, F. Lisbona, Time dependent singularly perturbed coupled reaction-diffusion systems: a high accurate uniformly convergent method, *submitted.*

[4] S.C. Cowin, Bone poroelasiticity, *Journal of Biomechanics* **32** (1999) 217–238.

[5] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, G.I. Shishkin, Robust computational techniques for boundary layers, Chapman & Hall (2000).

[6] J.L. Gracia, F. Lisbona, A uniformly convergent scheme for a system of reaction–diffusion equations. *submitted.*

[7] T. Linß, N. Madden, An improved error estimate for a numerical method for a system of coupled singularly perturbed reaction–diffusion equations, *Comp. Meth. Appl. Math.* **3** (2003) 417–423.

[8] T. Linß, N. Madden, Accurate solution of a system of coupled singularly perturbed reaction-diffusion equations, *Computing* **73** (2004) 121–133.

[9] N. Madden, M. Stynes, A uniformly convergent numerical method for a coupled system of two singularly perturbed linear reaction-diffusion problems, *IMA J. Numer. Anal.* **23** (2003) 627–644.

[10] S. Matthews, J.J.H. Miller, E. O'Riordan, G.I. Shishkin, A parameter robust numerical method for a system of singularly perturbed ordinary differenial equations. In: *Analytical and Numerical Methods for Convection–Dominated and Singularly Perturbed Problems* (J.J.H. Miller, G.I. Shishkin and L. Vulkov, eds.), Nova Science Publishers, New York, 2000, 219–224.

[11] S. Matthews, E. O'Riordan, G.I. Shishkin, A nunerical method for a system of singularly perturbed reaction–diffusion equations, *J. Comput. Appl. Math.* **145** (2002) 151–166.