

# Continuous and Discontinuous Finite Element Methods for Convection-Diffusion Problems: A Comparison

Andrea Cangiani\*, Emmanuil H. Georgoulis† and Max Jensen‡

June 30, 2006

## Abstract

We compare numerically the performance of a new continuous-discontinuous finite element method (CDFEM) for linear convection-diffusion equations with three well-known upwind finite element formulations, namely with the streamline upwind Petrov-Galerkin finite element method, the residual-free bubble method and the discontinuous Galerkin finite element method. The defining feature of the CDFEM is that it uses discontinuous approximation spaces in the vicinity of layers while continuous FEM approximation are employed elsewhere.

## 1 Introduction

Standard conforming finite element approximations of convection-dominated convection-diffusion problems often exhibit poor stability properties that manifest themselves as non-physical oscillations polluting the numerical solution. Various techniques have been proposed for the stabilization of finite element methods (FEMs) for convection-diffusion problems; for a complete survey see, for example, Morton [22] and Roos, Stynes and Tobiska [23]. Common techniques are *exponential fitting*, *symmetrization*, *upwinding* and *least squares regularization*. *Ad hoc* meshing, like *graded meshes* [24] and *Shishkin type meshes* [21], and *adaptive mesh refinement* (see, e.g., [12], [4], [1] and [16]) are also well-established branches of the subject. Furthermore, *bubble stabilization* [7, 14] and the closely related variational multiscale methods [18] constitute an illuminating way of reinterpreting many of the techniques just mentioned.

During the last decade, families of discontinuous Galerkin finite element methods (DGFEMs) have been proposed for the numerical solution of convection-diffusion problems, due to the many attractive properties they exhibit. In particular, DGFEMs admit good stability properties, they offer flexibility in the mesh design (irregular meshes are admissible) and in the handling of boundary conditions (Dirichlet boundary conditions are weakly imposed). Furthermore they are increasingly popular in the context of *hp*-adaptive algorithms.

Hovewer, discontinuous Galerkin methods involve more degrees of freedom than the same order conforming finite element schemes. Indeed, a discontinuous finite element space  $V_h$  of affine elements contains up to four times more degrees of freedom for two-dimensional problems and up to eight times more degrees of freedom for three-dimensional problems compared to the (unstable) standard finite element or SUPG formulations, without any improvement in the order of accuracy.

Therefore, a natural question arising is whether it is possible to reduce the additional degrees of freedom required by the DGFEMs without affecting their good stability properties. There have already been attempts in this direction which can be roughly grouped into two categories. The first category consists of the so-called multi-scale approach due to Hughes et. al. [9], in which the discontinuous finite element space is decomposed into continuous and discontinuous degrees of freedom, which are treated as coarse and fine spaces, respectively, in a multi-scale fashion. Local problems are solved on each element to project from the “fine” space to the “coarse” space. The second category revolves around the idea of identifying classes of elements which globally result in approximation spaces lying between the continuous and discontinuous finite element spaces, e.g. [3].

---

\*Departmento di Matematica, Universita’ di Pavia, Via Ferrata 1, 27100 Pavia, Italy, e-mail: [andrea.cangiani@unipv.it](mailto:andrea.cangiani@unipv.it)

†Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom, e-mail: [Emmanuil.Georgoulis@mcs.le.ac.uk](mailto:Emmanuil.Georgoulis@mcs.le.ac.uk)

‡Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany, e-mail: [jensenm@math.hu-berlin.de](mailto:jensenm@math.hu-berlin.de)

This work proposes a new finite element discretization which provides the advantages of discontinuous Galerkin methods while only requiring marginally more degrees of freedom than conforming schemes. The underlying observation is that for (linear) convection-diffusion problems stabilization is only required locally near layers of the solution. Taking into account that for physically relevant solutions layers make up only a relatively small part of the domain, we employ a standard unstabilised finite element scheme away from layers while investing in the more expensive DGFEM in the vicinity of layers.

The performance of this new continuous-discontinuous finite element method (CDFEM) is tested through numerical experiments and compared with the DG method and two well-known conforming methods, namely the SUPG and the RFB finite element methods. For completeness, the formulations of the latter methods are included along with a discussion on the computational characteristics of each of them.

This note is organized as follows. In Section 2 we state the model problem; Section 3 contains the function space framework and the construction of the finite element spaces. Sections 4, 5, and 6, contain brief descriptions of SUPG, RFB, and discontinuous Galerkin finite element methods. Section 7 introduces the new continuous-discontinuous finite element method and in Section 8 some numerical experiments are presented. Concluding remarks are given in Section 9.

## 2 Model Problem

Let  $\Omega$  be a bounded open polyhedral domain in  $\mathbb{R}^2$ , and let  $\Gamma_\partial$  signify the union of its one-dimensional open edges. We consider the steady state convection-diffusion equation

$$\mathcal{L}u \equiv -\epsilon \Delta u + \mathbf{b} \cdot \nabla u = f \quad \text{in } \Omega, \quad (1)$$

where  $f \in L^2(\Omega)$ , and  $\mathbf{b} = (b_1, b_2)^T$ , whose entries  $b_i, i = 1, 2$ , are Lipschitz continuous real-valued functions on  $\Omega$ . For simplicity of the presentation, we assume homogeneous Dirichlet boundary conditions on  $\partial\Omega$ .

## 3 Finite Element Spaces

We shall denote by  $H^s(\Omega)$  the standard Hilbertian Sobolev space of index  $s \geq 0$  of real-valued functions defined on  $\Omega$ .

Let  $\mathcal{T}$  be a subdivision of  $\Omega$  into disjoint open elements  $\kappa \in \mathcal{T}$  such that each edge of  $\kappa$  has at most one regular hanging node. We let  $h_\kappa := \text{diam}(\bar{\kappa})$ . We assume that the subdivision  $\mathcal{T}$  is constructed via mappings  $F_\kappa$  where  $F_\kappa : \hat{\kappa} := (-1, 1)^2 \rightarrow \kappa$  is a  $C^1$ -diffeomorphism with non-singular Jacobian. It is assumed that  $\bar{\Omega} = \cup_{\kappa \in \mathcal{T}} \bar{\kappa}$ .

By  $\Gamma$  we denote the union of all one-dimensional element faces associated with the subdivision  $\mathcal{T}$  including edges on the boundary. We assign to the subdivision  $\mathcal{T}$  the broken Sobolev space of composite order  $\mathbf{s} := \{s_\kappa : \kappa \in \mathcal{T}\}$

$$H^{\mathbf{s}}(\Omega, \mathcal{T}) := \{u \in L^2(\Omega) : u|_\kappa \in H^{s_\kappa}(\kappa) \text{ for all } \kappa \in \mathcal{T}\},$$

equipped with the standard broken Sobolev norm. When  $s_\kappa = s$  for all  $\kappa \in \mathcal{T}$ , we write  $H^s(\Omega, \mathcal{T})$ .

For a nonnegative integer  $p$  we denote by  $\mathcal{Q}_p(\hat{\kappa})$  the set of all tensor-product polynomials on  $\hat{\kappa}$  of degree  $p$  in each coordinate direction. For simplicity of the presentation we assume constant polynomial degree  $p \geq 1$  throughout the mesh. Then, the continuous and discontinuous finite element spaces are defined by

$$V_h^c := \{v \in C^0(\Omega) : v|_\kappa \circ F_\kappa \in \mathcal{Q}_p(\hat{\kappa}), \kappa \in \mathcal{T}\} \quad (2)$$

and

$$V_h^d := \{v \in L^2(\Omega) : v|_\kappa \circ F_\kappa \in \mathcal{Q}_p(\hat{\kappa}), \kappa \in \mathcal{T}\} \quad (3)$$

respectively. Note that  $V_h^c \subset H^1(\Omega)$  and  $V_h^d \subset H^1(\Omega, \mathcal{T})$ .

## 4 Streamline Upwind Petrov-Galerkin FEM

The weak formulation of the homogeneous problem (1) reads

$$\text{find } u \in H_0^1(\Omega) \text{ such that } B_c(u, v) = l(v) \quad \forall v \in H_0^1(\Omega), \quad (4)$$

where

$$B_c(u, v) := \int_\Omega (\epsilon \nabla u \cdot \nabla v + \mathbf{b} \cdot \nabla u) v \, dx, \quad \text{and} \quad l(v) := \int_\Omega f v \, dx. \quad (5)$$

The Galerkin FEM formulation is defined by restricting (11) onto the continuous finite element space  $V_h^c$ . Such method is unstable when the convective term dominates unless the mesh size is of the same order as the smaller scales present in the solution. Numerical instabilities are indeed observed when the *mesh Péclet number*  $Pe_\kappa := h_\kappa \|\mathbf{b}\|_{\infty, \kappa} / 2\varepsilon > 1$  for elements  $\kappa$  in the neighbours of boundary and internal layers.

The *streamline upwind Petrov–Galerkin* (SUPG) method was introduced by Hughes and Brooks [19] (see also Johnson and Nävert [20] and Hughes and Brooks [8]). In order to obtain a stable method, a diffusion term in the direction of convection is added to the standard Galerkin formulation. The SUPG-FEM reads: find  $u_h \in V_h^c$  such that

$$B_c(u_h, v_h) + \sum_{\kappa \in \mathcal{T}} \tau_\kappa (\mathcal{L}u_h, \mathbf{b} \cdot \nabla v_h)_\kappa = l(v_h) + \sum_{\kappa \in \mathcal{T}} \tau_\kappa (f, \mathbf{b} \cdot \nabla v_h)_\kappa \quad \forall v_h \in V_h^c; \quad (6)$$

The term  $(\mathbf{b} \cdot \nabla u_h, \mathbf{b} \cdot \nabla v_h)$  is added in order to suppress the nonphysical numerical oscillations which would otherwise arise. The addition of the term  $\sum_{\kappa \in \mathcal{T}} \tau_\kappa (f, \mathbf{b} \cdot \nabla v_h)_\kappa$  in the linear form ensures the consistency of the method.

In general, the SD-parameter  $\tau_\kappa$  depends both on the mesh size and on the mesh Péclet number. A typical choice for the SUPG method, taken from [23], is

$$\tau_\kappa = \begin{cases} \tau_0 h_\kappa & \text{if } Pe_\kappa > 1 \text{ (convection-dominated case)} \\ \tau_1 h_\kappa^2 \varepsilon & \text{if } Pe_\kappa \leq 1 \text{ (diffusion-dominated case)}, \end{cases} \quad (7)$$

Under appropriate assumptions the SUPG method satisfies with this choice of  $\tau_\kappa$  a global error estimate of the form

$$\|u - u_h\|_{SD} \leq C \left( \varepsilon^{1/2} + h^{1/2} \right) h^k |u|_{H^{k+1}(\Omega)}, \quad (8)$$

where the SD-norm is defined as

$$\|v\|_{SD} := \left( \varepsilon |v|_1^2 + \sum_{\kappa \in \mathcal{T}} \tau_\kappa \|\mathbf{b} \cdot \nabla v\|_{L^2(\kappa)}^2 \right)^{1/2}.$$

The definition of  $\tau_\kappa$  given by (7) states that where the problem is convection-dominated the SD-parameter should be proportional to the element size  $h_\kappa$ . Still the problem of identifying an optimal value of  $\tau_\kappa$ , in the sense of providing a sharp, non-oscillatory approximation in the layers, is left open. Indeed, the value for  $\tau_0$  and  $\tau_1$  is not implied by the *a priori* analysis.

The difficulty encountered with the problem of parameter identification may be seen as a consequence of the lack of physical justification of stabilized methods based on the notion of relevant scales.

Indeed, one of the reasons for the success of two-level or subgrid scale methods such as the variational multiscale method previously mentioned, the local Green's function approach, see [18], and the residual-free bubble method, is that they can provide the required theoretical foundation to classical stabilization techniques. The residual-free bubble method, as an example, is briefly described in the next section.

## 5 Residual-Free Bubble Method

The residual-free bubble method of Brezzi and Russo [7], and Franca and Russo [14], was analyzed in [6].

Let  $\Gamma$  signify the skeleton of the partition  $\mathcal{T}$ , i.e. the union of the boundaries of all elements in  $\kappa \in \mathcal{T}$ . The *residual free bubble* (RFB) space  $V_{RFB}$  is defined by augmenting  $V_h^c$  with the space of *bubbles*, which are the functions with support in  $\Omega \setminus \Gamma$ :

$$V_{RFB} = V_h^c + B_h,$$

where

$$B_h = \bigoplus_{\kappa \in \mathcal{T}} H_0^1(\kappa).$$

In the enriched space  $V_{RFB}$  fine scales of the boundary value problem can be resolved at the elemental level. The RFB method is the Galerkin formulation (11) on  $V_{RFB}$ , namely

$$\begin{cases} \text{find } u_{RFB} \in V_{RFB} \text{ such that} \\ B_c(u_{RFB}, v) = l(v) \quad \forall v \in V_{RFB}. \end{cases} \quad (9)$$

Starting from (9), a two-level procedure is obtained by splitting the solution into its polynomial component  $u_h \in V_h^c$  and bubble component  $u_b \in B_h$  and by testing *separately* in  $V_h^c$  and  $B_h$ . At the subgrid level the bubble component of the solution is obtained by solving the *bubble equation*

$$B_c(u_b, v) = l(v) - B_c(u_h, v) \quad \forall v \in B_h.$$

It is important that this can be done locally. Formally one has

$$u_b|_\kappa = \mathcal{L}_\kappa^{-1}(f - \mathcal{L}u_h)|_\kappa \quad \forall \kappa \in \mathcal{T}.$$

The second step consists of solving for the polynomial component, which satisfies the equation

$$B_c(u_h, v_h) + \sum_{\kappa \in \mathcal{T}} \int_\kappa \mathcal{L}_\kappa^{-1}(f - \mathcal{L}u_h) \mathcal{L}^* v_h dx = l(v_h) \quad \forall v_h \in V_h; \quad (10)$$

here  $\mathcal{L}^*$  is the differential operator adjoint to  $\mathcal{L}$ . Thus the RFB method consists of a local fine scale approximation and a global coarse scale approximation. We can also interpret the formulation (10) as a stabilised method in terms of the  $V_h^c$  finite element space only.

In practice, the actual computation of the bubble, hidden here in the formal local inversion of  $\mathcal{L}$ , is carried out numerically by introducing a subgrid. In this way a fully discrete procedure is obtained. The choice of the subgrid dictates which fine scales are incorporated into the coarse scale formulation. Cheap subgrid solves can be employed without compromising stability and accuracy of the stabilised formulation [7, 5]. Thus, in terms of computational cost, the stabilised RFB formulation (10) is comparable to the SUPG formulation. We refer, e.g., to [10] for details on the implementation of the method.

## 6 Discontinuous Galerkin Finite Element Method

The definition of the discontinuous Galerkin method requires the introduction of interelemental boundary operators. Let  $\kappa, \kappa'$  be two elements sharing a common face  $e := \bar{\kappa} \cap \bar{\kappa}'$ . Define the outward normal unit vectors  $\mathbf{n}^+$  and  $\mathbf{n}^-$  on  $e$  corresponding to  $\partial\kappa$  and  $\partial\kappa'$ , respectively. For functions  $q : \Omega \rightarrow \mathbb{R}$  and  $\phi : \Omega \rightarrow \mathbb{R}^2$  that may be discontinuous across  $\Gamma$ , we define the traces  $q^+ := q|_{\partial\kappa}$ ,  $q^- := q|_{\partial\kappa'}$  and  $\phi^+ := \phi|_{\partial\kappa}$ ,  $\phi^- := \phi|_{\partial\kappa'}$ . We then set

$$\{q\} := \frac{1}{2}(q^+ + q^-), \quad \{\phi\} := \frac{1}{2}(\phi^+ + \phi^-), \quad [q] := q^+ \mathbf{n}^+ + q^- \mathbf{n}^-, \quad [\phi] := \phi^+ \cdot \mathbf{n}^+ + \phi^- \cdot \mathbf{n}^-.$$

If  $e$  is instead an internal edge these definitions are modified to

$$\phi^+ := \phi|_{\partial\Omega}, \quad \{q\} := q^+, \quad \{\phi\} := \phi^+, \quad [q] := q^+ \mathbf{n}, \quad [\phi] := \phi^+ \cdot \mathbf{n}.$$

Further, we decompose the skeleton of  $\mathcal{T}$  into the subsets

$$\partial_- \kappa := \{x \in \partial\kappa : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0\}, \quad \partial_+ \kappa := \{x \in \partial\kappa : \mathbf{b}(x) \cdot \mathbf{n}(x) > 0\},$$

where  $\mathbf{n}(\cdot)$  denotes the unit outward normal vector function associated with the element  $\kappa$ . We call  $\partial_- \kappa$  and  $\partial_+ \kappa$  the *inflow* and *outflow* parts of  $\partial\kappa$  respectively.

Then, for every element  $\kappa \in \mathcal{T}$ , we denote by  $u_\kappa^+$  the trace of a function  $u$  on  $\partial\kappa$  taken from within the element  $\kappa$  (interior trace). We also define the exterior trace  $u_\kappa^-$  of  $u \in H^1(\Omega, \mathcal{T})$  for almost all  $x \in \partial_- \kappa \setminus \Gamma_\partial$  to be the interior trace  $u_{\kappa'}^+$  of  $u$  on the element(s)  $\kappa'$  that share the edges contained in  $\partial_- \kappa \setminus \Gamma_\partial$  of the boundary of element  $\kappa$ . Then, the *upwind jump* of  $u$  across  $e \subset \partial_- \kappa \setminus \Gamma_\partial$  is defined by

$$[u]_e := u_\kappa^+ - u_\kappa^-,$$

and  $[u]_e = u^+$  for  $e \subset \Gamma_\partial$ . We note that this definition of jump is not the same as the one above; here the sign of the jump depends on the direction of the flow, whereas  $[\cdot]$  depends only on the element-numbering.

We note that the subscript in this definition will be suppressed when no ambiguity can occur.

The broken weak formulation of the homogeneous problem (1), from which the interior penalty DGFEM emerges, reads

$$\text{find } u \in A \text{ such that } B(u, v) = l(v) \quad \forall v \in H^{3/2+\varepsilon}(\Omega, \mathcal{T}), \quad (11)$$

where

$$A := \{u \in H^{3/2+\varepsilon}(\Omega, \mathcal{T}) : u, \nabla u \cdot \nu \text{ are continuous across } \Gamma_{\text{int}}\},$$

$$\begin{aligned}
B(u, v) := & \sum_{\kappa \in \mathcal{T}} \left( \int_{\kappa} (\epsilon \nabla u \cdot \nabla v + (\mathbf{b} \cdot \nabla u) v) \, dx - \int_{\partial_{-\kappa}} (\mathbf{b} \cdot \mathbf{n}) [u] v^+ \, ds \right) \\
& + \int_{\Gamma} (\theta \{\epsilon \nabla v\} \cdot [u] - \{\epsilon \nabla u\} \cdot [v] + \sigma [u] \cdot [v]) \, ds,
\end{aligned}$$

and  $l(\cdot)$  as above, for  $\theta \in \{-1, 0, 1\}$ . The function  $\sigma$  is hereby defined as

$$\sigma|_e := C_\sigma \frac{\epsilon p^2}{h_\perp^e},$$

where  $h_\perp^e = \frac{|\kappa| + |\kappa'|}{2|e|}$ , for  $e \subset \bar{\kappa} \cap \bar{\kappa}'$ , and  $C_\sigma$  is a sufficiently large positive constant, and  $\epsilon > 0$ .

The *interior penalty DGFEM* for the homogeneous problem (1) is defined by:

$$\text{find } u_{\text{DG}} \in V_h^d \text{ such that } B(u_{\text{DG}}, v) = l(v) \quad \forall v \in V_h^d. \quad (12)$$

We refer to the method with  $\theta = -1$  as the *symmetric interior penalty DGFEM (SIPG)*, as the *incomplete interior penalty DGFEM (IIPG)* and for  $\theta = 1$  as the *non-symmetric interior penalty DGFEM (NIPG)*. This terminology stems from the fact that when  $\mathbf{b} \equiv \vec{0}$ , the bilinear form  $B(\cdot, \cdot)$  is symmetric if and only if  $\theta = -1$ . Various types of error analysis for the variants of interior penalty DGFEMs can be found in [2, 17, 15]. See also the references therein.

It is widely observed in numerical experiments that discontinuous Galerkin finite element methods exhibit good stability properties in the presence of sharp gradients of the solution. This can be explained by the essentially parameter-free upwinding in form of the upwinded jump included in the second integral on the right-hand side of (12)).

We further note that the only user-defined parameter in the DGFEM formulation is the coefficient of the discontinuity-penalization function  $\sigma$ . However, various numerical experiments have shown that the stability and accuracy of the method is only mildly dependent on  $\sigma$  ( e.g. cf. [11]).

However, it needs to be taken into account that the approximation spaces of discontinuous Galerkin methods have more degrees of freedom than corresponding continuous spaces. We remark that due to the enlargement of the approximation space the function  $\bar{\mathbf{b}} \cdot \nabla v_h$  for any piecewise constant  $\bar{\mathbf{b}}$  is an admissible test function. This term, of similar structure as the SUPG stabilization, plays an important role in the proof of stability of the DGFEM [17, 9].

As pointed out in the introduction, the discontinuous finite element space  $V_h^d$  contains for bilinear elements up to four times more degrees of freedom for two-dimensional problems and up to eight times more degrees of freedom for three-dimensional problems compared to the continuous FEM formulations.

Hence, the question arises as how to design a stable essentially parameter-free method which has less computational overhead than the DGFEM. In the next section we propose a new finite element method that blends continuous and discontinuous approximations in order to achieve the desired accuracy and stability with reduced computational cost.

## 7 Continuous-Discontinuous FEM

The domain  $\Omega$  is now subdivided into two parts  $\Omega_c$  and  $\Omega_d := \Omega \setminus \Omega_c$  such that all elements  $\kappa \in \mathcal{T}$  are subsets of either  $\Omega_c$  or  $\Omega_d$ . We define the finite element space

$$V_h^d(\Omega_c) := \{v_h \in V_h^d \text{ such that } [v_h]_e = 0, \text{ for all } e \subset \Omega_c\},$$

i.e., the space of element-wise polynomials that are continuous across the interfaces in  $\Omega_c$ . Notice that we have

$$V_h^c = V_h^d(\Omega) \text{ and } V_h^d = V_h^d(\emptyset).$$

The *continuous-discontinuous FEM (CDFEM)* for the homogeneous problem (1) is

$$\text{find } u_{\text{CD}} \in V_h^d(\Omega_c) \text{ such that } B(u_{\text{CD}}, v) = l(v) \quad \forall v \in V_h^d(\Omega_c), \quad (13)$$

where  $B$  and  $l$  are the bilinear and linear form of the DG method restricted to  $V_h^d(\Omega_c)$ .

As for the DG method we refer to the CDFEM with  $\theta = -1$  as the *symmetric interior penalty CDFEM*, for  $\theta = 0$  the DGFEM is referred as the *incomplete interior penalty CDFEM* whereas for  $\theta = 1$  the DGFEM will be referred to as the *non-symmetric interior penalty CDFEM*.

Clearly, the finite element space  $V_h^d(\Omega_c)$  has substantially smaller dimension than  $V_h^d$  when  $\Omega_c$  is large. Motivated by the hypothesis that stabilization is only required when sharp gradients are present in the solution, we can define  $\Omega_c$  to contain all the elements  $\kappa \in \mathcal{T}$  that are located away from neighbourhoods of boundary and interior layers. That way, the CDFEM offers the stabilization

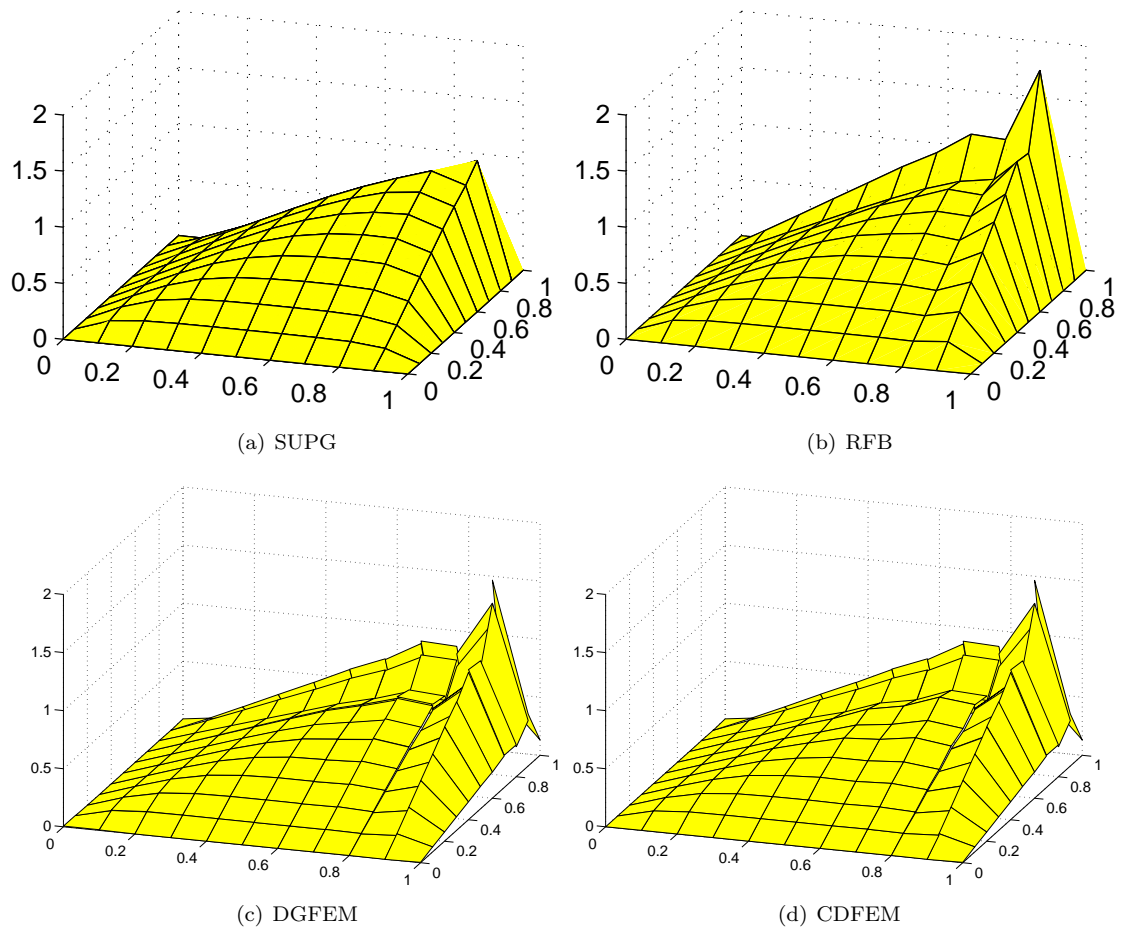


Figure 1: Example with  $f = 1$ ,  $\epsilon = 10^{-2}$  on a  $10 \times 10$  grid.

advantages of DGFEM without the extra degrees of freedom of DGFEM in subregions of the computational domain where the solution is smooth independent of  $\epsilon$ . Moreover, considering that for convection-diffusion problems these sharp gradients and layers are of  $(d-1)$ -dimensional nature, the potential computational gain of this approach is evident.

In the next section, we compare the stability of SUPG, RFB, DGFEM and CDFEM numerically. For a proof of stability of the CDFEM for convection-diffusion problems can be found in [11].

## 8 Numerical Experiments

We consider two numerical examples with constant wind  $\mathbf{b}$ : one with homogeneous boundary conditions and one with non-homogeneous boundary conditions. In particular, we consider (1) on  $\Omega = (0, 1)^2$  with  $\epsilon = 10^{-2}, 10^{-3}$  and  $\mathbf{b} = (1, 1)^T$  for two different choices of  $f$  and Dirichlet data.

In the first numerical example, we have  $u = 0$  on  $\partial\Omega$  and  $f = 1$ . The solution exhibits boundary layers along  $x = 1$  and  $y = 1$ . These layers become steeper as  $\epsilon \rightarrow 0$ . We consider a  $10 \times 10$  uniform subdivision of the computational domain in rectangular elements and consider piecewise bilinear finite elements. For CDFEM we set  $\Omega_c = (0, 0.8)^2$ . In Figure 1 the computed solutions are given. We observe that the CDFEM does not suffer from increased numerical oscillations compared to DGFEM. For comparison, the RFB and SUPG solutions are also shown. The SUPG parameter  $\tau_0$  was set to the value  $(1 - 1/Pe_\kappa)/2$  proposed in [13]. As for the RFB method, only the piecewise bilinear part of the solution is plotted.

In the second numerical example Dirichlet boundary conditions and the forcing function  $f$  are chosen such that the analytical solution is

$$u(x, y) = x + y(1 - x) + \frac{e^{-\frac{1}{\epsilon}} - e^{-\frac{(1-x)(1-y)}{\epsilon}}}{1 - e^{-\frac{1}{\epsilon}}}.$$

As before the solution exhibits boundary layer behaviour along  $x = 1$  and  $y = 1$ , and the layers

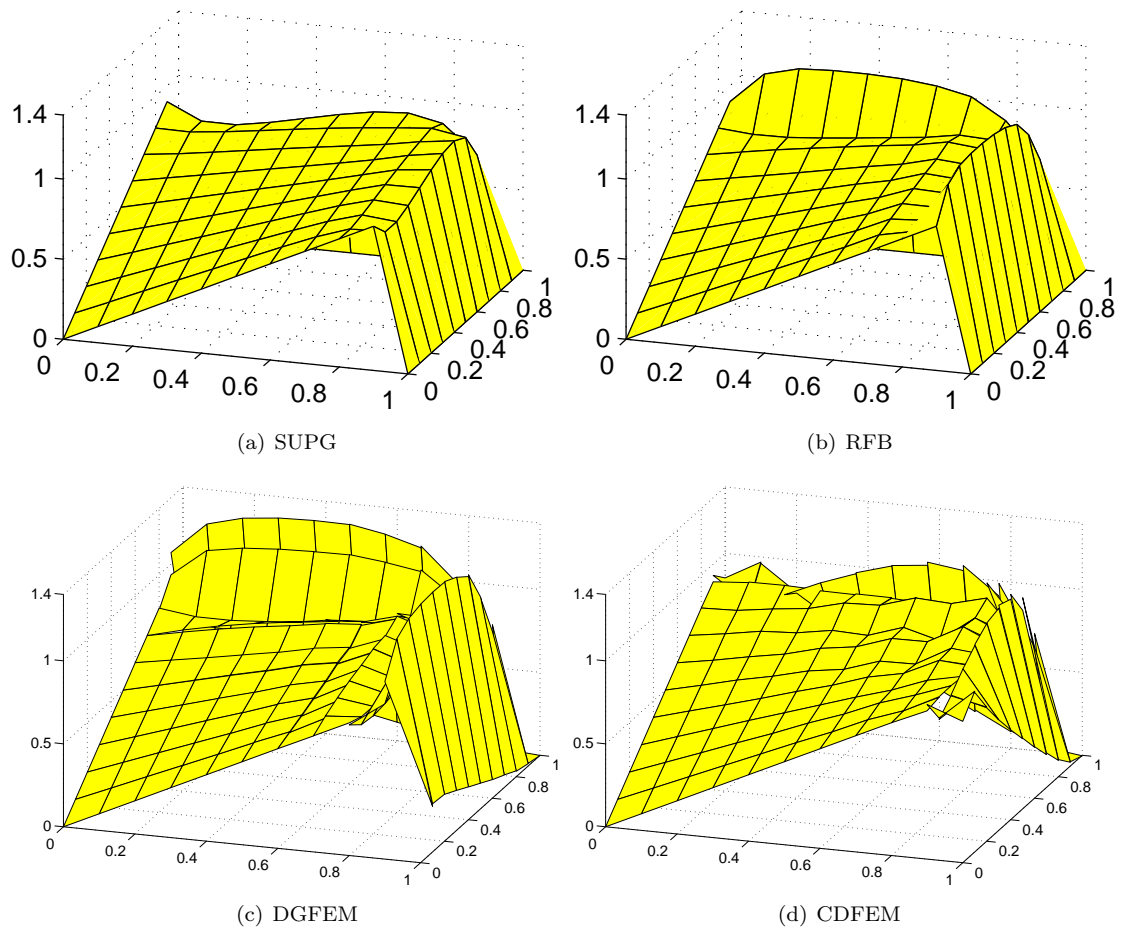


Figure 2: Example with non-homogeneous boundary conditions,  $\epsilon = 10^{-2}$  on a  $10 \times 10$  grid.

become steeper as  $\epsilon \rightarrow 0$ .

Figure 2 shows that again CDFEM, with  $\Omega_c = (0, 0.8)^2$ , delivers solutions comparable to those of the discontinuous Galerkin method.

## 9 Concluding Remarks

We compared numerically the performance of three well-known stable finite element formulations (namely SUPG, RFB, DGFEM) to the new CD finite element method. The CDFEM is a continuous Galerkin method in smooth parts of the domain while a Galerkin method of discontinuous type in the vicinity of internal and boundary layers on non-resolving grids. While in the presentation of the method given here the layer location has to be known a priori to set up the finite element space, it is our aim to control the composition of the approximation space automatically by adaptive algorithms driven by a posteriori knowledge. Hereby one not only wants to increase the accuracy of the numerical method, but also to ensure the stability properties on non-resolving grids without investing in unnecessary degrees of freedom; we refer to [11] for details.

## References

- [1] AINSWORTH, M., AND ODEN, J. T. *A posteriori error estimation in finite element analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2000.
- [2] ARNOLD, D. N. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.* 19 (1982), 742–760.
- [3] BECKER, R., BURMAN, E., AND HANSBO, P. LARSON, M. A reduced p1-discontinuous Galerkin method. *EPFL Preprint 05* (2004).

- [4] BECKER, R., AND RANNACHER, R. A feed-back approach to error control in finite element methods: basic analysis and examples. *East-West J. Numer. Math.* 4, 4 (1996), 237–264.
- [5] BREZZI, F., HAUKE, G., MARINI, L. D., AND SANGALLI, G. Link-cutting bubbles for the stabilization of convection-diffusion-reaction problems. *Math. Models Methods Appl. Sci.* 13, 3 (2003), 445–461. Dedicated to Jim Douglas, Jr. on the occasion of his 75th birthday.
- [6] BREZZI, F., MARINI, D., AND SÜLI, E. Residual-free bubbles for advection-diffusion problems: the general error analysis. *Numer. Math.* 85, 1 (2000), 31–47.
- [7] BREZZI, F., AND RUSSO, A. Choosing bubbles for advection-diffusion problems. *Math. Models Methods Appl. Sci.* 4, 4 (1994), 571–587.
- [8] BROOKS, A. N., AND HUGHES, T. J. R. Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.* 32, 1-3 (1982), 199–259. FENOMECH '81, Part I (Stuttgart, 1981).
- [9] BUFFA, A., HUGHES, T., AND SANGALLI, G. Analysis of a multiscale discontinuous galerkin method for convection diffusion problems. *ICES Report 05-40 (2005), to appear on SIAM J. Numer. Anal.*
- [10] CANGIANI, A. The residual-free bubble method for problems with multiple scales. *D.Phil. Thesis, University of Oxford* (2004).
- [11] CANGIANI, A., GEORGIOULIS, E. H., AND JENSEN, M. Continuous-discontinuous finite element methods for convection-diffusion problems. *in preparation*.
- [12] ERIKSSON, K., ESTEP, D., HANSBO, P., AND JOHNSON, C. Introduction to adaptive methods for differential equations. In *Acta numerica, 1995*, Acta Numer. Cambridge Univ. Press, Cambridge, 1995, pp. 105–158.
- [13] FISCHER, B., RAMAGE, A., SILVESTER, D. J., AND WATHEN, A. J. On parameter choice and iterative convergence for stabilised discretisations of advection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.* 179, 1-2 (1999), 179–195.
- [14] FRANCA, L. P., AND RUSSO, A. Deriving upwinding, mass lumping and selective reduced integration by residual-free bubbles. *Appl. Math. Lett.* 9, 5 (1996), 83–88.
- [15] GEORGIOULIS, E. H., AND SÜLI, E. Optimal error estimates for the  $hp$ -version interior penalty discontinuous Galerkin finite element method. *IMA J. Numer. Anal.* 25, 1 (2005), 205–220.
- [16] GILES, M. B., AND SÜLI, E. Adjoint methods for PDEs: a posteriori error analysis and post-processing by duality. *Acta Numer.* 11 (2002), 145–236.
- [17] HOUSTON, P., SCHWAB, C., AND SÜLI, E. Discontinuous  $hp$ -finite element methods for advection-diffusion-reaction problems. *SIAM J. Numer. Anal.* 39, 6 (2002), 2133–2163 (electronic).
- [18] HUGHES, T. J. R. Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origins of stabilized methods. *Comput. Methods Appl. Mech. Engrg.* 127, 1-4 (1995), 387–401.
- [19] HUGHES, T. J. R., AND BROOKS, A. A multidimensional upwind scheme with no crosswind diffusion. In *Finite element methods for convection dominated flows (Papers, Winter Ann. Meeting Amer. Soc. Mech. Engrs., New York, 1979)*, vol. 34 of AMD. Amer. Soc. Mech. Engrs. (ASME), New York, 1979, pp. 19–35.
- [20] JOHNSON, C., AND NÄVERT, U. An analysis of some finite element methods for advection-diffusion problems. In *Analytical and numerical approaches to asymptotic problems in analysis (Proc. Conf., Univ. Nijmegen, Nijmegen, 1980)*, vol. 47 of *North-Holland Math. Stud.* North-Holland, Amsterdam, 1981, pp. 99–116.
- [21] MADDEN, N., AND STYNES, M. Efficient generation of Shishkin meshes in solving convection-diffusion problems. *Preprint of the Department of Mathematics, University College, Cork, Ireland no. 1995-2* (1995).
- [22] MORTON, K. W. *Numerical solution of convection-diffusion problems*, vol. 12 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.
- [23] ROOS, H.-G., STYNES, M., AND TOBISKA, L. *Numerical Methods for Singularly Perturbed Differential Equations*. Springer-Verlag, Berlin, 1996.
- [24] SCHWAB, C., AND SURI, M. The  $p$  and  $hp$  versions of the finite element method for problems with boundary layers. *Math. Comp.* 65, 216 (1996), 1403–1429.