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Boundary layers for the Navier-Stokes equations : asymptotic analysis.

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1. Introduction

In this article, we simplify and improve the asymptotic analysis of the solutions of the Navier-Stokes problem given in [8]. This problem is characterized by the presence of a small term which corresponds to the inverse of Reynolds number $\mathfrak{Re}$ very large for the Navier-Stokes equations ($\mathfrak{Re} >> 1$). This small parameter is called the viscosity and denoted hereafter ε ($\varepsilon = 1/\mathfrak{Re}$). Thus, when the viscosity ε goes to zero, a particular phenomenon named *boundary layers* occurs in the interior or near the boundary of the domain considered for the Navier-Sotkes problem. The comprehension of these boundary layers makes the subject of many scientific works. For the mathematical point of view, see for example [4], [5], [6], [7], and [9] among many others.

We consider the Navier-Stokes equations in a channel $\Omega_\infty = \mathbb{R}^2 \times (0, h)$ with a permeable boundary, making the boundaries $z = 0, h$, non-characteristic. More precisely we have

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon + (u^\varepsilon.\nabla)\ u^\varepsilon + \nabla p^\varepsilon = f, \quad \text{in } \ \Omega_\infty, \\ \text{div } u^\varepsilon = 0, \quad \text{in } \ \Omega_\infty, \\ u^\varepsilon = (0, 0, -U), \quad \text{on } \ \Gamma_\infty, \\ u^\varepsilon \ \text{ is periodic in the } x \text{ and } y \text{ directions with periods } \ L_1, L_2, \\ u^\varepsilon|_{t=0} = u_0, \end{cases} \tag{1}$$

which is equivalent to

$$\begin{cases} \dfrac{\partial v^\varepsilon}{\partial t} - \varepsilon \Delta v^\varepsilon - U D_3 v^\varepsilon + (v^\varepsilon.\nabla)\ v^\varepsilon + \nabla p^\varepsilon = f, \quad \text{in } \ \Omega_\infty, \\ \text{div } v^\varepsilon = 0, \quad \text{in } \ \Omega_\infty, \\ v^\varepsilon = 0, \quad \text{on } \ \Gamma_\infty, \\ v^\varepsilon \ \text{ is periodic in the } x \text{ and } y \text{ directions with periods } \ L_1, L_2, \\ v^\varepsilon|_{t=0} = v_0. \end{cases} \tag{2}$$

Here $\Gamma_\infty = \partial\Omega_\infty = \mathbb{R}^2 \times \{0, h\}$ and we introduce also Ω and Γ :

$$\Omega = (0, L_1) \times (0, L_2) \times (0, h), \quad \Gamma = (0, L_1) \times (0, L_2) \times \{0, h\}.$$

We assume that f and u_0 are given functions as regular as necessary in the channel Ω_∞, and that U is a given constant; at the price of long technicalities, we can also consider the case where U is nonconstant everywhere.

The main objective of this work is to give a simple representation of the solution u^ε in some sobolev spaces to be specified later on. Mathematically, this is equivalent to give an asymptotic

expansion of v^ε when ε approches zero. Thus, firstly we must determine the limit solution and then study the convergence between v^ε and its limit denoted by v^0.
We recall here the limit problem which is the Euler problem and its solution v^0 satisfies :

$$\left\{\begin{array}{l} \dfrac{\partial v^0}{\partial t} - UD_3 v^0 + (v^0.\nabla)\ v^0 + \nabla p^0 = f, \quad \text{in } \ \Omega_\infty, \\ \text{div } v^0 = 0, \quad \text{in } \ \Omega_\infty, \\ v_3^0 = 0, \quad \text{on } \ \Gamma_0, \\ v^0 = 0, \quad \text{on } \ \Gamma_h, \\ v^0 \ \text{ is periodic in the } x \text{ and } y, \text{ directions with periods } \ L_1, L_2, \\ v^0|_{t=0} = v_0. \end{array}\right. \tag{3}$$

It is obvious that we can not expect a convergence result between v^ε and v^0 (in $\mathbf{H^1(\Omega)}$ for example) and more precisely we are leading with a boundary layer problem close to some parts of the boundary Γ. In order to understand this fact, we introduce a corrector term. The technique of correctors consists a fundamental tool in the theory of singular perturbations. See for instance [6] and [1] for the definition and the approches related to the use of correctors. This development can be extended to higher order in ε giving thus a complete asymptotic expansion. This is the main objective of our work.

The article is organized as follows : in Section 2., we deal with the linearized Navier-Stokes problem which will be useful to understand the nonlinear NS; we state and prove a convergence result giving an asymptotic expansion of the solution at all the orders. The last section (Section 3.) is devoted to the study of the full NS problem for which a simple representation of its solution is explicitly given. The proofs of our results are not given in details. For that purpose, we address the reader to [3] where we can find these results and their complete proofs besides other questions treated in this article.

2. The linear Navier-Stokes problem.

In this paragraph we study the linear NS problem. Besides its intrinsic interest, it will be helpful for the derivation of the corrector in the (full) nonlinear NS problem. We note that even for the linear NS problem (with characteristic boundary), the question of asymptotic analysis of the solution is still open. This will be the object of a forthcoming work; see [2].

Let us recall now the system satisfied by the linear NS solution :

$$\left\{\begin{array}{l} \dfrac{\partial v^\varepsilon}{\partial t} - \varepsilon \Delta v^\varepsilon - UD_3 v^\varepsilon + \nabla p^\varepsilon = f, \quad \text{in } \ \Omega_\infty, \\ \text{div } v^\varepsilon = 0, \quad \text{in } \ \Omega_\infty, \\ v^\varepsilon = 0, \quad \text{on } \ \Gamma_\infty, \\ v^\varepsilon \ \text{ is periodic in the } x \text{ and } y \text{ directions with periods } \ L_1, L_2, \\ v^0|_{t=0} = v_0. \end{array}\right. \tag{4}$$

So, we have the following theorem :

Theorem 2..1 *For each $N \geq 1$, there exists $C > 0$ and for all $k \in [0, N]$ an explicit given function $\overline{\theta}^{k,\varepsilon}$ such that :*

$$\|v^\varepsilon - \sum_{k=0}^{N} \varepsilon^k (v^k + \overline{\theta}^{k,\varepsilon} + \varepsilon \varphi^k)\|_{L^\infty(0,T;\mathbf{L^2(\Omega)})} \ \leq \ C\ \varepsilon^{N+1}, \tag{5}$$

$$\|v^\varepsilon - \sum_{k=0}^{N} \varepsilon^k (v^k + \overline{\theta}^{k,\varepsilon} + \varepsilon \varphi^k)\|_{L^2(0,T;\mathbf{H^1(\Omega)})} \ \leq \ C\ \varepsilon^{N+1/2}, \tag{6}$$

where $\mathbf{L^2(\Omega)} = (L^2(\Omega))^3, \mathbf{H^1(\Omega)} = (H^1(\Omega))^3$, φ^k *is a known function independent of* ε *and* C *denotes a constant which depends on the data (and* N*) but not on* ε*. Here* v^k *denotes the solution of the limit problem (*$\varepsilon = 0$*) at order* k*.*

Sketch of the Proof : **Step 1.** We first write the limit solution at order zero which is the Euler problem solution. More precisely, it follows from this work that the limit solution, denoted by v^0, is solution (for all time) of the following system :

$$\left\{\begin{array}{l} \dfrac{\partial v^0}{\partial t} - UD_3 v^0 + \nabla p^0 = f, \quad \text{ in } \Omega_\infty, \\ \text{div } v^0 = 0, \quad \text{ in } \Omega_\infty, \\ v_3^0 = 0, \quad \text{ on } \Gamma_0, \\ v^0 = 0, \quad \text{ on } \Gamma_h, \\ v^0 \text{ is periodic in the } x \text{ and } y, \text{ directions with periods } L_1, L_2, \\ v^0|_{t=0} = v_0, \end{array}\right. \tag{7}$$

We note that the problem of existence and uniqueness of v^0 is trivial.
Now, we observe the difference between the boundary values of v_τ^ε and v_τ^0 on the boundary Γ_0 (see $(4)_3$ and $(7)_3$); they do not match. Hence, the corrector function recovering this variation is given by the following system :

$$\left\{\begin{array}{l} -\varepsilon D_3^2 \theta^{0,\varepsilon} - UD_3 \theta^{0,\varepsilon} = 0, \quad \text{ on } (0,h), \\ \text{div } \theta^{0,\varepsilon} = 0, \quad \text{ in } \Omega_\infty, \\ \theta^{0,\varepsilon} = -\gamma_0 v^0, \quad \text{ on } \Gamma_0, \\ \theta^{0,\varepsilon} = 0, \quad \text{ on } \Gamma_h, \end{array}\right. \tag{8}$$

where $D_3 = \partial/\partial z$ and γ_0 (resp. γ_h) is the trace on $z = 0$ (respectively on $z = h$).
Our method here is quit different of that proposed in [8] since we give a simple representation of the corrector and we extend the asymptotic expansion of the solution of the linear NS problem at all orders. More precisely, we solve approximatively the system 8; we derive the tangential component of $\theta^{0,\varepsilon}$ and then by using the incompressibility condition we deduce its normal component. In order to simplify as much as possible the expression of our corrector, we neglect all the exponentially small terms ($e.s.t.$) in the expression of the corrector $\theta^{0,\varepsilon}$ and we consider its approximation, denoted $\overline{\theta}^{0,\varepsilon}$, as the final form of our corrector.
This simplification in our method generates some losses in the desired boundary conditions. Most of them are $e.s.t.$; they will be treated for all the orders together at the last step. But, the boundary value of the normal component of the corrector $\theta^{0,\varepsilon}$ is about $O(\varepsilon)$. Hence, we treat it differently by considering an additional corrector denoted by $\varphi^{0,\varepsilon} = \varepsilon\varphi^0$, which is solution of the following system :

$$\left\{\begin{array}{l} -\Delta\varphi^0 + \nabla\pi^0 = 0, \quad \text{ in } \Omega_\infty, \\ \text{div } \varphi^0 = 0, \quad \text{ in } \Omega_\infty, \\ \varphi_\tau^0 = 0, \quad \text{ on } \Gamma_0 \cup \Gamma_h, \\ \varphi_n^0 = -\dfrac{1}{\varepsilon}\gamma_0 \overline{\theta}_n^{0,\varepsilon} = \dfrac{1}{U}\text{div}_\tau(\gamma_0 v_\tau^0), \quad \text{ on } \Gamma_0, \\ \varphi_n^0 = 0, \quad \text{ on } \Gamma_h. \end{array}\right. \tag{9}$$

It is easy to obtain an approximation $\overline{\theta}^{0,\varepsilon}$ of the corrector $\theta^{0,\varepsilon}$ as described previously. We have thus :

$$\left\{\begin{array}{rcl} \theta^{0,\varepsilon} &=& \overline{\theta}^{0,\varepsilon} + e.s.t., \\ \overline{\theta}_\tau^{0,\varepsilon} &=& a_0^0 \mathrm{e}^{-Uz/\varepsilon}, \qquad \overline{\theta}_n^{0,\varepsilon} = \varepsilon b_0^0 \mathrm{e}^{-Uz/\varepsilon}, \\ a_0^0 &=& -\gamma_0 v_\tau^0, \qquad\qquad b_0^0 = -\dfrac{1}{U}\text{div}_\tau(\gamma_0 v_\tau^0). \end{array}\right. \tag{10}$$

Step 2. We proceed now with the construction of the corrector at order 1. we note that this order is different of the zeroth order and the orders higher than 2. This disparity takes place in the equation of the corrector and in the mode equation called also the Prandtl equation which reads as follows :

$$\left\{\begin{array}{l} \dfrac{\partial v^1}{\partial t} - UD_3 v^1 + \nabla p^1 = -\dfrac{\partial \varphi^0}{\partial t} + UD_3\varphi^0 + \Delta v^0, \quad \text{in } \Omega_\infty, \\ \text{div } v^1 = 0, \quad \text{in } \Omega_\infty, \\ v_3^1 = 0, \quad \text{on } \Gamma_0, \\ v^1 = 0, \quad \text{on } \Gamma_h, \\ v^1 \text{ is periodic in } x \text{ and } y. \end{array}\right. \tag{11}$$

The corrector $\theta^{1,\varepsilon}$ satisfies the following system :

$$\left\{\begin{array}{l} -\varepsilon D_3^2\theta^{1,\varepsilon} - UD_3\theta^{1,\varepsilon} = -\dfrac{1}{\varepsilon}\dfrac{\partial \overline{\theta}^{0,\varepsilon}}{\partial t}, \quad \text{in } \Omega_\infty, \\ \text{div } \theta^{1,\varepsilon} = 0, \quad \text{in } \Omega_\infty, \\ \theta^{1,\varepsilon} = -v^1, \quad \text{on } \Gamma_0 \cup \Gamma_h. \end{array}\right. \tag{12}$$

Identically as in Step 1, an additional corrector $\varphi^{1,\varepsilon} = \varepsilon\varphi^1$ is introduced :

$$\left\{\begin{array}{l} -\Delta\varphi^1 + \nabla\pi^1 = 0, \quad \text{in } \Omega, \\ \text{div } \varphi^1 = 0, \quad \text{in } \Omega, \\ \varphi_\tau^1 = 0, \quad \text{on } \Gamma_0 \cup \Gamma_h, \\ \varphi_n^1 = -\dfrac{1}{\varepsilon}\gamma_0\overline{\theta}_n^{1,\varepsilon} = -b_0^1, \quad \text{on } \Gamma_0, \\ \varphi_n^1 = 0, \quad \text{on } \Gamma_h. \end{array}\right. \tag{13}$$

Step 3. At this stage, we consider $N \geq 2$, and assume that all the orders preceding the order N are treated. Hence, the equations of the mode v^N have the following form :

$$\left\{\begin{array}{l} \dfrac{\partial v^N}{\partial t} - UD_3 v^N + \nabla p^N \\ \qquad = -\dfrac{\partial \varphi^{N-1}}{\partial t} + UD_3\varphi^{N-1} + \Delta v^{N-1}, \quad \text{in } \Omega, \\ \text{div } v^N = 0, \quad \text{in } \Omega, \\ v_3^N = 0, \quad \text{on } \Gamma_0, \\ v^N = 0, \quad \text{on } \Gamma_h, \\ v^N \text{ is periodic in } x \text{ and } y. \end{array}\right. \tag{14}$$

The final form of the corrector at this order is an approximation of $\theta^{N,\varepsilon}$ solution of

$$\left\{\begin{array}{l} -\varepsilon D_3^2\theta_\tau^{N,\varepsilon} - UD_3\theta_\tau^{N,\varepsilon} = -\dfrac{1}{\varepsilon}\Big\{\dfrac{\partial \overline{\theta}_\tau^{N-1,\varepsilon}}{\partial t} - \Delta\overline{\theta}_\tau^{N-2,\varepsilon}\Big\}, \quad \text{on } (0,h), \\ \text{div } \theta^{N,\varepsilon} = 0, \quad \text{in } \Omega, \\ \theta_\tau^{N,\varepsilon} = -v_\tau^N, \quad \text{on } \Gamma_0 \cup \Gamma_h, \end{array}\right. \tag{15}$$

and the additional corrector $\varphi^{N,\varepsilon} = \varepsilon\varphi^N$ verifies :

$$\left\{\begin{array}{l} -\Delta\varphi^N + \nabla\pi^N = 0, \quad \text{in } \Omega, \\ \text{div } \varphi^N = 0, \quad \text{in } \Omega, \\ \varphi_\tau^N = 0, \quad \text{on } \Gamma_0 \cup \Gamma_h, \\ \varphi_n^N = -\dfrac{1}{\varepsilon}\gamma_0\overline{\theta}_n^{N,\varepsilon} = -b_0^N, \quad \text{on } \Gamma_0, \\ \varphi_n^N = 0, \quad \text{on } \Gamma_h. \end{array}\right. \tag{16}$$

Finally, to conclude the estimates stated in Theorem 2..1 we introduce

$$\left\{ \begin{array}{rcl} w^{N,\varepsilon} & = & v^\varepsilon - \displaystyle\sum_{k=0}^{N} \varepsilon^k (v^k + \overline{\theta}^{k,\varepsilon} + \varphi^{k,\varepsilon} + \widetilde{\varphi}^{k,\varepsilon}), \\ p^\varepsilon & = & \displaystyle\sum_{k=0}^{N} \varepsilon^k (p^k + \pi^{k,\varepsilon}), \end{array} \right. \tag{17}$$

where $\widetilde{\varphi}^{k,\varepsilon}$ is the corrector introduced to recover the *e.s.t.* losses. More precisely, it is defined as follows :

$$\left\{ \begin{array}{l} -\varepsilon\,\Delta\widetilde{\varphi}^{N,\varepsilon} + \nabla\widetilde{\pi}^{N,\varepsilon} = 0, \quad \text{in } \ \Omega, \\ \text{div } \widetilde{\varphi}^{N,\varepsilon} = 0, \quad \text{in } \ \Omega, \\ \widetilde{\varphi}_\tau^{N,\varepsilon} = 0, \quad \text{on } \ \Gamma_0, \\ \widetilde{\varphi}_\tau^{N,\varepsilon} = -\dfrac{1}{\varepsilon^N} \displaystyle\sum_{k=0}^{N} \varepsilon^k \gamma_h \overline{\theta}_\tau^{k,\varepsilon} = O(\varepsilon^{-2N}\,\mathrm{e}^{-Uh/\varepsilon}), \quad \text{on } \ \Gamma_h, \\ \widetilde{\varphi}_n^{N,\varepsilon} = 0, \quad \text{on } \ \Gamma_0, \\ \widetilde{\varphi}_n^{N,\varepsilon} = -\dfrac{1}{\varepsilon^N} \displaystyle\sum_{k=0}^{N} \varepsilon^k \gamma_h \overline{\theta}_n^{k,\varepsilon} = O(\varepsilon^{-2N+1}\,\mathrm{e}^{-Uh/\varepsilon}), \quad \text{on } \ \Gamma_h. \end{array} \right. \tag{18}$$

Then, we write the equations and boundary conditions verified by $w^{N,\varepsilon}$ and p^ε. Thereafter, we multiply the equation of $w^{N,\varepsilon}$ by $w^{N,\varepsilon}$ and integrate over Ω, we obtain an energy inequality. By the use of Hardy's inequality, we deduce the desired result.

3. The full nonlinear Navier-Stokes problem.

Firstly, we note that the equations corresponding to this problem are given in the introduction by the system (2).
In the same way, our aim here is to derive a complete asymptotic expansion of the solution v^ε when $\varepsilon \to 0$. We limit ourselves to the order 0 and 1. But, we believe that this work can be extended to higher orders.

Hence, The second result in this article concerns the solution of the full nonlinear problem (2) and more precisely we prove analogous results to the NS linear problem unless that the convergence results are only valid until a fixed time :

Theorem 3..1 *For v^ε solution of the Navier-Stokes problem (2), there exist a time $T_* > 0$, corrector functions $\overline{\theta}^{0,\varepsilon}$ and $\overline{\theta}^{1,\varepsilon}$ explicitly given, and a constant $\kappa > 0$ depending on the data but not on ε, such that :*

$$\|v^\varepsilon - (v^0 + \overline{\theta}^{0,\varepsilon} + \varepsilon\varphi^0) - \varepsilon(v^1 + \overline{\theta}^{1,\varepsilon} + \varepsilon\varphi^1)\|_{L^\infty(0,T_*;\,\mathbf{L^2(\Omega)})} \leq \kappa\,\varepsilon^2, \tag{19}$$

$$\big\|v^\varepsilon - (v^0 + \overline{\theta}^{0,\varepsilon} + \varepsilon\varphi^0) - \varepsilon(v^1 + \overline{\theta}^{1,\varepsilon} + \varepsilon\varphi^1)\big\|_{L^2(0,T_*;\,\mathbf{H^1(\Omega)})} \leq \kappa\,\varepsilon^{3/2}. \tag{20}$$

Sketch of the Proof . Notice that the limit problem at order 0 is already given by (3) and that all the additional correctors $\varphi^{N,\varepsilon}$ keep the same definitions as in the linear case. Also, we keep the corrector $\overline{\theta}^{0,\varepsilon}$. But, the mode v^1 and the corrector of order 1 are here different.

Thus, the mode v^1 is solution of

$$\left\{\begin{array}{l}\dfrac{\partial v^1}{\partial t} - UD_3v^1 + (v^1.\nabla)\ v^0 + (v^0.\nabla)\ v^1 + \nabla p^1 \\ \qquad = -\dfrac{\partial \varphi^0}{\partial t} + UD_3\varphi^0 - (v^0.\nabla)\ \varphi^0 - (\varphi^0.\nabla)\ v^0 + \Delta v^0, \quad \text{in } \Omega_\infty, \\ \text{div } v^1 = 0, \quad \text{in } \Omega_\infty, \\ v_3^1 = 0, \quad \text{on } \Gamma_0, \\ v^1 = 0, \quad \text{on } \Gamma_h, \\ v^1 \text{ is periodic in the } x \text{ and } y, \text{ directions with periods } L_1, L_2.\end{array}\right. \tag{21}$$

The corresponding corrector $\theta^{1,\varepsilon}$ is chosen as the solution of :

$$\left\{\begin{array}{l}-\varepsilon D_3^2\theta_\tau^{1,\varepsilon} - UD_3\theta_\tau^{1,\varepsilon} = -\dfrac{1}{\varepsilon}[\dfrac{\partial \overline{\theta}_\tau^{0,\varepsilon}}{\partial t} + (\overline{\theta}_\tau^{0,\varepsilon}.\nabla_\tau)\ v_\tau^0 + (v^0.\nabla)\overline{\theta}_\tau^{0,\varepsilon}] - \\ \qquad\qquad -\varphi_3^0 D_3\overline{\theta}_\tau^{0,\varepsilon} - \dfrac{1}{\varepsilon}(\overline{\theta}^{0,\varepsilon}.\nabla)\ \overline{\theta}_\tau^{0,\varepsilon}, \quad \text{in } \Omega_\infty, \\ \text{div } \theta^{1,\varepsilon} = 0, \quad \text{in } \Omega_\infty, \\ \theta^{1,\varepsilon} = -v^1, \quad \text{on } \Gamma_0 \cup \Gamma_h.\end{array}\right. \tag{22}$$

Contrary to the linear case, it is important here to emphasize that equation $(22)_1$ is valid only for the tangential component of the corrector $\theta^{1,\varepsilon}$. The equation satisfied by the normal component $\theta_n^{1,\varepsilon}$ is different and it will be derived later on. It will be slightly different than for the tangential corrector equation since the nonlinear term is not divergence free.
Of course, we derive an approximation $\overline{\theta}^{1,\varepsilon}$ of $\theta^{1,\varepsilon}$ up to *e.s.t.* which will be considered for the final form of the corrector at order 1.
Now, the normal component of $\theta^{1,\varepsilon}$ satisfies :

$$\left\{\begin{array}{l}-\varepsilon D_3^2\overline{\theta}_n^{1,\varepsilon} - UD_3\overline{\theta}_n^{1,\varepsilon} \\ \quad = -\dfrac{1}{2U}\text{div}_\tau[((\gamma_0 v_\tau^0).\nabla_\tau)(\gamma_0 v_\tau^0) + \text{div}_\tau(\gamma_0 v_\tau^0)\gamma_0 v_\tau^0]\ \mathrm{e}^{-2Uz/\varepsilon} + \\ \quad + \dfrac{1}{\varepsilon}\displaystyle\int_h^z \mathrm{e}^{-U\zeta/\varepsilon}\text{div}_\tau\Big[[U\varphi_3^0\gamma_0 v_\tau^0 - ((\gamma_0 v_\tau^0).\nabla_\tau)v_\tau^0](x,y,\zeta)\Big]\mathrm{d}\zeta - \\ \quad -\dfrac{1}{\varepsilon}\dfrac{\partial \overline{\theta}_n^{0,\varepsilon}}{\partial t} - \dfrac{1}{\varepsilon}v_n^0 D_3\overline{\theta}^{0,\varepsilon}, \\ |\ \gamma_0\overline{\theta}_n^{1,\varepsilon}| \leq c\ \varepsilon, \quad \text{on } \Gamma_0, \\ \overline{\theta}_n^{1,\varepsilon} = O(\varepsilon\ \mathrm{e}^{-Uh/\varepsilon}), \quad \text{on } \Gamma_h.\end{array}\right. \tag{23}$$

Finally, to recover all the *e.s.t.* losses we introduce the global additional corrector $\widetilde{\varphi}^{1,\varepsilon}$. We define then the following quantity :

$$w^{1,\varepsilon} = v^\varepsilon - (v^0 + \overline{\theta}^{0,\varepsilon} + \varepsilon\varphi^0) - \varepsilon(v^1 + \overline{\theta}^{1,\varepsilon} + \widetilde{\varphi}^{1,\varepsilon}),$$

for which we write the corresponding equations and boundary conditions, then multiply by itself and integrate over Ω. After some calculations this yields :

$$\frac{d}{dt}\|w^{1,\varepsilon}\|^2_{\mathbf{L^2(\Omega)}} + \varepsilon\|\nabla w^{1,\varepsilon}\|^2_{\mathbf{L^2(\Omega)}} \leq \ \leq \kappa\varepsilon^4 + c\ \|w^{1,\varepsilon}\|^2_{\mathbf{L^2(\Omega)}}, \tag{24}$$

to which we apply the Gronwall inequality to conclude the proof of Theorem 3..1. □

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