Int. Conference on Boundary and Interior Layers
BAIL 2006
G. Lube, G. Rapin (Eds)
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A Difference Scheme of Improved Accuracy for a Quasilinear Singularly Perturbed Elliptic Convection-Diffusion Equation in the Case of the Third Kind Boundary Condition *

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1. Introduction

In the case of boundary value problems for elliptic convection-diffusion equations, the order of the ε-uniform convergence of well known special methods based on classical grid approximations does not exceed 1. To improve accuracy of discrete solutions for *regular* boundary value problems, the Richardson method (see, e.g., [1] and also the bibliography therein) turned out to be very effective. The same method was successfully applied for improvement of the ε-uniform convergence order of solutions for *linear singularly perturbed* boundary value problems (see, e.g., [2]–[5] and also the bibliography therein). Using new approaches based on the Richardson technique, ε-uniformly convergent finite difference schemes with improved accuracy were constructed in the case of the Dirichlet problem for *quasilinear* singularly perturbed equations of parabolic type [7] and elliptic type [6]) (reaction-diffusion on a strip).

In the present paper we study a mixed boundary value problem for the quasilinear singularly perturbed elliptic convection-diffusion equation on a vertical strip. *The singularly perturbed third kind boundary condition* admitting both Dirichlet and Neumann conditions is given on that part of the domain boundary in a neighbourhood of which a boundary layer appears. For such a problem, solutions of difference schemes constructed on the basis of classical approximations, in general, are not ε-uniformly bounded. We construct a base (nonlinear) scheme on a piecewise uniform mesh condensing in the boundary layer that converges ε-uniformly at the rate $\mathcal{O}\left(N_1^{-1} \ln N_1 + N_2^{-1}\right)$, where N_1+1 and N_2+1 are the number of nodes in meshes with respect to the x_1-axis and on a unit interval on the x_2-axis, respectively. On the basis of this nonlinear base scheme, using the Richardson technique, we construct a linearized iterative scheme convergent ε-uniformly at the rate $\mathcal{O}\left(N_1^{-2} \ln^2 N_1 + N_2^{-2} + q^t\right)$, here t is the number of iterations, $q < 1$. The use of upper and lower solutions of the iterative Richardson scheme as a stopping criterion allows us during the computational process to define a current iteration under which the same ε-uniform accuracy of the solution is achieved as for the nonlinear Richardson scheme $\mathcal{O}\left(N_1^{-2} \ln^2 N_1 + N_2^{-2}\right)$.

2. Problem formulation

On a vertical strip $\overline{D}$

$$D = \{x : 0 < x_1 < d, \quad x_2 \in I\!R\}, \tag{2.1}$$

we consider the mixed boundary value problem for the quasilinear singularly perturbed elliptic convection-diffusion equation in that case, when the one of the boundary conditions is of the

*This research was supported in part by the Russian Foundation for Basic Research (grants No. 04-01-00578, 04–01–89007–NWO_a), by the Dutch Research Organisation NWO (grant No. 047.016.008), and also the Boole Centre for Research in Informatics at the National University of Ireland, Cork.

first kind and the second one is of the third kind that is singularly perturbed, [1]

$$L_{(2.2)}\big(u(x)\big) \equiv L^2_{(2.2)}\, u(x) - f\big(x,\, u(x)\big) = 0, \quad x \in D, \tag{2.2}$$
$$l\, u(x) = \psi(x), \quad x \in \Gamma_1, \qquad u(x) = \varphi(x), \quad x \in \Gamma_2.$$

Here $\Gamma = \overline{D} \setminus D$, $\Gamma = \Gamma_1 \cup \Gamma_2$, Γ_1 and Γ_2 are the left and right parts of the boundary Γ,

$$L^2_{(2.2)} = L^{(2)}_{(2.2)} + L^{(1)}_{(2.2)}, \quad L^{(2)}_{(2.2)} \equiv \varepsilon \sum_{s=1,2} a_s(x) \frac{\partial^2}{\partial x_s^2}, \quad L^{(1)}_{(2.2)} \equiv \sum_{s=1,2} b_s(x) \frac{\partial}{\partial x_s} - c(x), \quad x \in D,$$

$$l \equiv -\varepsilon\, \alpha^R(x)\, \frac{\partial}{\partial x_1} + \beta^R(x), \quad x \in \Gamma_1,$$

the functions $a_s(x)$, $b_s(x)$, $c(x)$, $f(x,\, u)$ and $\alpha^R(x)$, $\beta^R(x)$, $\psi(x)$, $\varphi(x)$ are assumed to be sufficiently smooth on $\overline{D}$, $\overline{D} \times I\!R$ Γ_1, Γ_2 respectively, moreover,[2]

$$a_0 \le a_s(x) \le a^0, \quad b_0 \le b_s(x) \le b^0, \quad s = 1, 2, \quad |c(x)| \le c^0, \quad x \in \overline{D}; \tag{2.3a}$$
$$|f(x,\, u)| \le M, \quad c_1 \le c(x) + \frac{\partial}{\partial u} f(x,\, u) \le c^1, \quad (x,\, u) \in \overline{D} \times I\!R;$$
$$0 \le \alpha^R(x), \quad \beta^R(x) \le M, \quad \alpha^R(x) + \beta^R(x) \ge m, \quad |\psi(x)| \le M, \quad x \in \Gamma_1;$$
$$|\varphi(x)| \le M, \quad x \in \Gamma_2; \quad a_0,\, b_0,\, c_1 > 0;$$

the perturbation parameter ε takes arbitrary values in the half-open interval $(0, 1]$.

For $\alpha^R(x) = 0$, $x \in \Gamma_1$ problem (2.2), (2.1) is a Dirichlet problem; for $\beta^R(x) = 0$, $x \in \Gamma_1$ is a mixed problem with Neumann conditions on the set Γ_1. For simplicity, we assume that the following condition is satisfied

$$\textit{either } \alpha^R(x) = 0, \quad \textit{or } \alpha^R(x) \ge m, \quad x \in \Gamma_1. \tag{2.3b}$$

When the parameter ε tends to zero, a boundary layer appears in a neighbourhood of the set Γ_1 (the part of the boundary Γ, through which characteristics of the reduced equation passing through points $x \in D$, leave the set D).

Our **aim** is for the boundary value problem (2.2), (2.1) with using a Richardson technique, to construct a difference scheme convergent ε-uniformly with the accuracy order more than one.

3. Base scheme for problem (2.2), (2.1)

We give *a-priori* estimates of solutions and derivatives for boundary value problem (2.2), (2.1). We represent the solution of the problem as the sum of functions

$$u(x) = U(x) + V(x), \quad x \in \overline{D}, \tag{3.1}$$

where $U(x)$ and $V(x)$ are the regular and singular parts of the solution.

For $U(x)$, $V(x)$, using the technique from [6, 7], we obtain the estimates

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}}\, U(x) \right| \le M\, [1 + \varepsilon^{n+1-k}], \tag{3.2a}$$

$$\left| \frac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2}}\, V(x) \right| \le M\, [\varepsilon^{-k_1} + \varepsilon^{1-k}]\, \exp(-m\, \varepsilon^{-1}\, x_1), \quad x \in \overline{D}, \quad k \le K, \tag{3.2b}$$

[1] Throughout the paper, the notation $L_{(j.k)}$ ($M_{(j.k)}$, $G_{h(j.k)}$) means that these operators (constants, grids) are introduced in formula $(j.k)$.

[2] Throughout this paper, M, M_i (or m) denote sufficiently large (small) positive constants that do not depend on ε and on the discretization parameters.

where m is an arbitrary number in the interval $(0, m_0)$, $m_0 = \min_{\overline{D}} [a_1^{-1}(x) b_1(x)]$; $K = n + 2$ for sufficient smoothness of the data of boundary value problem (2.2), (2.1).

Theorem 1 *Let the data of the boundary value problem* (2.2), (2.1) *satisfy the condition* $a_s,\, b_s,\, c \in C^{3n+2+\alpha}(\overline{D})$, $f \in C^{3n+2+\alpha}(\overline{D} \times I\!R)$, α^R, β^R, $\psi \in C^{n+2+\alpha}(\Gamma_1)$, $\varphi \in C^{3n+2+\alpha}(\Gamma_2)$, $s = 1, 2$, $n \geq 1$, $\alpha > 0$. *Then for the solution of the boundary value problem and its component in the representation* (3.1) *the estimates* (3.2), *where* $K = n + 2$, *are satisfied.*

First we give ε-uniformly convergent finite difference scheme constructing on the base of classical approximation of problem (2.2), (2.1). We will use the solutions of the base scheme for construction of discrete solutions with improved accuracy order.

On the set $\overline{D}$ we introduce the rectangular mesh

$$\overline{D}_h = \overline{\omega}_1 \times \omega_2, \tag{3.3}$$

where $\overline{\omega}_1$ and ω_2 are arbitrary, in general, nonuniform meshes on the interval $[0, d]$ and at the x_2-axis respectively. Let $h_s^i = x_s^{i+1} - x_s^i$, x_s^i, $x_s^{i+1} \in \overline{\omega}_1$ for $s = 1$ and x_s^i, $x_s^{i+1} \in \omega_2$ for $s = 2$; let $h_s = \max_i h_s^i$, $h = \max_s h_s$. Assume that $h \leq M\, N^{-1}$, where $N = \min[N_1, N_2]$; $N_1 + 1$ and $N_2 + 1$ are the number of nodes in the mesh $\overline{\omega}_1$ and the minimal number of nodes in the mesh ω_2 on a unit interval.

Problem (2.2), (2.1) is approximated by the finite difference scheme

$$\Lambda\left(z(x)\right) \equiv \Lambda^2\, z(x) - f(x,\, z(x)) = 0, \quad x \in D_h, \tag{3.4}$$

$$\lambda\, z(x) = \psi(x), \quad x \in \Gamma_{1h}. \qquad z(x) = \varphi(x), \quad x \in \Gamma_{2h}.$$

Here $D_h = D \bigcap \overline{D}_h$, $\Gamma_{ih} = \Gamma_i \bigcap \overline{D}_h$, $i = 1, 2$,

$$\Lambda^2\, z(x) \equiv \left\{ \varepsilon \sum_{s=1,2} a_s(x)\, \delta_{\overline{xs}\widehat{xs}} + \sum_{s=1,2} b_s(x)\, \delta_{xs} - c(x) \right\} z(x), \quad x \in D_h,$$

$$\lambda\, z(x) \equiv \left\{ -\varepsilon\, \alpha^R(x)\, \delta_{x_1} + \beta^R(x) \right\} z(x), \quad x \in \Gamma_{1h},$$

$\delta_{xs}\, z(x)$ and $\delta_{\overline{xs}\widehat{xs}}$ are the first (forward) and second difference derivatives; for example, $\delta_{\overline{x1}\,\widehat{x1}}\, z(x) = 2\,(h_1^i + h_1^{i-1})^{-1}\, [\delta_{x1} z(x) - \delta_{\overline{x1}} z(x)]$, $x = (x_1^i, x_2)$.

Scheme (3.4), (3.3) is monotone ε-uniformly.

Lemma 1 *Solutions of difference schemes* (3.4), (3.3) *are not ε-uniformly bounded. In the case of the condition $\beta^R(x) = 0$, $x \in \Gamma_1$, the condition $N_1^{-1} = \mathcal{O}(\varepsilon)$ is sufficient for the boundedness of the discrete solutions; under the condition $\beta^R(x) \geq m$, $x \in \Gamma_1$, the discrete solutions are ε-uniformly bounded.*

Let us consider a scheme on *piecewise uniform* meshes.

On the set $\overline{D}$ we construct the mesh

$$\overline{D}_h = \overline{\omega}_1^* \times \omega_2, \tag{3.5a}$$

Here $\omega_2 = \omega_2^u$ is a *uniform* mesh, $\overline{\omega}_1^*$ is a mesh with a piecewise constant step-size. When constructing the mesh $\overline{\omega}_1^*$, the interval $[0, d]$ is divided into two parts $[0, \sigma]$, $[\sigma, d]$, σ is a parameter in the interval $(0, d)$. In each interval the step-sizes are constant and equal to $h_1^{(1)} = 2\,\sigma\, N_1^{-1}$ in $[0, \sigma]$ and $h_1^{(2)} = 2\,(d - \sigma)\, N_1^{-1}$ in $[\sigma, d]$. The parameter σ is defined by

$$\sigma = \sigma(\varepsilon, N_1, d;\, l, m) = \min[2^{-1}\, d,\; l\, m^{-1}\, \varepsilon\, \ln N_1], \tag{3.5b}$$

where $m = m_{(3.2)}$, $l > 0$ is a mesh parameter; $N = \min[N_1,\, N_2]$. The mesh $\overline{\omega}_1^*$, and hence the mesh $\overline{D}_h = \overline{D}_h(l)$ are constructed.

For solutions of boundary value problem (2.2), (2.1) we use the scheme (3.4) on the mesh

$$\overline{D}_h = \overline{D}_{h(3.5)}(l = 1). \tag{3.6}$$

Lemma 2 *Solutions of difference schemes* (3.4), (3.6) *are ε-uniformly bounded.*

For solutions of difference scheme (3.4), (3.6), i.e, a nonlinear base scheme, we obtain the ε-uniform estimate

$$|u(x) - z(x)| \le M \left[N_1^{-1} \ln N_1 + N_2^{-1}\right], \quad x \in \overline{D}_h. \tag{3.7}$$

Theorem 2 *Let solutions of boundary value problem* (2.2), (2.1) *satisfy a priori estimates* (3.2) *for $K = 3$. Then the solution of nonlinear base difference scheme* (3.4), (3.6) *for $N \to \infty$ converges ε-uniformly to the solution of the boundary value problem at the rate $\mathcal{O}\left(N_1^{-1} \ln N_1 + N_2^{-1}\right)$. For the discrete solution the error estimate* (3.7) *is valid.*

4. Richardson method for problem (2.2), (2.1)

On the set $\overline{D}$ we construct meshes

$$\overline{D}_h^i = \overline{\omega}_1^{*i} \times \omega_2^i, \quad i = 1, 2, \tag{4.1a}$$

uniform in x_2 and *piecewise-uniform* in x_1. Here $\overline{D}_h^2$ is $\overline{D}_{h(3.5a)}$, where

$$\sigma = \sigma_{(3.5b)}(\varepsilon,\, N_1,\, l) \ \ for \ \ l \ge 2; \tag{4.2}$$

$\overline{D}_h^1$ is a "coarse" mesh. For the parameters σ^i, which define piecewise uniform meshes $\overline{\omega}_1^{*i} = \overline{\omega}_1^{*i}(\sigma^i)$, we impose the condition $\sigma^1 = \sigma^2$, where $\sigma^2 = \sigma_{(4.2)}$, i.e., intervals on which the meshes $\overline{\omega}_1^{*1}$ and $\overline{\omega}_1^{*2}$ have a constant step-size, are the same. Step-sizes in the mesh $\overline{\omega}_1^{*1}$ on the intervals $[0, \sigma]$, $[\sigma, d]$ are k times larger than step-sizes in the mesh $\overline{\omega}_1^{*2}$, and step-sizes in the mesh ω_2^1 are k times larger than step-sizes in the mesh ω_2^2; $k^{-1} N_1 + 1$ and $k^{-1} N_2 + 1$ are the number of nodes in the mesh $\overline{\omega}_1^{*1}$ and in the mesh ω_2^1 on a unit interval, respectively. Let

$$\overline{D}_h^0 = \overline{D}_h^1 \bigcap \overline{D}_h^2. \tag{4.1b}$$

$\overline{D}_h^0 = \overline{D}_h^1$ if k is integer, $(k \ge 2)$; $\overline{D}_h^0 \ne \overline{D}_h^1$ if k is noninteger.

Let $z^i(x)$, $x \in \overline{D}_h^i$, $i = 1,\, 2$ be solutions of the difference schemes

$$\Lambda_{(3.4)}(z^i(x)) = 0, \quad x \in D_h^i, \tag{4.3a}$$
$$\lambda_{(3.4)}\, z^i(x) = \psi(x), \quad x \in \Gamma_{1h}^i, \quad z^i(x) = \varphi(x), \quad x \in \Gamma_{2h}^i, \quad i = 1, 2.$$

Assume

$$z^0(x) = \gamma\, z^1(x) + (1 - \gamma)\, z^2(x), \quad x \in \overline{D}_h^0, \quad \gamma = \gamma(k) = -(k-1)^{-1}. \tag{4.3b}$$

We call the function $z^0(x)$, $x \in \overline{D}_h^0$ the solution of the difference scheme (4.3), (4.1), i.e. the scheme based on the Richardson method on two embedded meshes; the functions $z^1(x)$, $x \in \overline{D}_h^1$ and $z^2(x)$, $x \in \overline{D}_h^2$ are called the components generating the solution of the difference scheme (4.3), (4.1).

For justification of convergence to Richardson scheme (4.3), (4.1) under condition (4.2), we apply a technique similar to one used in [4, 6]. It is suitable to consider a problem solution in the form of a decomposition. Let us construct expansions for solutions of the difference scheme (3.4), (4.1a) under the condition (4.2).

To the decomposition

$$u(x) = U(x) + V(x), \quad x \in \overline{D} \tag{4.4a}$$

of the solution of boundary value problem (2.2), (2.1) (see the representation (3.1)) the following discrete decomposition of the solution of difference scheme (3.4), (3.5), (4.2) corresponds:

$$z(x) = z_U(x) + z_V(x), \quad x \in \overline{D}_h. \tag{4.4b}$$

The functions $z_U(x)$, $z_V(x)$ in the representation (4.4b) are solutions of the problems

$$\Lambda_{(3.4)}\big(z_U(x)\big) = 0, \quad x \in D_h,$$
$$\lambda_{(3.4)}\, z_U(x) = l_{(2.2)}\, U(x), \quad x \in \Gamma_{1h}, \quad z_U(x) = U(x), \quad x \in \Gamma_{2h};$$
$$\Lambda^2_{(3.4)}\, z_V(x) - \big[f\big(x,\, z_U(x) + z_V(x)\big) - f\big(x,\, z_U(x)\big)\big] = 0, \quad x \in D_h,$$
$$\lambda_{(3.4)}\, z_V(x) = \psi(x) - l_{(2.2)}\, U(x), \quad x \in \Gamma_{1h}, \quad z_V(x) = V(x), \quad x \in \Gamma_{2h}.$$

From the representation (4.4) and the expansion of its components it follows

$$z(x) = u(x) + N_1^{-1}\,[u_1^0(x) + u_1^1(x)] + N_2^{-1}\,[u_2^0(x) + u_2^1(x)] + \rho_u(x), \quad x \in \overline{D}_h. \tag{4.5}$$

For the components in expansion (4.5) for the function $z(x)$ the following estimates hold

$$|u_i^0(x)| \;\le\; M \ln N_1, \quad |u_i^1(x)| \le M\,\varepsilon \ln N_1, \quad x \in \overline{D}, \quad i = 1, 2,$$
$$|\rho_u(x)| \;\le\; M\,\big[\,N_1^{-2}\ln^2 N_1 + N_2^{-2}\,\big], \quad x \in \overline{D}_h.$$

Thus, for the function $z^0_{(4.3b)}(x)$, $x \in \overline{D}{}^0_h$, we obtain the estimate

$$|u(x) - z^0(x)| \le M\,\big[\,N_1^{-2}\ln^2 N_1 + N_2^{-2}\,\big], \quad x \in \overline{D}{}^0_h. \tag{4.6}$$

Theorem 3 *Let solutions of the boundary value problem* (2.2), (2.1) *satisfy a priori estimates* (3.2) *for* $K = 7$. *Then the function* $z^0_{(4.3b)}(x)$, $x \in \overline{D}{}^0_h$, *i.e. the solution of the Richardson scheme* (4.3), (4.1) *converges for* $N \to \infty$ *to the solution of boundary value problem* (2.2), (2.1) *ε-uniformly at the rate* $\mathcal{O}\big(N_1^{-2}\ln^2 N_1 + N_2^{-2}\big)$; *for the function* $z(x)$, $x \in \overline{D}_h$ *the expansion* (4.5) *is valid, and for the function* $z^0(x)$, $x \in \overline{D}{}^0_h$ *the error estimate* (4.6) *is valid.*

5. Linearized iterative base scheme

On mesh (3.3) we consider an iterative monotone two-level difference scheme in which the nonlinear term of the differential equation is computed using the sought function from the previous iterative level. To the boundary value problem (2.2), (2.1) corresponds the difference scheme

$$\Lambda_{(5.1)}(z(x,\, t)) \equiv \Lambda^2_{(3.4)} z(x,\, t) - p\,\delta_{\overline{t}}\, z(x,\, t) - f(x,\, \check{z}(x,\, t)) = 0, \quad (x,\, t) \in G_h, \tag{5.1a}$$
$$\lambda_{(3.4)}\, z(x,\, t) = \psi(x), \quad (x,\, t) \in S_{1h}, \quad z(x,\, t) = \varphi(x), \quad (x,\, t) \in S_h \setminus S_{1h}.$$

Here

$$\overline{G}_h = \overline{G}_h(\overline{D}_h) = G_h \bigcup S_h, \quad \overline{G}_h = \overline{D}_h \times \overline{\omega}_0, \quad G_h = D_h \times \omega_0, \quad S_{1h} = \Gamma_{1h} \times \omega_0, \tag{5.1b}$$

$\overline{\omega}_0$ is a *uniform* mesh on the semiaxis $t \ge 0$ with the step-size $h_t = 1$, the variable $t \in \overline{\omega}_0$ defines the number of iteration; $S_h = S_h^L \cup S_{h0}$, $S_h^L = \Gamma_h \times \omega_0$ is the lateral part of the boundary; $\delta_{\overline{t}}\, z(x,\, t) = h_t^{-1}\,[z(x,\, t) - \check{z}(x,\, t)]$, $\check{z}(x,\, t) = z(x,\, t - h_t)$, $(x,\, t) \in G_h$; the coefficient p satisfies the condition

$$p - \frac{\partial}{\partial u} f(x,\, u) \ge p_0, \quad (x,\, u) \in \overline{D} \times I\!R, \quad p_0 > 0, \tag{5.1c}$$

ensuring the monotonicity of the difference scheme. The function $\varphi_{(5.1)}(x)$, $x \in \overline{D}$ on the boundary Γ_2 satisfies the condition $\varphi_{(5.1)}(x) = \varphi_{(2.2)}(x)$, $x \in \Gamma_2$, moreover, $\varphi_{(5.1)}(x)$, $x \in \overline{D}$ is sufficiently arbitrary bounded function. We call the function $z(x,\, t)$, $(x,\, t) \in \overline{G}_h$, where

$\overline{G}_h$ is generated by the mesh $\overline{D}_{h(3.3)}$, the solution of the linearized iterative difference scheme (5.1), (3.3).

In the case of schemes (3.4), (3.6) and (5.1), (3.6), using the majorant function technique, we find the following estimate for their solutions:

$$|z(x) - z(x,\,t)| \le M\, q^{\,t}, \quad (x,\,t) \in \overline{G}_h, \tag{5.2}$$

where $z(x) = z_{(3.4;3.6)}(x)$, $z(x,\,t) = z_{(5.1;3.6)}(x,\,t)$; $q \le q_0 \equiv p^0(c_{10} + p^0)^{-1}$, $p^0 = \max\Big(p - \dfrac{\partial}{\partial u}\, f(x,\,u)\Big)$, $c_{10} = \min\Big(c(x) + \dfrac{\partial}{\partial u}\, f(x,\,u)\Big)$, $(x,\,u) \in \overline{D} \times I\!R$.

In the case of the mesh (3.6) we obtain the estimate

$$|u(x) - z(x,\,t)| \le M\ \left[N_1^{-1}\ln N_1 + N_2^{-1} + q^{\,t}\right], \quad (x,\,t) \in \overline{G}_h, \tag{5.3}$$

where $q \le q_{0(5.2)}$. Difference scheme (5.1), (3.6) converges ε-uniformly as $N_1,\, N_2,\, t \to \infty$.

Theorem 4 *Let hypothesis of Theorem* 2 *be fulfilled. Then the solution of the linearized iterative difference scheme* (5.1), (3.6) *for* $N_1,\, N_2,\, t \to \infty$ *converges to the solution of the boundary value problem* (2.2), (2.1) ε*-uniformly at the rate* $\mathcal{O}\left(N_1^{-1}\ln N_1 + N_2^{-1} + q_0^t\right)$, *where* $q_0 = q_{0(5.2)}$. *For the discrete solutions the error estimates* (5.2), (5.3) *are valid.*

6. Linearized iterative scheme of improved accuracy

We now give a linearized iterative difference scheme of improved accuracy which is constructed using a Richardson technique.

On the meshes

$$\overline{G}{}_h^i = \overline{D}{}_h^i \times \overline{\omega}_0, \quad i = 1, 2, \tag{6.1a}$$

where $\overline{D}{}_h^i = \overline{D}{}_{h(4.1)}^i$, $\overline{\omega}_0 = \overline{\omega}_{0(5.1)}$, we consider the functions $z^i(x,\,t)$, $(x,\,t) \in \overline{G}{}_h^i$, $i = 1, 2$, i.e. solutions of the iterative schemes

$$\Lambda_{(5.1)}(z^i(x,\,t)) = 0, \quad (x,\,t) \in G_h^i \tag{6.1b}$$

$$\lambda_{(3.4)}\, z^i(x,\,t) = \psi(x), \quad (x,\,t) \in S_{1h}^i, \quad z^i(x,\,t) = \varphi(x), \quad (x,\,t) \in S_h^i \setminus S_{1h}^i, \quad i = 1, 2;$$

here $\varphi(x) = \varphi_{(5.1)}(x)$, $(x,\,t) \in S_h^i$. Note that the solutions $z^i(x,\,t)$ of difference scheme (6.1b), (6.1a) are ε-uniformly bounded.

On the set

$$\overline{G}{}_h^0 \equiv \overline{G}{}_h^1 \bigcap \overline{G}{}_h^2 = \overline{D}{}_h^0 \times \overline{\omega}_0, \tag{6.1c}$$

where $\overline{D}{}_h^0 = \overline{D}{}_{h(4.1)}^0$, we define the function

$$z^{\,0}(x,\,t) = \gamma\, z^{\,1}(x,\,t) + (1-\gamma)\, z^{\,2}(x,\,t), \quad (x,\,t) \in \overline{G}{}_h^0, \tag{6.1d}$$

where $\gamma = \gamma_{(4.3)}$. We call the function $z^{\,0}(x,\,t)$, $(x,\,t) \in \overline{G}{}_h^0$, $\overline{G}{}_h^0 = \overline{G}{}_h^0(\overline{D}{}_{h(3.3)}^0)$ the solution of the linearized difference scheme (6.1), (4.1), i.e. linearized iterative scheme on the base of the Richardson method on two embedded meshes (meshes $\overline{D}{}_h^1$ and $\overline{D}{}_h^2$).

For the function $z^{\,0}(x,\,t)$, by virtue of estimate (5.2), we have

$$|z^{\,0}(x) - z^{\,0}(x,\,t)| \le M\, q^{\,t}, \quad (x,\,t) \in \overline{G}{}_h^0, \tag{6.2}$$

where $z^{\,0}(x)$, $x \in \overline{D}{}_h^0$ is the solution of nonlinear improved Richardson difference scheme (4.3), (4.1), $q \le q_{0(5.2)}$. Taking into account estimate (4.6), we find

$$|u(x) - z^{\,0}(x,\,t)| \le M\,[N_1^{-2}\ln^2 N_1 + N_2^{-2} + q^{\,t}], \quad (x,\,t) \in \overline{G}{}_h^0, \quad q \le q_{0(5.2)}. \tag{6.3}$$

Theorem 5 *Let hypothesis of Theorem* 3 *be fulfilled. Then the solution of the linearized iterative difference scheme* (6.1), (4.1) *for* N_1, N_2, $t \to \infty$ *converges to the solution of the boundary value problem* (2.2), (2.1) ε*-uniformly at the rate* $\mathcal{O}\left(N_1^{-2}\ln^2 N_1 + N_2^{-2} + q_0^t\right)$, *where* $q_0 = q_{0(5.2)}$. *For the discrete solutions the error estimates* (6.2), (6.3) *are valid.*

We consider how to use the upper and lower solutions for estimation of solutions of the nonlinear Richardson difference scheme.

We will denote by $z^{(1)i}(x,\,t)$, $z^{(2)i}(x,\,t)$, $(x,\,t) \in \overline{G}_h^i$, $i = 1,\,2$ the solutions of problem (5.1) on the mesh $\overline{D}_{h(4.1)}^i$, satisfying at the "initial moment" the condition

$$z^{(1)i}(x,\,0) \le z^{i}(x) \le z^{(2)i}(x,\,0), \quad x \in \overline{D}_h^i, \quad i = 1,\,2, \tag{6.4}$$

where $z^i(x)$, $x \in \overline{D}_h^i$ is the solution of nonlinear base difference scheme (3.4) on the meshes $\overline{D}_h^i$, $i = 1,\,2$. For the functions $z^i(x)$, $x \in \overline{D}_h^i$, the estimate holds

$$z^{(1)i}(x,\,t) \le z^{i}(x) \le z^{(2)i}(x,\,t), \quad (x,\,t) \in \overline{G}_h^i, \quad i = 1,\,2,$$

moreover, $\max_x |z^{(j)i}(x,\,t) - z^i(x)| \to 0, \quad x \in \overline{D}_h^i$ *for* $t \to \infty, \quad i,j = 1,\,2.$

We call such functions $z^{(1)i}(x,\,t) \quad z^{(2)i}(x,\,t)$, $(x,\,t) \in \overline{G}_h^i$ the lower and upper solutions of nonlinear base difference scheme (3.4) on the meshes $\overline{D}_h^i$, $i = 1,\,2$ from (4.1). On the basis of the functions $z^{(j)i}(x,\,t)$, $(x,\,t) \in \overline{G}_h^i$, $i,j = 1,\,2$, we construct "improved" lower and upper solutions, i.e. the lower and upper solutions for the function $z^0(x)$, $x \in \overline{D}_h^0$, i.e. the solution of the difference scheme (4.3), (4.1).

Introduce the functions $z^{[1]0}(x,\,t)$, $z^{[2]0}(x,\,t)$, $(x,\,t) \in \overline{G}_h^0$, assuming

$$z^{[1]0}(x,\,t) = \gamma\, z^{(2)1}(x,\,t) + (1-\gamma)\, z^{(1)2}(x,\,t),$$

$$z^{[2]0}(x,\,t) = \gamma\, z^{(1)1}(x,\,t) + (1-\gamma)\, z^{(2)2}(x,\,t), \quad (x,\,t) \in \overline{G}_h^0, \quad \gamma = \gamma_{(4.3)}.$$

For the functions $z^{[1]0}(x,\,t)$, $z^{[2]0}(x,\,t)$, the estimates are valid

$$z^{[1]0}(x,\,t) \le z^0(x) \le z^{[2]0}(x,\,t), \quad (x,\,t) \in \overline{G}_h^0,$$

moreover, $\max_x |z^{[j]0}(x,\,t) - z^0(x)| \to 0, \quad x \in \overline{D}_h^0$ *for* $t \to \infty, \quad j = 1,\,2.$

We call the functions $z^{[1]0}(x,\,t) \quad z^{[2]0}(x,\,t)$, $(x,\,t) \in \overline{G}_h^0$ with such conditions the lower and upper, respectively, solutions of the scheme (4.3), (4.1), i.e. nonlinear Richardson difference scheme of improved accuracy.

Note that $0 \le z^{[2]0}(x,\,t) - z^{[1]0}(x,\,t) \le M\,q^t, \quad (x,\,t) \in \overline{G}_h^0$, where $q \le q_{0(5.2)}$.

We will use the upper and lower solutions of improved nonlinear scheme (4.3), (4.1) in order to define the number of iterations ensuring the same accuracy of linearized iterative solutions as it is for the scheme (4.3), (4.1).

We choose the value T, i.e. the number of iterations in the scheme (6.1), (4.1), (6.4), so that the error of the solution of the scheme (4.3), (4.1) and a difference between the solution of the iterative scheme (6.1), (4.1) and the solution of the nonlinear scheme (4.3), (4.1) were commensurable. We call the solution of the iterative scheme for $t = T$ the solution, consistent with respect to accuracy of the improved nonlinear scheme (4.3), (4.1).

We define the value T by the relations

$$\max_{\overline{D}_h^0}[z^{[2]0}(x,\,t) - z^{[1]0}(x,\,t)] > M_1\,[N_1^{-2}\ln^2 N_1 + N_2^{-2}], \tag{6.5}$$

$$\max_{\overline{D}_h^0}[z^{[2]0}(x,\,T) - z^{[1]0}(x,\,T)] \le M_1\,[N_1^{-2}\ln^2 N_1 + N_2^{-2}], \quad x \in \overline{D}_h^0, \quad t < T.$$

The functions $z^{[j]0}_{(6.1;4.1)}(x,\,T)$, $x \in \overline{D}^{\,0}_h$, $j = 1, 2$, are (upper for $j = 2$ and lower for $j = 1$) the consistent solutions of scheme (6.1), (4.1), (6.4), (6.5), i.e. consistent with respect to accuracy of the improved nonlinear scheme (4.3), (4.1).

For the consistent solution of the linearized iterative difference scheme (6.1), (4.1), (6.4), (6.5) the estimate is valid

$$|u(x) - z^{[j]0}(x,\,T)| \le M_2\,[N_1^{-2}\,\ln^2 N_1 + N_2^{-2}], \quad x \in \overline{D}^{\,0}_h, \quad j = 1, 2; \tag{6.6a}$$

and also, for the number of iterations T the following estimate holds

$$T \le M_3\,\left(\ln q_0^{-1}\right)^{-1}\,\ln N, \tag{6.6b}$$

where $T = T_{(6.5)}$, $q_0 = q_{0(5.2)}$, constants $M_{1(6.5)}$, $M_{2(6.6)}$, $M_{3(6.6)}$ are independent of q_0; the value T is defined according to the relations (6.5).

Theorem 6 *Let hypothesis of Theorem* 3 *be fulfilled. Then the solution of the linearized iterative difference scheme* (6.1), (4.1), (6.4), (6.5) *for* N_1, $N_2 \to \infty$ *converges to the solution of the boundary value problem* (2.2), (2.1) *ε-uniformly at the rate* $\mathcal{O}\left(N_1^{-2}\,\ln^2 N_1 + N_2^{-2}\right)$. *For the discrete solutions the error estimates* (6.6) *are valid.*

Thus, by virtue of ε-uniform estimate (6.6b) for the number of required iterations, the iterative method of improved accuracy (6.1), (4.1), (6.4), (6.5) turns out to be close with respect to computational expenses to the method (3.4), (3.6) for solving of a linear problem, i.e. the linear problem (2.2), (2.1), where $f(x,t,u)$ is $f(x,t)$, convergent ε-uniformly at the rate $\mathcal{O}\left(N_1^{-1}\,\ln N_1 + N_2^{-1}\right)$.

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