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Parameter–Uniform Method for a Singularly Perturbed Parabolic Equation Modelling the Black–Scholes equation in the Presence of Interior and Boundary Layers *

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1. Introduction

Mathematical modeling in financial mathematics leads to the Cauchy problem for the parabolic Black-Scholes equation [9] with respect to the value $C = C(S, t')$, i.e., a European call option, where S and t' are the current values of the underlying asset and time. Along with the solution $C = C(S, t')$ itself, the first partial derivative $(\partial/\partial S)\, C(S, t')$ of the solution is of interest. The change of variables leads to the Cauchy problem for the dimensionless parabolic equation, i.e., the singularly perturbed equation with the perturbation parameter $\varepsilon = 2^{-1}\,\sigma^2\, r^{-1}$, $\varepsilon \in (0, 1]$; σ and r are some financial parameters (the volatility and the interest rate, respectively). Already for finite values of the parameter ε, the solution of the Cauchy problem has singularities of different types that are generated by the unboundedness of the domain where the problem is defined, the discontinuity of the first derivative of the initial function and its unbounded growth at infinity. For small values of the parameter ε, an additional singularity arises, such as an interior (moving in time) layer. In this problem, primarily, we are interested in approximations to both the solution and its first order derivative in a neighbourhood of the interior layer generated by the piecewise smooth initial function.

In the present paper, in order to construct adequate grid approximations for the singularity of the interior layer type, we consider, instead of the Cauchy problem for the dimensionless Black-Scholes equation, a "simpler" singularly perturbed boundary value problem with a piecewise smooth initial condition, i.e., the problem (2.2), (2.1) (see its formulation in Section 2). In this boundary value problem in a bounded domain, except the interior layer, an additional singularity appears, namely, a boundary layer with the typical width of ε; the typical width of the interior layer is $\varepsilon^{1/2}$. Moreover, the singularity of the boundary layer is stronger than that of the interior layer, which makes it difficult to construct and study special numerical methods suitable for the adequate description of the singularity of the interior layer type. Using the method of special meshes that condense in a neighbourhood of the boundary layer and the method of additive splitting of the singularity of the interior layer type, a special finite difference scheme is designed that make it possible to approximate ε-uniformly the solution of the boundary value problem on the whole domain, its first order derivative in x on the whole domain, except the discontinuity point, however, outside a neighbourhood of the boundary layer, and also the normalized derivative (the first order spatial derivative multiplied by the parameter ε) in a finite neighbourhood of the boundary layer.

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Boundary value problems in bounded domains for parabolic equations with a discontinuous initial condition have been studied in [1, 5, 7]; however, an approximation of the derivative itself was not considered. A boundary value problem on an interval for singularly perturbed parabolic convection-diffusion equations with a piecewise smooth initial condition has been considered in [8]; approximations of the solution and the derivative were investigated. Here, in contrast to those papers, a finite difference scheme based on the solution decomposition method is constructed that allows us to resolve each singularity of the problem separately.

2. Problem Formulation. Aim of Research

On the set $\overline{G}$ with the boundary S,

$$\overline{G} = G \cup S, \quad G = D \times (0, T], \quad D = \{x : \ x \in (-d, d)\}, \tag{2.1}$$

we consider the Dirichlet problem for the singularly perturbed parabolic convection-diffusion equation *

$$L_{(2.2a)}\, u(x,t) = f(x,t), \quad (x,t) \in G, \qquad u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.2a}$$

Here $L_{(2.2)} \equiv \varepsilon\, a \dfrac{\partial^2}{\partial x^2} + b \dfrac{\partial}{\partial x} - c - q \dfrac{\partial}{\partial t}$, $a,\ b,\ q > 0,\ c \geq 0$, the right-hand side $f(x,t)$ is a sufficiently smooth function on $\overline{G}$; the parameter ε takes arbitrary values in the half-open interval $(0, 1]$. The boundary function $\varphi(x,t)$ is sufficiently smooth on the sets $\overline{S}_0^{\,-}$, $\overline{S}_0^{\,+}$, $\overline{S}^{\,L}$ and continuous on S. The first order derivative in x of the function $\varphi(x,t)$ has a jump discontinuity at the point $S^{(*)} = \{(0, 0)\}$, which is defined by the relation

$$\left[\frac{\partial}{\partial x}\, \varphi(x,t)\right] \equiv \lim_{x_1 \searrow x} \frac{\partial}{\partial x} \varphi(x_1, t) - \lim_{x_1 \nearrow x} \frac{\partial}{\partial x} \varphi(x_1, t) \neq 0, \quad (x,t) \in S^{(*)}. \tag{2.2b}$$

Here $S_0^- = \{(x,t) : \ x \in [-d, 0),\ t = 0\}$, $S_0^+ = \{(x,t) : \ x \in (0, d],\ t = 0\}$, $S_0 = \overline{S}_0^{\,-} \bigcup \overline{S}_0^{\,+}$, S_0 and S^L are the lower and lateral parts of the boundary S, $S^{\,L} = \Gamma \times (0, T]$, $\Gamma = \overline{D} \setminus D$.

Under the condition $a = c = p = 1$, $b = 1 - \varepsilon$, $f(x,t) = 0$, $(x,t) \in \overline{G}$, the equation (2.2a) becomes the dimensionless Black-Scholes one.

For simplicity, we assume that compatibility conditions [2] are fulfilled on the set $S_* = S_0 \bigcap \overline{S}^{\,L}$. Let $\overline{G}^{\,\delta}$ be the δ-neighbourhood of the set S_*, i.e., $\overline{G}^{\,\delta} = \{(x,t) : \ r\big((x,t),\, S_*\big) \leq \delta\}$, where $r\big((x,t),\, S_*\big)$ is the distance from the point (x,t) to the set S_*. Suppose

$$u \in C^{l+\alpha,(l+\alpha)/2}(\overline{G}^{\,\delta}), \quad l \geq 2, \quad \alpha \in (0, 1). \tag{2.3}$$

It follows from [2] that, under the condition (2.3), the solution of the problem (for sufficiently smooth functions $f(x,t)$ on $\overline{G}$ and $\varphi(x,t)$ on $\overline{S}_0^{\,-}$, $\overline{S}_0^{\,+}$, $\overline{S}^{\,L}$) is smooth on the set $\overline{G}^{\,*} = \overline{G} \setminus S^{(*)}$, i.e., $u \in C^{\,l+\alpha,(l+\alpha)/2}(\overline{G}^{\,*})$. The derivative $p(x,t) = (\partial/\partial x)u(x,t)$ is continuous on $\overline{G}^{\,*}$, bounded on $\overline{G}^{\,*}$ for fixed values of ε and has a discontinuity on the set $S^{(*)}_{(2.2b)}$. We are interested in approximations to the solution $u(x,t)$, $(x,t) \in \overline{G}$, and its derivative $p(x,t)$, $(x,t) \in \overline{G}^{\,*}$. Let us describe the behaviour of the solution and derivatives more precisely.

Let $S^{\,L} = S^{\,l} \bigcup S^{\,r}$, $S^{\,l}$ and $S^{\,r}$ be the left and right parts of the boundary $S^{\,L}$, and let

$$S^{\,\gamma} = \{(x,t) : \ x = \gamma(t),\ (x,t) \in \overline{G}\}, \quad \gamma(t) = -b\, q^{-1}\, t, \quad t \geq 0$$

be the characteristic of the reduced equation passing through the point $(0, 0)$. When the parameter ε tends to zero, boundary and interior layers with the typical width scales ε and $\varepsilon^{1/2}$, respectively, appear in a neighbourhood of the sets $S^{\,l}$ and S^{γ}; as opposed to the boundary layer, the interior layer is weak (the first order derivative in x of the interior-layer function is bounded

* The notation $L_{(j.k)}$ $(m_{(j.k)},\ M_{(j.k)},\ G_{h(j.k)})$ means that these symbols are introduced in formula $(j.k)$.

ε-uniformly). For simplicity, we assume that the characteristic S^γ does not meet the boundary S^l. The derivative $p(x,t)$ in a neighbourhood of the set S^l grows without bound as $\varepsilon \to 0$. It is convenient to consider the quantity $P(x,t) = \varepsilon\,(\partial/\partial x)u(x,t)$, i.e., the normalized first derivative in x, in the m-neighbourhood of the set S^l instead of the derivative $p(x,t)$, because only this quantity is bounded ε-uniformly. The quantity $P(x,t)$ will be called the diffusion flux (or briefly, the flux). Outside a neighbourhood of the set S^l, the derivative $p(x,t)$ is bounded ε-uniformly on $\overline{G}^*$. For small values of the parameter ε, the derivative $p(x,t)$ is more "informative" (on the set where it is bounded) than the flux $P(x,t)$.

It is well known that even for singularly perturbed problems with sufficiently smooth data, solutions of classical finite difference schemes do not converge ε-uniformly; for small values of the parameter ε, errors in the discrete solutions are commensurable with the actual solutions of the differential problem. The diffusion fluxes obtained on the basis of such schemes also do not converge ε-uniformly. Due to this it would be interesting to construct a difference scheme that allows us to approximate ε-uniformly both the solution on the whole domain $\overline{G}$ and diffusion fluxes in this domain excluding the discontinuity point $S^{(*)}$, also to determine conditions under which the boundary layer does not appear, and for such a problem, to find the ε-uniform approximation of the derivative in x on the set $\overline{G}^*$.

Definition. Let [†]

$$\overline{G}_0^* = \overline{G}_0^*(m) = \overline{G}^* \cap \{x \ge -d + m\} \tag{2.4}$$

be the set $\overline{G}^*$ excluding an m-neighbourhood of the set $\overline{S}^l$ (the m-neighbourhood of the boundary layer). If the interpolants constructed using the solution of some finite difference scheme converge on $\overline{G}$ ε-uniformly, we say that the discrete solution (the difference scheme) converges on $\overline{G}$ uniformly with respect to the parameter ε (or, briefly, ε-uniformly in $C(\overline{G})$). If, moreover, the interpolants of the diffusion fluxes (or, respectively, the first order derivatives in x) converge ε-uniformly in $\overline{G}^*$ (ε-uniformly in $\overline{G}_0^*$), we say that the difference scheme converges ε-uniformly in $C^{1(n)}(\overline{G}^*)$ (ε-uniformly in $C^1(\overline{G}_0^*)$).

Our aim is to construct a difference scheme for problem (2.2), (2.1) that converges ε-uniformly in $C^{1(n)}(\overline{G}^*) \cap C^1(\overline{G}_0^*)$, and also to determine conditions under which the boundary layer does not appear, and in this case to construct a difference scheme that converges ε-uniformly in $C^1(\overline{G}^*)$. Some preliminary results related to this problem are given in [3].

3. *A priori* estimates of the solution and derivatives

For the solution of the boundary value problem (2.2), (2.1) and its derivatives, we give *a priori* estimates used in the constructions (the more detailed derivation can be found in [8]).

We represent the set $\overline{G}$ as the sum of overlapping sets

$$\overline{G} = \bigcup_j \overline{G}^j, \quad j = 1,\, 2,\, 3, \tag{3.1}$$

where

$$G^1 = G^1(m^1) = \{(x,t):\ |x-\gamma(t)| < m^1,\ \ t \in (0,T]\},$$
$$G^2 = G^2(m^2) = \{(x,t):\ x \in (-d, -d+m^2),\ \ t \in (0,T]\},$$
$$G^3 = G^3(m^3) = G \setminus \{G^1(m^3) \bigcup G^2(m^3)\}, \quad m^3 < m^1,\, m^2,$$

G^1 and G^2 are neighbourhoods of the interior and boundary layers, respectively.

The solution on the set $\overline{G}^3$ is smooth; we have the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u(x,t)\right| \le M, \quad (x,t) \in \overline{G}^3, \ \ k + 2k_0 \le K. \tag{3.2}$$

[†] Throughout this paper, M, M_i (or m) denote sufficiently large (small) positive constants that do not depend on ε and on the discretization parameters.

The value K is defined by the problem data; and $K \geq 4$.

We represent the solution on the set $\overline{G}^{\,2}$ as the decomposition into two functions

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}^{\,2}, \tag{3.3}$$

where $U(x,t)$ and $V(x,t)$ are the *regular and singular parts* of the solution; $V(x,t)$ is the *boundary layer function.* For the functions $U(x,t)$ and $V(x,t)$, the following estimates are valid:

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U(x,\,t) \right| \leq M, \quad \left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, V(x,\,t) \right| \leq M\,\varepsilon^{-k} \exp\Big(-m\,\varepsilon^{-1}\, r\big((x,t),\,\overline{S}^l\big)\Big), \tag{3.4}$$

$$(x,t) \in \overline{G}^2, \quad k + 2\,k_0 \leq K,$$

where $r\big((x,t),\,\overline{S}^{\,l}\big)$ is the distance from the point (x,t) to the set $\overline{S}^{\,l}$, m is an arbitrary constant from the interval $(0,\,m_0)$, $m_0 = a^{-1}\,b$.

On the set $\overline{G}^{\,1}$, we have the representation

$$u(x,t) = U^1(x,t) + W^1(x,t), \quad (x,\,t) \in \overline{G}^{\,1}, \tag{3.5a}$$

where $U^1(x,t)$ and $W^1(x,t)$ are the regular and singular parts; $W^1(x,t)$ is the *interior layer function*

$$W^1(x,t) = 2^{-1} \left[\frac{\partial}{\partial x}\, \varphi(0,\,0) \right] \Big\{ \big(x - \gamma(t)\big)\, v\Big(2^{-1}\,\varepsilon^{-1/2}\,a^{-1/2}\,q^{1/2}\,\big(x-\gamma(t)\big) t^{-1/2} \Big) + \tag{3.5b}$$

$$+ 2\,\pi^{-1/2}\,\varepsilon^{1/2}\,a^{1/2}\,q^{-1/2}\,t^{1/2}\,\exp\Big(-4^{-1}\,\varepsilon^{-1}\,a^{-1}\,q\,\big(x - \gamma(t)\big)^2\,t^{-1} \Big) \Big\} \exp(-\alpha\,t), \;\; (x,t) \in I\!R \times [0,\,T].$$

For the components in representation (3.5), we have the estimates (similar to those in [8])

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U^1(x,t) \right| \leq M\,\big[1 + \varepsilon^{(2-k-k_0)/2}\,\rho^{\,2-k-k_0} + \varepsilon^{\,(2-k)/2}\,\rho^{\,2-k-2k_0} \big], \quad (x,t) \in \overline{G}^{\,1},$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, W^1(x,t) \right| \leq M\,\big[1 + \varepsilon^{\,(1-k-k_0)/2}\,\rho^{\,1-k-k_0} + \varepsilon^{\,(1-k)/2}\,\rho^{\,1-k-2k_0} \big] \times \tag{3.6}$$

$$\times \exp(-m\,\varepsilon^{-1/2}\,|x - \gamma(t)|\,), \quad (x,t) \in \overline{G}; \quad k + 2k_0 \leq\; K,$$

where $\rho = \rho(x,t;\,\varepsilon) = \varepsilon^{-1/2}\,\big|x - \gamma(t)\big| + t^{1/2}$, and m is an arbitrary constant.

Theorem 1 *Let the data of the boundary value problem* (2.2), (2.1) *satisfy the condition* $f \in C^{\,l,l/2}(\overline{G})$, $\varphi \in C^{\,l}(\overline{S}_0^{\,-}) \bigcap C^{\,l}(\overline{S}_0^{\,+}) \bigcap C^{\,l/2}(\overline{S}^{\,L}) \bigcap C(S)$, *and let the condition* (2.3) *hold for the solution of this problem, moreover,* $l = K$. *Then the solution of the boundary value problem and its components in representations* (3.3), (3.5) *satisfy the estimates* (3.2), (3.4), *and* (3.6).

Remark 1 Let us give the condition under which the boundary layer does not arise.

Let us define the sets

$$G^4 = G^4(m) = \big\{ (x,t);\; x \,>\, \gamma(t) - \gamma\,(T) - d + m \big\}, \quad \overline{G}^{\,4} = G^{\,4} \bigcup S^{\,4}, \tag{3.7a}$$

$$\overline{G}^{\,5} = G^{\,5} \bigcup S^{\,5}, \quad G^{\,5} = G^{\,5}(m) = G \setminus \overline{G}^{\,4}(m); \quad m \,<\, d + \gamma\,(T). \tag{3.7b}$$

Let the functions $f(x,t)$ and $\varphi(x,t)$ satisfy the conditions

$$f(x,t) = 0, \quad (x,\,t) \in \overline{G}^{\,4}, \qquad \varphi(x,t) = 0, \quad (x,\,t) \in\, S \bigcap \overline{G}^{\,4}. \tag{3.8}$$

Then the boundary layer is absent, i.e., the singular component is absent in representation (3.3), and $u(x,t) = U(x,t)$ on the set $\overline{G}^{\,2}$. For the solution $u(x,t)$, the estimate (3.4) holds, moreover, $\left| \dfrac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u(x,t) \right| \leq\, M\,\varepsilon^{K_1}$, $(x,t) \in \overline{G}^{\,2}$, $k + 2k_0 \leq K$, where $\overline{G}^{\,2} = \overline{G}^{\,2}_{(3.1)}(m^2)$, $m^2 < m_{(3.1)}$, and the constant K_1 can be chosen sufficiently large. ■

4. Classical grid approximations on uniform and piecewise uniform meshes

4.1. Let us consider a difference scheme based on classical approximations of problem (2.2), (2.1). On the set $\overline{G}_{(2.1)}$, we introduce the rectangular mesh

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega} \times \overline{\omega}_0, \tag{4.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are meshes on the intervals $[-d,\, d]$ and $[0,\, T]$, respectively; the mesh $\overline{\omega}$ has an arbitrary distribution of nodes satisfying only the condition $h \le MN^{-1}$, where $h = \max_i h^i$, $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}$; the mesh $\overline{\omega}_0$ is uniform with the step-size $h_0 = TN_0^{-1}$. Here $N+1$ and N_0+1 are the numbers of nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$, respectively.

We approximate the boundary value problem (2.2) by the difference scheme [4]

$$\Lambda_{(4.2)}\, z(x,t) = f(x,t), \quad (x,t) \in G_h, \qquad z(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \tag{4.2}$$

Here $\Lambda_{(4.2)} \equiv \varepsilon\, a\, \delta_{\overline{x}\widehat{x}} + b\, \delta_x - c - q\, \delta_{\overline{t}}$, $\delta_{\overline{x}\widehat{x}}\, z(x,t)$ is the second difference derivative on a nonuniform mesh. On the uniform mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{4.3}$$

the scheme (4.2), (4.3) converges under the condition $N^{-1} \ll \varepsilon$:

$$|u(x,t) - z(x,t)| \le M\left[(\varepsilon + N^{-1})^{-1} N^{-1} + N^{-1/2} + N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h. \tag{4.4}$$

Under the condition (3.8), when in (3.3)

$$V(x,t) = 0, \quad (x,t) \in \overline{G}^{\,2}, \tag{4.5}$$

the scheme (4.2), (4.3) converges ε-uniformly:

$$|u(x,t) - z(x,t)| \le M\left[N^{-1/2} + N_0^{-1/2}\right], \quad (x,t) \in \overline{G}_h. \tag{4.6}$$

If, except (3.8) in (3.5), the following condition holds:

$$W^1(x,t) = 0, \quad (x,t) \in \overline{G}^{\,1}, \tag{4.7}$$

i.e., when $\left[\dfrac{\partial}{\partial x}\, \varphi(x,t)\right] = 0$, $(x,t) \in S^{(*)}$, we have the ε-uniform estimate for the solution of the difference scheme (4.2), (4.3):

$$|u(x,t) - z(x,t)| \le M\left[N^{-1} + N_0^{-1+\nu_0}\right], \quad (x,t) \in \overline{G}_h, \tag{4.8}$$

where ν_0 is an arbitrary constant in the interval (0, 1).

Theorem 2 *Let the solution of problem* (2.2) *and its components in* (3.3), (3.5) *satisfy the estimates* (3.2), (3.4), (3.6) *for* $K = 4$. *Then the difference scheme* (4.2), (4.3) *converges under the condition* $N^{-1} \ll \varepsilon$. *In the case of the condition* (4.5), *the scheme* (4.2), (4.3) *converges* ε*-uniformly. The discrete solutions satisfy the estimate* (4.4)*; and, in the case of the condition* (4.5), *the estimate* (4.6) *holds, and under conditions* (4.5) *and* (4.7), *the estimate* (4.8) *is valid.*

4.2. We now consider the problem when the solution has a boundary layer. On the set $\overline{G}$, we construct the mesh condensing in a neighbourhood of the boundary layer (similar to that constructed in [6]–[8]):

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega}^{\,*} \times \overline{\omega}_0, \tag{4.9a}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(4.1)}$, $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a piecewise uniform mesh on $[-d,\, d]$, and σ is a parameter depending on ε and N. We choose the quantity σ satisfying the condition

$$\sigma = \sigma(\varepsilon,\, N) = \min\,[\,\beta,\, 2\,m^{-1}\,\varepsilon\,\ln N\,], \tag{4.9b}$$

where β is an arbitrary number in the half-open interval $(0,\, d]$, and $m = m_{(3.4)}$. The interval $[-d,\, d]$ is divided into two parts: $[-d,\, -d+\sigma]$ and $[-d+\sigma,\, d]$; on each part, the step-size is constant and is equal to $h^{(1)} = 2\,d\,\sigma\,\beta^{-1}\,N^{-1}$ on the subinterval $[-d, -d+\sigma]$ and to $h^{(2)} = 2\,d\,(2\,d-\sigma)\,(2\,d-\beta)^{-1}\,N^{-1}$ on the subinterval $[-d+\sigma, d]$, $\sigma \le d$.

On the mesh (4.9), the scheme (4.2), (4.9) converges ε-uniformly:

$$|\; u(x,t) - z(x,t)\; | \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \quad (x,t) \in \overline{G}_h. \tag{4.10}$$

Theorem 3 *Let the assumptions of Theorem* 2 *be fulfilled. Then the difference scheme* (4.2), (4.9) *converges ε-uniformly with the error estimate* (4.10).

4.3. We consider the approximation of the functions $u(x,t)$, $p(x,t)$, $P(x,t)$, $(x,t) \in \overline{G}$, using the interpolants constructed on the basis of the functions $z(x,t)$, $p^h(x,t)$, $P^h(x,t)$.

Let $z(x,t)$, $(x,t) \in \overline{G}_h$, be a solution of some scheme. For the function $z(x,t)$, we construct its extension $\overline{z}(x,t)$ to $\overline{G}$; $\overline{z}(x,t)$ is a bilinear interpolant on the elementary rectangles generated by the lines that pass through the nodes of the mesh $\overline{G}_h$ in parallel to the coordinate axes. Further, we construct the interpolant $\overline{p}^h(x,t)$, $(x,t) \in \overline{G}$, for the discrete derivative $p^h(x,t)$, $(x,t) \in \overline{G}_h$, $x \ne d$. At the interior points of the elementary rectangles, we assume $\overline{p}^h(x,t) = \overline{p}^h_z(x,t) = (\partial/\partial x)\,\overline{z}(x,t)$; the function $\overline{p}^h(x,t)$ is continuous on the upper and on the lower sides of the rectangles, and it is defined according to continuity on the left sides of the elementary rectangles. But if the rectangles are adjacent, by their right sides, to the set $\overline{S}^r$, then we also define the function $\overline{p}^h(x,t)$ on these sides according to continuity. Hence, we have constructed the function $\overline{p}^h(x,t)$, $(x,t) \in \overline{G}$. The interpolant $\overline{p}^h(x,t)$, in general, has discontinuities on the lines that are parallel to the t-axis and pass through the nodes of the mesh G_h. We define the interpolant of the diffusion flux by the relation $\overline{P}^h(x,t) = \overline{P}^h_z(x,t) = \varepsilon\,\overline{p}^h(x,t)$, $(x,t) \in \overline{G}$.

In the case of the difference scheme (4.2), (4.3) under conditions (4.5) and (4.7), we have the following estimates for the interpolants:

$$|\; p(x,t) - \overline{p}^{\,h}(x,t)\; | \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \quad (x,t) \in \overline{G}, \quad x \ge -d + \beta_0; \tag{4.11a}$$

$$|\; P(x,t) - \overline{P}^{\,h}(x,t)\; | \le M\,[\,N^{-1/2} + N_0^{-1/2}\,], \quad (x,t) \in \overline{G}, \tag{4.11b}$$

where $\beta_0 = \beta_{0(4.9b)}$, $M_{(4.11a)} = M(\beta_0)$, and also the estimate similar to (4.8):

$$|\; u(x,t) - \overline{z}(x,t)\; | \le M\,[\,N^{-1} + N_0^{-1+\nu_0}\,], \quad (x,t) \in \overline{G}, \quad \nu_0 = \nu_{0(4.8)}. \tag{4.11c}$$

Theorem 4 *Let the assumptions of Theorem* 2 *be fulfilled for $K = 6$. Then, under the conditions* (4.5), (4.7), *the difference scheme* (4.2), (4.3) *approximates the solution of problem* (2.2), (2.1), *its derivative and the diffusion flux ε-uniformly with the error estimates* (4.11).

5. Solution decomposition scheme approximating the derivative $p(x,t)$

5.1. We represent the solution of problem (2.2), (2.1) as the sum of functions

$$u(x,t) = u_1(x,t) + u_2(x,t), \quad (x,\, t) \in \overline{G}. \tag{5.1a}$$

Here $u_1(x,t)$ and $u_2(x,t)$ are components of the solution of the boundary value problem (2.2), (2.1), including singularities of the boundary and interior layers types, respectively. We call the functions $u_1(x,t)$ and $u_2(x,t)$ the components containing the boundary and interior layers, respectively, or, briefly, the *boundary and interior layer components.*

We represent the *interior layer component* $u_2(x,t)$ as the sum of functions

$$u_2(x,t) = u_2^1(x,t) + u_2^2(x,t), \quad (x,\, t) \in \overline{G}, \tag{5.1b}$$

where $u_2^1(x,t)$ and $u_2^2(x,t)$ are the *regular and singular parts* of the function $u_2(x,t)$;

$$u_2^2(x,t) = W^1_{(3.5b)}(x,t), \quad (x,\, t) \in \overline{G}.$$

The functions $u_1(x,t)$ and $u_2^1(x,t)$ are solutions of the following problems

$$L\, u_2^1(x,t) = f_2(x,t), \quad (x,\, t) \in G, \qquad u_2^1(x,t) = \varphi_2(x,t), \quad (x,\, t) \in S; \tag{5.1c}$$

$$L\, u_1(x,t) = f_1(x,t), \quad (x,\, t) \in G, \qquad u_1(x,t) = \varphi_1(x,t), \quad (x,\, t) \in S. \tag{5.1d}$$

The functions $f_i(x,t)$, $\varphi_i(x,t)$, $i = 1, 2$, are defined by the relations

$$f_2(x,t) = f(x,t)\,\eta\,(x,t), \qquad f_1(x,t) = f(x,t) - f_2(x,t), \quad (x,t) \in \overline{G}; \tag{5.1e}$$

$$\varphi_2(x,t) = (\varphi(x,t) - u_2^2(x,t))\,\eta\,(x,t), \quad \varphi_1(x,t) = \varphi(x,t) - \varphi_2(x,t) - u_2^2(x,t), \quad (x,t) \in S.$$

Here $\eta\,(x,t)$, $(x,t) \in \overline{G}$, is a sufficiently smooth function that vanishes in a neighbourhood of the boundary layer

$$\left.\begin{aligned} &\eta\,(x,t) = 0, \quad (x,t) \in \overline{G}{}^5_{(3.7b)}(2^{-1}\, m_1) \\ &\eta\,(x,t) = 1, \quad (x,t) \in \overline{G}{}^4_{(3.7a)}(m_1) \end{aligned}\right\}, \quad 0 \le \eta(x,t) \le 1, \quad (x,t) \in \overline{G},$$

where m_1 is an arbitrary number in the interval $\big(0,\, 2^{-1}\,(d + \gamma(T))\big)$.

To solve problem (5.1d), we use the difference scheme on the piecewise uniform mesh (4.9):

$$\Lambda_{(4.2)} z_1(x,\, t) = f_1(x,t), \quad (x,t) \in\, G_{h(4.9)}, \qquad z_1(x,t) = \varphi_1(x,t), \quad (x,t) \in\, S_h. \tag{5.2a}$$

To solve problem (5.1c), we use the difference scheme on the uniform mesh (4.3):

$$\Lambda_{(4.2)} z_2^1(x,\, t) = f_2(x,t), \quad (x,t) \in\, G_{h(4.3)}, \qquad z_2^1(x,t) = \varphi_2(x,t), \quad (x,t) \in\, S_h. \tag{5.2b}$$

Further, we construct the special interpolants into which the singular part $u_2^2(x,t)$, i.e., the function of the interior layer type, enters in the explicit form as follows:

$$u_0^h(x,t) = \overline{z}_1(x,t) + u_2^h(x,t), \quad u_2^h(x,t) = \overline{z}_2^1(x,t) + u_2^2(x,t), \qquad (x,t) \in \overline{G}; \tag{5.2c}$$

$$p_0^h(x,t) = \overline{p}^{\,h}_{z_1}(x,t) + p_2^h(x,t), \quad p_2^h(x,t) = \overline{p}^{\,h}_{z_2^1}(x,t) + \frac{\partial}{\partial x}\, u_2^2(x,t), \quad (x,t) \in \overline{G}^{\,*}; \tag{5.2d}$$

$$P_0^h(x,t) = \varepsilon\, p_0^h(x,t), \quad (x,t) \in \overline{G}^{\,*}, \tag{5.2e}$$

where $\overline{z}_1(x,t)$, $\overline{p}^{\,h}_{z_1}(x,t)$ and $\overline{z}_2^1(x,t)$, $\overline{p}^{\,h}_{z_2^1}(x,t)$ are bilinear interpolants that are constructed using the functions $z_1(x,t)$, $(x,t) \in \overline{G}_{h(4.9)}$ and $z_2^1(x,t)$, $(x,t) \in \overline{G}_{h(4.3)}$ $\big($similarly to the construction of interpolants in Subsection 4.3$\big)$. The use of the interpolants allows us to find the solution on the set $\overline{G}$, its first derivative in x and the diffusion flux on the set $\overline{G}^{\,*}$.

The function $u_0^h(x,t)$, $(x,t) \in \overline{G}$, is called the solution of the difference scheme (5.2), (4.3), (4.9), and the functions $p_0^h(x,t)$ and $P_0^h(x,t)$, $(x,t) \in \overline{G}^{\,*}$, are called the derivative and the diffusion flux, respectively, corresponding to this scheme. The scheme (5.2), (4.3), (4.9) is the solution decomposition scheme with the additive splitting of a singularity of the interior-layer type $\big($briefly, we call this scheme the singularity splitting scheme$\big)$.

5.2. Let us give estimates on the solutions and derivatives for the schemes constructed above. In the case of scheme (5.2), (4.3), (4.9), we have the estimates

$$\left|u(x,t) - u_0^h(x,t)\right| \le M\left[N^{-1}\ln N + N_0^{-1+\nu_0}\right], \quad (x,\, t) \in \overline{G}; \tag{5.3a}$$

$$\left|P(x,t) - P_0^h(x,t)\right| \le M\left[N^{-1/2} + N_0^{-1/2}\right], \qquad (x,\, t) \in \overline{G}^*; \tag{5.3b}$$

$$\left|p(x,t) - p_0^h(x,t)\right| \le M\left[N^{-1/2} + N_0^{-1/2}\right], \qquad (x,\, t) \in \overline{G}_0^*, \tag{5.3c}$$

where $\overline{G}_0^* = \overline{G}_{0(2.4)}^*(m)$, m is an arbitrary sufficiently small constant, and $M_{(5.3c)} = M(m)$. The interior layer component $u_{2(5.1a)}(x,t)$ and its derivative in x satisfy the estimates

$$\left|u_2(x,t) - u_2^h(x,t)\right| \le M\left[N^{-1} + N_0^{-1+\nu_0}\right], \quad (x,\, t) \in \overline{G}, \tag{5.4}$$

$$\left|p_2(x,t) - p_2^h(x,t)\right| \le M\left[N^{-1/2} + N_0^{-1/2}\right], \quad (x,\, t) \in \overline{G}^*,$$

where $p_2(x,t) = \dfrac{\partial}{\partial x}u_2(x,t)$, $p_2^h(x,t) = p_{2(5.2\text{d})}^h(x,t)$. In (5.3) and (5.4), $\nu_0 = \nu_{0(4.8)}$.

Thus, the interior layer component converges ε-uniformly in $C^1(\overline{G}^*)$.

Theorem 5 *Let the assumptions of Theorem* 2 *be fulfilled for* $K = 6$. *Then the difference scheme* (5.2), (4.3), (4.9) *approximates the solution of the problem* (2.2), (2.1), *the derivative* $p(x,t)$, *the diffusion flux* $P(x,t)$ *and also the interior layer component* $u_{2(5.1a)}(x,t)$ *and its derivative* $p_2(x,t)$ ε*-uniformly with the error estimates* (5.3) *and* (5.4), *respectively.*

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