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Boundary Layer Solution For laminar flow through a Loosely curved Pipe by Using Stokes Expansion

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1-Introduction

We consider fully developed steady laminar flow through a toroidal pipe of small curvature ratios .The solution is expanded up to 40 terms by computer in powers of Dean Number. The major conclusion of this investigation is that the friction ratio in a loosely coiled pipe grows asymptotically as the 1/4 power of the similarity parameter and not as the 1/2 power as previously deduced from boundary-layer analysis. This work confirmed the results obtained by [1]. The goal of this analysis is to provide as complete a description as possible of the flow. The analysis yields a solution for all values of Reynolds number from zero to infinity in a continuous fashion.

The utility of the Stokes series has been transformed by the advent of the computer. Researchers believe that only the first few terms of the series could be determined numerically without excessive labors; but now-particularly in simpler geometry –we can compute dozens or hundreds of terms. From those we can estimate accurately the radius of convergence, and then attempt to extend the range of utility by analytic continuation. The ultimate achievement is to extract from the Stokes series for small value the boundary-layer solution for the similarity parameter tending to infinity

The paradox concerns the discrepancy between the solution obtained using the extended Stokes series method [1] and that obtained using boundary layer techniques [2],[3],[4],[5],[6]and experimental work of [7],[8],[9],[10]and numerical work of [11]for the ratio of the friction factors in coiled tubes to that in straight one in steady, fully developed laminar flow. In the ensuing debates several papers of [12],[13].[14],[15],[16].[17],[18].[19],[20],[21],[22],[23] various explanations and new evidence have been given; however, the paradox , the resolution of which is important still remain as a open problem. The author in [20] raise the possibility of cause of this paradox is the use of only 24 terms to estimate the asymptotic limit obtained by [1]. In this paper we extend the Stokes series from 24 terms used by [1] to 40 terms. We confirm the major result of the friction ratio in a loosely coiled pipe grows asymptotically as the 1/4 power of the similarity parameter and not as the 1/2 power and that confirm the result of [1].

What is particularly important in problems of this type is the presence of analyticity. Not every stokes expansion, for examples that of the flow past sphere as described by the full Navier-Stokes equation are analytic in Reynolds number. In this case, method of matched asymptotic expansions is required and can be automated

2-Statement of Problem

We adopt Dean's co-ordinate system(r, θ) reference [24]and his normalization: Lengths are referred to the radius a of the pipe, and the velocity w down the pipe to the maximum speed $W_0 = Ga^2/4\mu$ in a straight pipe under the same axial pressure gradient $G = L^{-1}\ \partial p/\partial\phi$ but the stream function ψ for the secondary motion is referred to kinematics viscosity ν (which mean that transverse velocities are referred to ν /a).then in the approximations of negligible helicity and loose coiling the Navier-Stokes equations for incompressible fluid reduce to([24], Dean' equation (15)-(18))

$$\nabla^2 W + 4 = \frac{1}{r}\left(-\frac{\partial\psi}{\partial\theta}\frac{\partial W}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial W}{\partial\theta} \right) \qquad (1)$$

$$\nabla^4\psi + KW\left(\cos\theta\frac{\partial W}{\partial r} - \frac{\sin\theta}{r}\frac{\partial W}{\partial\theta} \right) = \frac{1}{r}\left(-\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta} \right)\nabla^2\psi \qquad (2)$$

With boundary condition: $W = \psi = \dfrac{\partial\psi}{\partial r} = 0$ at $r = 1$

Where $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2}\left(\dfrac{\partial^2}{\partial\theta^2}\right)$ $\qquad K = 2\left(\dfrac{W_0 a}{\nu}\right)^2 \dfrac{a}{L} = \dfrac{G^2 a^7}{8\mu^2\nu^2 L}$.(3)

3. Series Derivation and Computer Extension

If the curvature ratio is small or $\frac{a}{L}$ is small, so that K is small, the nonlinear terms on the right hand side of Equation (1) and (2) are negligible and the first approximation is linear and gives $W_1 = 1 - r^2$. One can systematically improve on this approximation by expanding in powers of K according to:

$$W = W_1 + \left(\frac{K}{576}\right)W_2 + \left(\frac{K}{576}\right)^2 W_3 + \cdots = \sum_{n=1}^{\infty} W_n \left(\frac{K}{576}\right)^{n-1} \qquad (4)$$

$$\psi = \psi_1\left(\frac{K}{576}\right) + \left(\frac{K}{576}\right)^2\psi_2 + \left(\frac{K}{576}\right)^3\psi_3 + \cdots = \sum_{n=1}^{\infty}\psi_n\left(\frac{K}{576}\right)^n \qquad (5)$$

$$\psi_1 = \left(4r - 9r^3 + 6r^5 - r^7\right)\sin(\theta)$$

$$\psi_2 = \left(\frac{4979\ r^2}{5600} - \frac{255\ r^4}{112} + \frac{81 r^6}{40} - \frac{4 r^8}{5} + \frac{3 r^{10}}{16} - \frac{9 r^{12}}{350} + \frac{r^{14}}{1120} \right)\sin(2\theta)$$

$$W(1) = \left(\frac{19 r}{40} - r^3 + \frac{3 r^5}{4} - \frac{r^7}{4} + \frac{r^9}{40} \right)\cos(\theta)$$

$$W_2 = -\frac{4119}{44800} + \frac{19r^2}{40} - \frac{331r^4}{320} + \frac{99r^6}{80} - \frac{569r^8}{640} + \frac{157r^{10}}{400} - \frac{33r^{12}}{320} + \frac{r^{14}}{70} - \frac{r^{16}}{1280} + \left(\frac{14569r^2}{141120} - \frac{2297r^4}{8400} + \frac{1317r^6}{4480} - \frac{139r^8}{800} + \frac{61r^{10}}{960} - \frac{87r^{12}}{5600} + \frac{47r^{14}}{22400} - \frac{r^{16}}{8820}\right)\cos(2\theta)$$

We have found eight terms exactly by means of symbolic language. However, for saving space only we report the result for the flux ratio as defined as the ratio of the flux F_c through a curved pipe to the flux F_s through a stationary pipe with the same pressure gradient. Its series has the form:

$$\frac{F_c}{F_s} = \sum_0^\infty a_n \left(\frac{K}{576}\right)^{2n} = 1 - \frac{1541}{50400}\left(\frac{K}{576}\right)^2 + \frac{6471982981}{542442700800}\left(\frac{K}{576}\right)^4 - \frac{80321152554888274264316429}{12198321972347948236800000000}\left(\frac{K}{576}\right)^6 + \ldots \quad (6)$$

$$\left(\frac{F_c}{F_s}\right)^{-1} = \sum_0^\infty b_n \left(\frac{K}{576}\right)^{2n} = 1 + \frac{1541}{50400}\left(\frac{K}{576}\right)^2 + \ldots. \quad (7)$$

Following the recipes given by [15] and [21], we have also obtained 40 terms of the series by writing a FORTRAN program in quadruple precision. The results for the coefficients a_n are listed as below:

n	a_n	n	a_n	n	a_n	N	a_n
1	$1.0000000000 \times 10^{0}$	6	$-2.9771922933 \times 10^{-3}$	11	$9.1763240107 \times 10^{-4}$	16	$4.2658069115 \times 10^{-4}$
2	$-3.0575396825 \times 10^{-2}$	7	$2.2124224081 \times 10^{-3}$	12	$7.7126893272 \times 10^{-4}$	17	$3.748929969 \times 10^{-4}$
3	$1.1931182725 \times 10^{-2}$	8	$-1.7101745006 \times 10^{-3}$	13	$6.5605790575 \times 10^{-4}$	18	$3.313561134 \times 10^{-4}$
4	$-6.5846066973 \times 10^{-3}$	9	$1.3610000850 \times 10^{-3}$	14	$5.6371324890 \times 10^{-4}$	19	$2.943633637 \times 10^{-4}$
5	$4.2384990979 \times 10^{-3}$	10	$1.1076549324 \times 10^{-3}$	15	$4.8855590554 \times 10^{-4}$	20	$2.626867853 \times 10^{-4}$

This series seem to converge to a value $(0.9668584)^{-1}$ for the reciprocal of the radius of convergence, K_0^{-1}. The leading singularity for the reciprocal series $\left(\frac{F_c}{F_s}\right)^{-1}$ is a square-root type on the negative real axis.

4. Extension of the range of validity

In this problem the nearest singularity is on the negative real axis of K^2 has no physical significance, and therefore unnecessarily limits the range of applicability of the series. The Euler transformation is one way to analytically continue a series or extend the range of convergence for physical K. We use the new variable δ, which is defined as:

$$\delta = \frac{(K/576)^2}{(K/576)^2 + D},$$

Which maps the nearest singularity to infinity? The transformed series has fixed signs, indicating that the nearest singularity is on the positive real x-axis. The Domb-Sykes plots described in [16] to determine its exponent and corresponding radius of convergence clearly indicate that the radius of Convergence is equal to one. This nearest singularity at δ =1 Corresponds to $K^2 = \infty$ in the original variables. Thus we have been successful in extending the series up to an infinite value of Reynolds number. A singularity at δ =1 is confirmed by the Associated Neville tables and other available devices.

5. The exponent at infinity

Estimation of the exponent α of the singularity at $K = \infty$ is the most important part of the analysis. In previous problems, this step has proven to be the hardest, and care must be taken. The Domb-Sykes plot clearly indicates that is nearly linear, and suggests that the series converges for $0 \le \delta \le 1$ with a limiting singularity of the form:

$$\frac{F_r}{F_s} \sim C(1-\delta)^{\alpha} \quad \text{As} \quad \delta \to 1$$

In the original variable, this is:

$$\frac{F_r}{F_s} \sim CK^{-2\alpha} \quad \text{As} \quad K \to \infty$$

Or Here, α and C are the important parameters that we seek. We try to determine the value of α as accurately as possible. By using usual devices such as Domb-Sykes plot and its associated Neville tables described in [16] all shows the value close to -.05

6. Páde approximation

The final method for determining the exponent α is an independent analysis of the original flux ratio series (9), Known as the Pade approximant. This method does not necessarily require any information about the radius of convergence. The Pade approximants provide an approximation that is invariant under an Euler transformation of the independent variables. The theory of Pade approximants may be found in [17]). Briefly stated, the $[m/n]$ Pade approximant is the ratio P (Z)/Q (Z) of polynomials P and Q of degree m and n, respectively, that, when expanded, agrees with the given series through terms of degree m+n, and normalized by P(Q)=1. Such rational fractions are known to have remarkable properties of analytic continuation ([19]). The coefficients of the power series must be known to degree (m+n).and .by equating like power of g(Z) and P(Z)/Q(Z), the linear system of m+n+1 equation must be solved to obtain the coefficients in the functional form P(Z)/Q(Z). A pade approximant can only indicate a singularity by the poles that are the zeros of its denominator. One way to use the Pade approximant is to apply it by taking the logarithmic derivative of the original series (9). We do this by forming the n/(n+1) approximants, whose denominators are of one degree higher than the numerators. The value of α can be determined by taking the ratio of the two highest degree coefficients in the numerator and the denominator, respectively. When we form the ratios 1/2,2/3,3/4,5/6 ,… of the Pade approximants, the value α goes as: **-0.0565685,** -0.0572354, -0.0572357, -0.0582652, -0.0525516, -0.0544958, -0.0547399, -0.0548389, -0.0549789, -0.0559719, -0.0533766, -0.0537541, -0.0537595, -0.0543275, -0.0542895, -0.0543502,-0.0543381, -0.0544779.

This sequence is likely approaching 0.055 and thus is consistent with the value that has already been discussed. It is typical of Pade approximants that this sequence is irregular. However, it is very unlikely that it approaches the value 1/10 of existing boundary-layer analysis, and very plausible that it approaches 1/18. To summarize, almost all the techniques of analysis indicate that the singularity at infinity is located at one in the Euler-transformed plane to at least four-digit accuracy. However, the exponent related to that singularity cannot be absolutely determined by available methods. Taken together, all these techniques strongly suggest that the exponent α is surely within the range of .05 and .055. Pade approximants strongly suggest that the exponent is clearly close to 1/18 which is the value corresponding to -1/9 in the original form.

7. The secondary singularity at $k = \infty$ and conclusion

In this problem the nearest singularity is on the negative real axis of K^2 has no physical significance, and

Therefore unnecessarily limits the range of applicability of the series. The Euler transformation is one way to analytically continue a series or extend the range of convergence for physical K. All the analysis for finding boundary layer singularity by having more term is consistent with what appeared in [1]. We omit details for lack of space. Following [1] we finally get the asymptotic relationship between our parameter K and White's k [7] it is based on the actual mean velocity W_m down the pipe. Their k is defined in our terms as:

$$k=\frac{2W_m a}{\nu}\left(\frac{a}{L}\right)^{\frac{1}{2}} \quad \text{or} \quad k = R\left(\frac{a}{L}\right)^{\frac{1}{2}}$$

then $$k=1.38K^{\frac{2}{5}} \qquad (8)$$

The flux ratio decays as:

$$\frac{F_c}{F_s}\sim 2.12k^{-\frac{1}{4}} \qquad (9)$$

And the friction ratio grows asymptotically as:

$$\frac{f_c}{f_s}=\left(\frac{F_c}{F_s}\right)^{-1}\sim 0.47k^{\frac{1}{4}} \qquad (10)$$

Fig 1 is a comparison of previous experimental and boundary-layer results with the semi-numerical result obtained in this research.

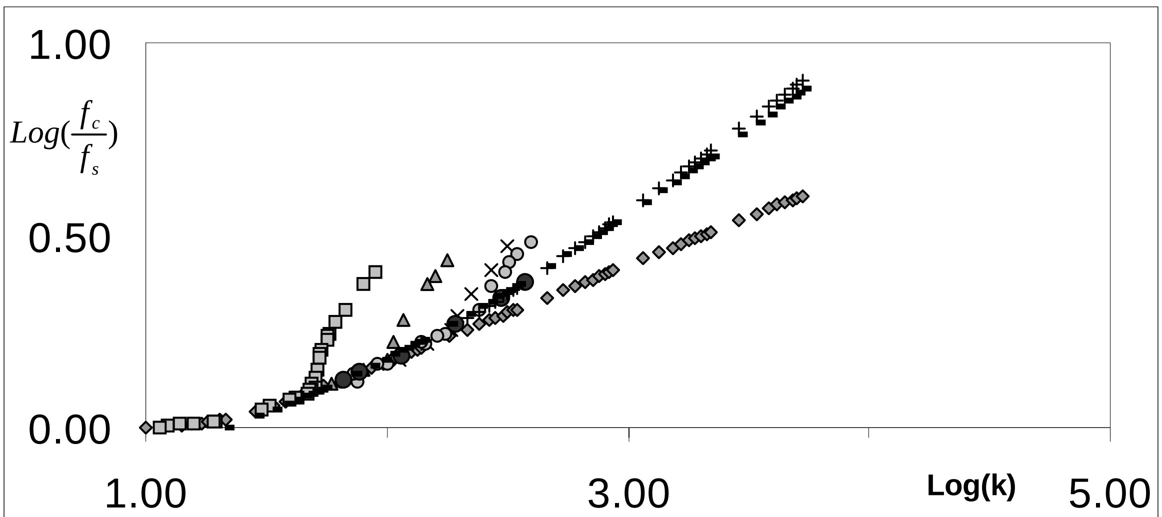

Fig1. Experimental friction ratios for various curvature ratios

3. Discussion

This work is concerned with the problem of flow through a loosely coiled pipe. Then by using symbolic language MACSYMA we have given some exact terms of their corresponding Stokes expansion. Then we did computer extension. Up to 40 terms and our analysis persist the same conclusion as previously reported in [1] which had only 24 terms, a major difference between our results and those of other analyses is in the asymptotic behavior of the friction factor as the similarity parameter $K = R^2(\frac{a}{L})$ and hence k increases. The prevailing opinion has been that the relationship goes as $f_r / f_s \sim k^{1/2}$, whereas we find that $f_r / f_s \sim k^{1/4}$.

As can be seen in figure 1, this difference is not significant until k is greater than 8000. Below that is little discrepancy between the present work and others. It can be explained by the fact that experiments require a finite amount of curvature, whereas this investigation considers the limit as the coiling ratio goes to zero. It would be helpful to obtain experimental data for this curved pipe problem when a/L is very small in order to see if our analysis is valid for considerably lower values of k.

As regards the difference between the asymptotic behaviors for large values of k, $f_r / f_s \sim k^{1/4}$ versus $f_r / f_s \sim k^{1/2}$, the following remarks can be made. Our expansion is based on the double limit:

$$\left.\begin{array}{l} R \to 0 \\ \frac{a}{L} \to \infty \end{array}\right\} K = R^2 \ (\frac{a}{L})\, fixed.$$

From an expansion for small K the limiting case as K goes to infinity has been exerted; however, the experiments are based upon $R \to \infty$ and $\frac{a}{L}$ fixed (and small). In other words the asymptotic behavior may depend on the manner in which the similarity parameter tends to infinity. The effect of diameter ratios on the relation between the friction factor and the Dean number has been investigated numerically by [16] and is found to be negligibly small. They concluded the friction factor ratio is relatively insensitive to the diameter ratios at a given Dean number. However in 1988 [18] for corresponding curved pipe problem tried to make experiment to match the series extension, would have to satisfy that is the flow be fully developed at high Dean number namely bigger than 500 and laminar as well as a/L less than .03. But his result adds another mystery to this paradox and they obtained data, which lay closer to the present method.

An alternative explanation of the departure of the experiments from our curve might be that for more tightly coiled pipes the steady laminar flow is succeeded not by turbulent flow, but by an intermediate regime of unsteady laminar motion, with higher friction. Taylor [25] expressed the possibility that there may exist an intermediate flow regime between the laminar and turbulent ranges. He observed a transition from a steady laminar flow to a laminar vibrating flow as the speed increased. The onset of turbulence accrued only at a significantly higher speed.

As far as uniqueness is concerned, for this problem a second solution has been observed. [13] And some other researchers have been able to obtain dual solutions in a coiled pipe using a series truncation or finite-difference methods. This phenomenon has also been observed in the flow through a curved semicircular and rectangular duct see [14],[17] respectively. By existence of such a duality, the fact (which has long been established numerically) that the laminar flow in a curved duct is composed of a main flow in the axial direction with a superimposed secondary flow having two counter-rotating vortices, should be reconsidered. Certainly the second solution having four counter-rotating vortices will raise the question about other branches of bifurcation. Moreover they have shown that this phenomenon has little effect on friction ratio.

Thus we have seen possible explanations for the discrepancy between our semi numerical results and the experiments regarding the exponent of the similarity parameter. For large values of Dean Number however, conventional boundary-layer theories, which are based on the same equations as ours, and the same assumption of tightly coiled pipes flow, are in agreement. So if the experimental graphs are not completely describing the flow field, then we also must find fault with the boundary-layer results. The authors in [2] believes that the boundary-layers on the two halves of the tours collide at the innermost circle, separate there, and form a re-entrant jet that moves outward through the core; but he makes no attempt to incorporate that phenomenon into his analysis. The model of [5] for the structure of the flow leads to the infinite thickness; or in other words, his model breaks down locally. On the other hand [4], believing that collision of the boundary-layer is unreasonable deliberately suppress it.[6],[22] assuming that a solution with attached boundary-layer exists, show that the velocity would vanish in non analytic fashion at the innermost triangle and circle respectively. Finally [3] claims that the boundary-layer separate from the wall at about 27 degree from inside. But does not consider how that would alter his assumed core flow. In this paper we have shown that what proposed by [20] as the cause of the discrepancy lies in the use of the only first 12 terms for corresponding curved problem is not correct. In this work we increase number of terms from 12 in [1] to 20 and still the major conclusion of the discrepancy between the solution obtained using the extended stoke series method and that obtained using other method persist as it was the case for the rotating pipe[26].

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