Global Interactive Boundary Layer (GIBL) for a Channel

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1. Introduction

We consider a laminar, steady, two-dimensional flow of an incompressible Newtonian fluid in a channel at high Reynolds number. When the walls are slightly deformed, adverse pressure gradients are generated and separation can occur. The analysis of the flow structure has been done essentially by Smith [7]. Later, a systematic asymptotic analysis has been performed by Saintlos and Mauss [6]. More recently, the modelling of channel flow has been examined by Lagrée et al. [5]. These analyses show that there is no external flow region and the asymptotic models for the flow perturbations are mainly based on an inviscid rotational core flow region together with boundary layers near the walls; a comprehensive discussion of this structure can be found in Sobey [8].

Here, we use the Successive Complementary Expansion Method, SCEM, in which we assume a uniformly valid approximation (UVA) based on generalized expansions. This method, developed by Cousteix and Mauss [1, 2], has been used by Dechaume et al. [4]. The first step consists of an inviscid approximation which applies far from the walls. This approximation must be improved near the walls by adding a correction which takes into account the effects of viscosity. Thanks to generalized expansions, a strong coupling occurs between the viscous and inviscid regions. This notion is called "interactive boundary layer" (IBL). This means that the effect of the boundary layer on the inviscid flow and the reciprocal effect are considered simultaneously. The construction of the UVA does not require any matching principle, only the boundary conditions of the problem are applied.

2. Formulation of the problem

Navier-Stokes dimensionless equations can be written

$$\mathbf{div}\,\overrightarrow{V}=0\ ,\quad (\mathbf{grad}\,\overrightarrow{V})\cdot\overrightarrow{V}=-\,\mathbf{grad}\,\Pi+\frac{1}{\mathcal{R}}\triangle\overrightarrow{V}\ . \tag{1}$$

where $\mathcal{R}$ is the Reynolds number. The basic plane Poiseuille flow is

$$v_{(x)}=u_0=\frac{1}{4}-y^2\ ,\quad v_{(y)}=0\ ,\quad \Pi=\Pi_0=-\frac{2x}{\mathcal{R}}+p_0\ . \tag{2}$$

The flow is perturbed, for instance, by indentations of the lower and upper walls such as

$$y_l=-\frac{1}{2}+\varepsilon F(x,\varepsilon)\ ,\quad y_u=\frac{1}{2}-\varepsilon G(x,\varepsilon)\ , \tag{3}$$

where ε is a small parameter (Fig. 1). If we seek a solution in the form

$$v_{(x)}=u_0(y)+\varepsilon u(x,y,\varepsilon)\ ,\quad v_{(y)}=\varepsilon v(x,y,\varepsilon)\ ,\quad \Pi-p_0=-\frac{2x}{\mathcal{R}}+\varepsilon p(x,y,\varepsilon)\ , \tag{4}$$

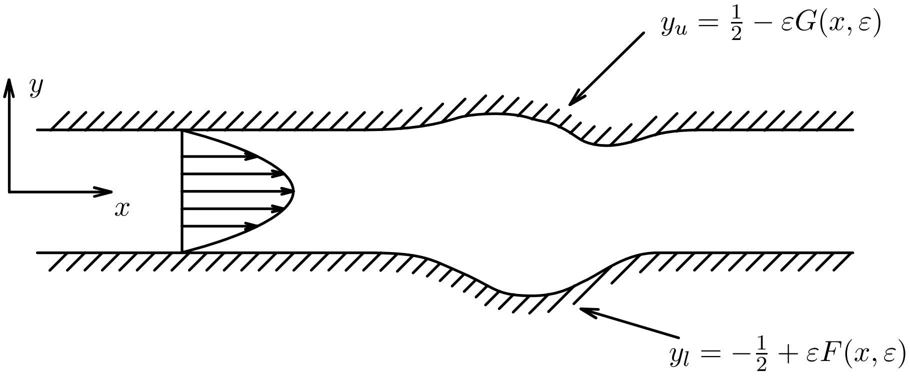

Figure 1: Flow in a two-dimensional channel with deformed walls. In this figure, all quantities are dimensionless

the Navier-Stokes equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{5a}$$

$$\varepsilon\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + u_0\frac{\partial u}{\partial x} + v\frac{\mathrm{d}u_0}{\mathrm{d}y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathcal{R}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\,, \tag{5b}$$

$$\varepsilon\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + u_0\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{1}{\mathcal{R}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\,. \tag{5c}$$

It is clear that, for high Reynolds numbers, the reduced equations are of first order leading to a singular perturbation. In the core flow, we are looking for approximations coming from asymptotic generalized expansions such as

$$u = u_1(x,y,\varepsilon) + \cdots\,, \quad v = v_1(x,y,\varepsilon) + \cdots\,, \quad p = p_1(x,y,\varepsilon) + \cdots\,. \tag{6}$$

Formally, neglecting terms of order $\mathrm{O}(\varepsilon, 1/\mathcal{R})$, for the core flow, we obtain

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0\,, \tag{7a}$$

$$u_0\frac{\partial u_1}{\partial x} + v_1\frac{\mathrm{d}u_0}{\mathrm{d}y} = -\frac{\partial p_1}{\partial x}\,, \tag{7b}$$

$$u_0\frac{\partial v_1}{\partial x} = -\frac{\partial p_1}{\partial y}\,. \tag{7c}$$

It is useful to note the behaviour of the solution of (7a–7c) in the vicinity of the walls. For instance, as $y \to -1/2$, we have

$$\begin{aligned} u_1 &= -2p_{10}\ln\left(\frac{1}{2}+y\right) + c_{10} + \cdots\,, \\ v_1 &= -p_{10x} + 2p_{10x}\left(\frac{1}{2}+y\right)\ln\left(\frac{1}{2}+y\right) - \left(\frac{1}{2}+y\right)(2p_{10x}+c_{10x}) + \cdots\,, \\ p_1 &= p_{10} + \frac{1}{2}\left(\frac{1}{2}+y\right)^2 p_{10xx} + \cdots\,. \end{aligned}$$

In the above equations, p_{10} and c_{10} are functions of x and ε. The letter x in index denotes a derivative with respect to the streamwise variable x.

3. Uniformly Valid Approximation

In order to satisfy the no-slip condition at the walls, two boundary layers are introduced in which the appropriate variables are

$$Y = \frac{\frac{1}{2}+y}{\varepsilon} , \quad (8a)$$

$$\widehat{Y} = \frac{\frac{1}{2}-y}{\varepsilon} . \quad (8b)$$

In terms of boundary layer variables, the boundary layer thicknesses are of order 1. Then, in the two boundary layers, we have $u_0 = \mathrm{O}(\varepsilon)$. In this way, u_0 and εu_1 are of the same order near the walls and the velocity $u_0 + \varepsilon u$ in (4) can be negative. According to SCEM, a UVA is obtained by complementing the core approximation

$$u = U_1(x, Y, \varepsilon) + \widehat{U}_1(x, \widehat{Y}, \varepsilon) + u_1(x, y, \varepsilon) , \quad (9a)$$

$$v = \varepsilon V_1(x, Y, \varepsilon) - \varepsilon \widehat{V}_1(x, \widehat{Y}, \varepsilon) + v_1(x, y, \varepsilon) , \quad (9b)$$

$$p = \Delta(\varepsilon) P_1(x, Y, \varepsilon) + \Delta(\varepsilon) \widehat{P}_1(x, \widehat{Y}, \varepsilon) + p_1(x, y, \varepsilon) , \quad (9c)$$

where the gauge function $\Delta(\varepsilon)$ is yet undetermined. Here, the quantities (u, v, p) do not represent the exact solution but only an approximate solution.

The form of approximation for v in (9b) is imposed by the continuity equation which must be non trivial

$$\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial Y} = 0 , \quad (10a)$$

$$\frac{\partial \widehat{U}_1}{\partial x} + \frac{\partial \widehat{V}_1}{\partial \widehat{Y}} = 0 . \quad (10b)$$

With this formulation, it is clear that, if (u_1, v_1) represent an approximation in the core of the flow, we have

$$Y \to \infty \quad : \quad U_1 \to 0 \quad , \quad V_1 \to 0 , \quad (11a)$$

$$\widehat{Y} \to \infty \quad : \quad \widehat{U}_1 \to 0 \quad , \quad \widehat{V}_1 \to 0 . \quad (11b)$$

Boundary conditions are required along the lower and upper walls of the channel, i.e. along the lines $Y = F(x, \varepsilon)$ and $\widehat{Y} = G(x, \varepsilon)$. Along these two walls, we have

$$Y = F(x, \varepsilon) \quad : \quad u_0 + \varepsilon u = 0 \quad , \quad v = 0 , \quad (12a)$$

$$\widehat{Y} = G(x, \varepsilon) \quad : \quad u_0 + \varepsilon u = 0 \quad , \quad v = 0 . \quad (12b)$$

With the approximation given by (9a, 9b), we have

$$Y = F(x, \varepsilon) \quad : \quad u_0 + \varepsilon U_1 + \varepsilon u_1 = 0 \quad , \quad \varepsilon V_1 + v_1 = 0 , \quad (13a)$$

$$\widehat{Y} = G(x, \varepsilon) \quad : \quad u_0 + \varepsilon \widehat{U}_1 + \varepsilon u_1 = 0 \quad , \quad -\varepsilon \widehat{V}_1 + v_1 = 0 . \quad (13b)$$

It is useful to note that, in contrast with external boundary layers, the terms u_1 and v_1 or their y-derivatives are singular in the vicinity of the walls. This shows the great advantage of SCEM since the UVAs for u and v are perfectly regular.

4. IBL Model for the Lower Wall

In order to obtain a UVA in the lower boundary layer and in the core flow, we set

$$\begin{aligned} u &= U_1(x,Y,\varepsilon) + u_1(x,y,\varepsilon) \ , & (14a)\\ v &= \varepsilon V_1(x,Y,\varepsilon) + v_1(x,y,\varepsilon) \ , & (14b)\\ p &= \Delta(\varepsilon) P_1(x,Y,\varepsilon) + p_1(x,y,\varepsilon) \ , & (14c) \end{aligned}$$

where, for the sake of simplicity of notation, the same notation (u, v, p) as for the preceding UVA given by (9a–9c) is used. In order to have the same order for the inertia terms and the viscous terms in the boundary layer, we take

$$\mathcal{R} = \frac{1}{\varepsilon^3} \ . \qquad (15)$$

In Navier-Stokes equations, it is necessary to keep terms which are apparently negligible in order to ensure that the behaviour at the wall is bounded. Now, it is essential to examine the pressure terms. From the condition on the transverse velocity given by (13a) and from (7c), in the boundary layer, we have $v_1 = \mathrm{O}(\varepsilon)$ and $\dfrac{\partial p_1}{\partial y} = \mathrm{O}(\varepsilon^2)$. Then, we must take $\Delta = \varepsilon^3$, otherwise the transverse momentum equation cannot be satisfied. Coming back to approximations (u, v, p) expressed by (14a–14c), we obtain

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \ , & (16a)\\ \varepsilon\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + u_0\frac{\partial u}{\partial x} + v\frac{\mathrm{d}u_0}{\mathrm{d}y} &= -\frac{\partial p_1}{\partial x} + \frac{1}{\mathcal{R}}\frac{\partial^2 u}{\partial y^2} \ . & (16b) \end{aligned}$$

Similar equations can be obtained for the upper boundary layer. These equations must be solved in association with the core flow equations. Therefore, it is clear that (16a, 16b) associated with the core flow equations give an *approximation valid in the whole channel*.

If necessary, the transverse momentum equation can be used to give the transverse pressure gradient $\dfrac{\partial p}{\partial y}$. We have

$$\varepsilon\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + u_0\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{1}{\mathcal{R}}\frac{\partial^2 v}{\partial y^2} \ . \qquad (17)$$

5. Global IBL Model

The generalized asymptotic expansions for the velocity components are given by

$$\begin{aligned} v_{(x)} &= u_0(y) + \varepsilon u(x,y,\varepsilon) + \cdots \ , & (18a)\\ v_{(y)} &= \varepsilon v(x,y,\varepsilon) + \cdots \ . & (18b) \end{aligned}$$

Let us remember that it is necessary to solve (16a, 16b) in association with the core flow equations (7a–7c).

Equations (16a, 16b) can be recast in the same form as Prandtl's equations if we set

$$\begin{aligned} \tilde{u} &= u_0 + \varepsilon u \ , & (19a)\\ \tilde{v} &= \varepsilon v \ , & (19b)\\ \tilde{p}_1 &= -\frac{2x}{\mathcal{R}} + \varepsilon p_1 + p_c \ , & (19c) \end{aligned}$$

where p_c is an arbitrary constant. Equations (16a, 16b) become

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0 \,, \tag{20a}$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y} = -\frac{\partial \tilde{p}_1}{\partial x} + \frac{1}{\mathcal{R}}\frac{\partial^2 \tilde{u}}{\partial y^2} \,. \tag{20b}$$

The above equations have the same form as Prandtl's equations, but the pressure is not constant in the y-direction.

These equations are associated with boundary conditions. At the walls, the no-slip conditions are $\tilde{u} = 0$ and $\tilde{v} = 0$. It is clear that the pressure gradient must be adjusted to ensure mass flow conservation in the channel. In addition, the solution for the core flow equations requires additional conditions.

6. Numerical Solution

In this section, we present a brief description of the numerical solution of the global IBL model.

To calculate the pressure and to produce the results discussed in Sect. 7., we use a simple approach suggested by Smith's theory and obtained in the case of longer wall deformations. The core flow equations are

$$\frac{\partial(\tilde{u}_1 - u_0)}{\partial x} + \frac{\partial \tilde{v}_1}{\partial y} = 0 \,, \tag{21a}$$

$$u_0\frac{\partial(\tilde{u}_1 - u_0)}{\partial x} + \tilde{v}_1\frac{\mathrm{d}u_0}{\mathrm{d}y} = 0 \,, \tag{21b}$$

$$u_0\frac{\partial \tilde{v}_1}{\partial x} = -\frac{\partial}{\partial y}\left(\tilde{p}_1 + \frac{2x}{\mathcal{R}}\right) \,. \tag{21c}$$

The solution is given by

$$\tilde{u}_1 - u_0 = \widetilde{A}(x)\frac{\mathrm{d}u_0}{\mathrm{d}y} \,, \tag{22a}$$

$$\tilde{v}_1 = -\frac{\mathrm{d}\widetilde{A}}{\mathrm{d}x}u_0 \,, \tag{22b}$$

$$\tilde{p}_1 + \frac{2x}{\mathcal{R}} = \widetilde{B}(x) + \frac{\mathrm{d}^2\widetilde{A}}{\mathrm{d}x^2}\int_0^y u_0^2(\eta)\,\mathrm{d}\eta \,, \tag{22c}$$

where η is an integration variable and the arbitrary constant in the pressure is absorbed in the function $\widetilde{B}(x)$ and

$$\tilde{u}_1 = u_0 + \varepsilon u_1 \,, \tag{23a}$$

$$\tilde{v}_1 = \varepsilon v_1 \,. \tag{23b}$$

Thus, the pressure is given by

$$\tilde{p}_1 + \frac{2x}{\mathcal{R}} = \widetilde{B}(x) + \frac{\mathrm{d}^2\widetilde{A}}{\mathrm{d}x^2}\left(\frac{y}{16} - \frac{y^3}{6} + \frac{y^5}{5}\right) \,. \tag{24}$$

In this formulation, the question is to determine the function $\widetilde{B}(x)$ and the so-called *displacement function* $\widetilde{A}(x)$. To this end, two conditions are used. The first one is to ensure mass flow conservation in the channel and the second one is given by

$$\tilde{v}(y_c) = \tilde{v}_1(y_c) \,, \tag{25}$$

where $y = y_c$ is a core line. This assumption is justified by the fact that the wall boundary layers are thin and that the flow perturbation in the core is inviscid; therefore the solution for $\tilde{u}$ and $\tilde{v}$, which is assumed to be a *uniformly valid approximation*, must agree with the solution for $\tilde{u}_1$ and $\tilde{v}_1$.

7. Comparison with Navier-Stokes Solutions

In order to assess the validity of the proposed global IBL model, comparisons with Navier-Stokes solutions are presented in this subsection. The IBL model is based on the system of generalized boundary layer equations (20a) and (20b) associated with (22c). The Navier-Stokes solutions were obtained by Dechaume who developed a highly accurate solver [3]. A spectral method based on Legendre polynomials has been implemented and the solution involves a domain decomposition of Dirichlet-Neuman type. A technique of velocity-pressure decoupling is used. For the time integration, the time derivatives are expressed by an implicit Euler scheme, the nonlinear terms and the pressure boundary conditions are extrapolated. The resulting linear systems are solved by successive diagonalisations.

For these comparisons, the flow is calculated in a channel whose upper wall is flat and the lower wall is deformed in the domain $-L/2 \leq x \leq L/2$ according to

$$y_l = -\frac{1}{2} + \frac{h}{2}\left[1 + \cos\left(\frac{2\pi x}{L}\right)\right] . \tag{26}$$

Outside the domain $-L/2 \leq x \leq L/2$, the lower wall is flat.

One case is presented here. The lower wall is deformed by a bump such as

$$h = 0.36 \quad , \quad L = 4 \quad , \quad \mathcal{R} = 1000 .$$

The characteristics of the wall indentation were chosen to produce a flow close to separation on the lower wall.

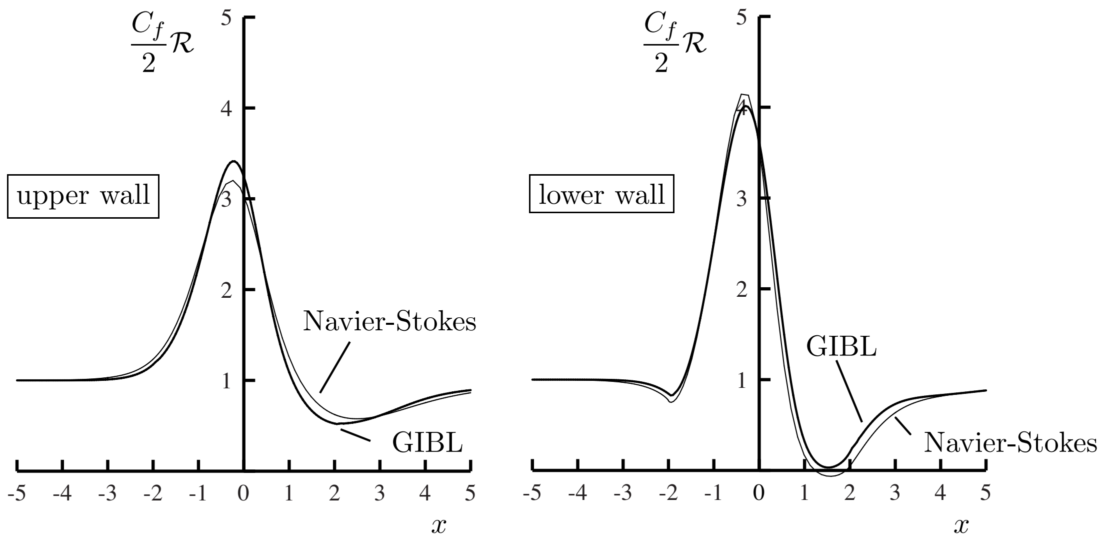

Figure 2: Comparisons between global IBL results and Navier-Stokes solutions ($\mathcal{R} = 1000$, $h = 0.36$, $L = 4$)

The skin-friction coefficient is defined by

$$C_f = \frac{2}{\mathcal{R}} \left| \frac{\partial \tilde{u}}{\partial y} \right|_{y=y(\text{wall})} .$$

The evolution of $\frac{C_f}{2}\mathcal{R}$ is plotted as function of x in Fig. 2. The overall agreement between the global IBL results and the Navier-Stokes solutions is very satisfactory. Considering that the Navier-Stokes results are reference solutions, the shape of the curves and the level of the skin-friction are well predicted by the global IBL model. Let us note that the asymptotic theory is established for large Reynolds numbers and for wall indentations whose height is small and length is large compared to the channel width. Even if these conditions are not satisfied a priori in this test case, the agreement of IBL results with Navier-Stokes solutions is strikingly good. When the flow perturbations induced by the wall deformation are weak, the cross-section pressure variations are very small and it is sufficient to assume that $\dfrac{\partial p}{\partial y} = 0$. Then, the evolutions of the skin-friction on the upper and lower walls are identical. For more severe wall deformations, the hypothesis of a constant pressure in a cross-section does not hold and (22c) can be used to calculate the pressure variations. In this case, for a non symmetric wall deformation, the skin-friction evolutions along the upper and lower walls are not the same.

8. Conclusion

Different approximations of Navier-Stokes equations for the study of high Reynolds number flows in a two-dimensional channel with deformed walls are obtained by applying SCEM.

The flow perturbations are described by an inviscid flow model in the core which is strongly coupled to generalized boundary layer equations valid in the whole channel. Finally, we obtain a global *interactive boundary layer* model. As in the study of external flows, SCEM proved to be a very fruitful tool for analyzing the flow structure.

A simplified model for the pressure variations has been implemented numerically. Essentially, this model for the pressure is based on Smith's theory which is the equivalent of the triple deck theory for external flows. IBL results obtained with this simplified pressure equation are in very good agreement with Smith's theory, at least as far as boundary layer characteristics are concerned, and also with the Navier-Stokes solutions. Even with relatively severe wall indentations and not very large Reynolds numbers, the global IBL model produces satisfactory results. It is expected that even better results can be obtained with a more refined model for the core flow.

It should be noted that SCEM offers interesting perspectives with the construction of a UVA. In fact, $(\tilde{u}_1, \tilde{v}_1)$ is not necessarily an approximation in the flow core whereas $(\tilde{u}, \tilde{v})$ gives a UVA.

References

[1] J. Cousteix and J. Mauss. Approximations of the Navier-Stokes equations for high Reynolds number flows past a solid wall. *Jour. Comp. and Appl. Math.*, 166(1):101–122, 2004.

[2] J. Cousteix and J. Mauss. *Analyse asymptotique et couche limite*, volume 56 of *Mathématiques et Applications*. Springer-Verlag, SMAI, Berlin, Heidelberg, 2006.

[3] A. Dechaume. *Analyse asymptotique et numérique des équations de Navier-Stokes : cas du canal.* PhD thesis, Université Paul Sabatier, Toulouse, 2006.

[4] A. Dechaume, J. Cousteix, and J. Mauss. An interactive boundary layer model compared to the triple deck theory. *Eur. J. of Mechanics B/Fluids*, 24:439–447, 2005.

[5] P.Y. Lagrée, E. Berger, M. Deverge, C. Vilain, and A. Hirschberg. Characterization of the pressure drop in a 2d symmetrical pipe: some asymptotical, numerical and experimental comparisons. *ZAMM Z. Ange. Math. Mech.*, 85(1):1–6, 2005.

[6] S. Saintlos and J. Mauss. Asymptotic modelling for separating boundary layers in a channel. *Int. J. Engng. Sci.*, 34(2):201–211, 1996.

[7] F.T. Smith. On the high Reynolds number theory of laminar flows. *IMA J. Appl. Math.*, 28(3):207–281, 1982.

[8] I.J. Sobey. *Introduction to interactive boundary layer theory.* Oxford Applied and Engineering Mathematics. Oxford University Press, Oxford, 2000.