QUALITATIVE ANALYSIS OF NAVIER-STOKES LAYERS IN VICINITY OF THEIR CRITICAL LINES

by Adriana Nastase
Aerodynamik des Fluges, RWTH-Aachen, Germany
Fax: 0049/241/809-2173, E-Mail: nastase@lafaero.rwth-aachen.de

1. Introduction

The starting point of this study are the three-dimensional Navier-Stokes partial differential equations (PDEs), as given in [1] and [2], without any simplifications. Own developed, zonal, spectral solutions for the compressible stationary Navier-Stokes layer (NSL) over flattened flying configurations (FCs), are here proposed, as in [3]-[5]. These solutions use the potential flow over the FCs at the NSL's edge, instead of the parallel flow used by Prandtl in his boundary layer theory. A new spectral coordinate η was introduced in the NSL:

$$\eta = \frac{x_3 - Z(x_1, x_2)}{\delta(x_1, x_2)} \qquad (0 \le \eta \le 1) \tag{1}$$

and the dimensionless axial, lateral and vertical velocities u_δ, v_δ and w_δ, the density function $R = ln\rho$ and the absolute temperature T on the upper NSL (which is here only considered) are expressed in the following spectral forms, as in [3]-[5], namely :

$$u_\delta = u_e \sum_{i=1}^{N} u_i \eta^i \ , \qquad v_\delta = v_e \sum_{i=1}^{N} v_i \eta^i \ , \qquad w_\delta = w_e \sum_{i=1}^{N} w_i \eta^i \ ,$$

$$R = R_w + (R_e - R_w) \sum_{i=1}^{N} r_i \eta^i \ , \qquad T = T_w + (T_e - T_w) \sum_{i=1}^{N} t_i \eta^i \ . \tag{2a-e}$$

Hereby u_e, v_e, w_e, R_e and T_e are the edge values, which can be obtained from the outer potential flow, R_w and T_w are the given values of R and T on the wall of the FC and u_i, v_i, w_i, r_i and t_i are the unknown spectral coefficients of the velocity's components, which are used to fulfill the PDEs of the NSL. The first and the second derivatives of the velocity's components u_δ, v_δ, w_δ are linear functions versus the spectral coefficients u_i, v_i, w_i of the velocity's components.

Further, an exponential law of the viscosity μ versus T and the physical equation of ideal gas for the pressure p are here used:

$$\mu = \mu_\infty \left[\frac{T}{T_\infty} \right]^{n_1} \quad , \qquad p \equiv R_g \rho T = R_g e^R T \quad . \tag{3a,b}$$

Here are : R_g the universal gas constant, T_∞ the absolute temperature of the undisturbed flow and n_1 the viscosity exponent.

The seven boundary conditions for the velocity's components, at the NSL's edge, are written here in original explicit form versus seven chosen coefficients, namely u_{N-2}, u_{N-1}, u_N, v_{N-2}, v_{N-1}, v_N and w_N, as in [3]-[4] :

$$u_{N-2} = \alpha_{0,N-2} + \sum_{i=1}^{N-3} \alpha_{i,N-2} u_i \quad , \qquad v_{N-2} = \alpha_{0,N-2} + \sum_{i=1}^{N-3} \alpha_{i,N-2} v_i \quad ,$$

$$u_{N-1} = \alpha_{0,N-1} + \sum_{i=1}^{N-3} \alpha_{i,N-1} u_i \quad , \qquad v_{N-1} = \alpha_{0,N-1} + \sum_{i=1}^{N-3} \alpha_{i,N-1} v_i \quad ,$$

$$u_N = \alpha_{0,N} + \sum_{i=1}^{N-3} \alpha_{i,N} u_i \quad , \qquad v_N = \alpha_{0,N} + \sum_{i=1}^{N-3} \alpha_{i,N} v_i \quad ,$$

$$w_N = \gamma_{0,N} + \sum_{i=1}^{N-1} \gamma_{i,N}\, w_i \quad . \tag{4a-g}$$

2. The NSL's Impulse Equations

If the spectral forms, given in (2a-c), are introduced in the impulse equations and the seven coefficients u_{N-2}, u_{N-1}, u_N, v_{N-2}, v_{N-1}, v_N and w_N , given in (4a-g), are eliminated, the impulse equations contain $3N-7$ variables.

The impulse equations of the NSL, which are PDEs of second order, are now considered. If the spectral forms of the velocity's components u_δ, v_δ and w_δ, given in (2a-c), are introduced in the impulse equations and the seven coefficients u_{N-2}, u_{N-1}, u_N, v_{N-2}, v_{N-1}, v_N and w_N , given in (4a-g), are eliminated, the impulse equations lead to three equivalent quadratical algebraic equations (QAEs) with slightly variable coefficients, versus the spectral coefficients u_i, v_i and w_i, as in [3], [5], namely:

$$\sum_{i=1}^{N-3} u_i \left[\sum_{j=1}^{N-3} (\tilde{A}_{ij}^{(1)} u_j + \tilde{B}_{ij}^{(1)} v_j) + \sum_{j=1}^{N-1} \tilde{C}_{ij}^{(1)} w_j \right] = \tilde{D}^{(1)} + \sum_{i=1}^{N-3} (\tilde{A}_i^{(1)} u_i + \tilde{B}_i^{(1)} v_i)$$

$$+ \sum_{i=1}^{N-1} \tilde{C}_i^{(1)} w_i \quad ,$$

$$\sum_{i=1}^{N-3} v_i \left[\sum_{j=1}^{N-3} (\tilde{A}_{ij}^{(2)} u_j + \tilde{B}_{ij}^{(2)} v_j) + \sum_{j=1}^{N-1} \tilde{C}_{ij}^{(2)} w_j \right] = \tilde{D}^{(2)} + \sum_{i=1}^{N-3} (\tilde{A}_i^{(2)} u_i + \tilde{B}_i^{(2)} v_i)$$

$$+ \sum_{i=1}^{N-1} \tilde{C}_i^{(2)} w_i \quad ,$$

$$\sum_{i=1}^{N-1} w_i \left[\sum_{j=1}^{N-3} (\tilde{A}_{ij}^{(3)} u_j + \tilde{B}_{ij}^{(3)} v_j) + \sum_{j=1}^{N-1} \tilde{C}_{ij}^{(3)} w_j \right] = \tilde{D}^{(3)} + \sum_{i=1}^{N-3} (\tilde{A}_i^{(3)} u_i + \tilde{B}_i^{(3)} v_i)$$

$$+ \sum_{i=1}^{N-1} \tilde{C}_i^{(3)} w_i \quad . \tag{5a-c}$$

The qualitative analysis for the QAE with variable coefficients is further used.

3. Reduction of Quadratical, Elliptical and Hyperbolical Algebraic Equations to Their Canonical Forms

Let us consider a QAE of elliptical and hyperbolical types, namely:

$$\sum_{i=1}^{M} \left[\sum_{j=1}^{M} a_{ij}\, x_i\, x_j + 2a_{i,M+1}\, x_i \right] + a = 0 \quad . \tag{6}$$

In this QAE the free term is $a = a_{M+1,M+1}$ and its discriminant δ is the following :

$$\delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1M} \\ a_{21} & a_{22} & \dots & a_{2M} \\ \vdots & \vdots & & \vdots \\ a_{M,1} & a_{M,2} & \dots & a_{M,M} \end{vmatrix} \quad . \tag{7}$$

For elliptical and hyperbolical QAEs the discriminant δ does not cancel ($\delta \neq 0$).

The great determinant Δ of the QAE (6) is:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1,M} & a_{1,M+1} \\ a_{21} & a_{22} & \dots & a_{2,M} & a_{2,M+1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{M,1} & a_{M,2} & \dots & a_{M,M} & a_{M,M+1} \\ a_{M+1,1} & a_{M+1,2} & \dots & a_{M+1,M} & a_{M+1,M+1} \end{vmatrix} \quad . \tag{8}$$

The eigenvalues λ_i, of the QAE (6), are obtained as solutions of its characteristical equation $\Delta_c = 0$. Hereby the characteristic determinant Δ_c is the following :

$$\Delta_c = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1,M} \\ a_{21} & a_{22} - \lambda & \dots & a_{2,M} \\ \vdots & \vdots & & \vdots \\ a_{M,1} & a_{M,2} & \dots & a_{M,M} - \lambda \end{vmatrix} \quad . \tag{9}$$

The performing of the qualitative analysis of a QAE is easier, when the QAE is written in a canonical form :

$$\sum_{i=1}^{M} \lambda_i \, x_i''^2 + a'' = 0 \quad , \tag{10}$$

after translation and rotation. In this canonical form of the QAEs the free term is $a'' = \Delta/\delta$.

4. Elliptical and Hyperbolical Quadratical Algebraic Equations with Variable Free Terms

In these equations the free terms are proportional to the gradients of the pressure p in the directions of the axes of coordinates $O\,x_i$ and, therefore, these terms have greater variations than the other coefficients of these equations. The influence of the variations of free term and of one of coefficients of the linear term of the QAE over the existence of real values of the spectral coefficients and the performing of the qualitative analysis of the asymptotical behaviours of the three-dimensional PDEs of the compressible NSLs in the vicinity of their singular points and lines, are treated here by using the qualitative analysis of the equivalent QAEs.

The visualizations of asymptotical behaviours of these equivalent QAEs are made in a here introduced "Euclidian M-orthogonal space of the NSL's free spectral coefficients", which are here treated as variables.

Further the assumption is made that all QAEs of the same type (elliptical or hyperbolical), of the same size (same space dimension M), with the same number of positive eigenvalues, for which the free terms a are varied from $-\infty$ to $+\infty$ and all the other coefficients are maintained constant, have, in the vicinity of their singular points, qualitatively, similar asymptotical behaviours.

The visualizations of the asymptotical behaviours of elliptical and hyperbolical QAEs with variable coefficients, are made versus their principal coordinates. Their critical points and lines are obtained by cancelling of their great determinants Δ_i.

The visualization of elliptical QAEs with variable free coefficients is made, for two-dimensional variables. The elliptical QAE, for $M = 2$, chosen as exemplification and visualization, as in [5], is :

$$F_1 \equiv 3x^2 + 5y^2 + 4xy - 6x - 3y + a = 0 \quad . \tag{11}$$

The canonical form of this equation, after the translation and rotation, is :

$$F_1 \equiv \lambda_1 x''^2 + \lambda_2 y''^2 + a'' = 0 \quad . \qquad (a'' = \Delta_1/\delta_1) \tag{12}$$

The discriminant δ_1 and the great determinant Δ_1 of the QAE (11) are:

$$\delta_1 = 11 \quad , \qquad \Delta_1 = \frac{1}{4}(44a - 135) \tag{13a,b}$$

and the equation $F_1 = 0$ is elliptical because the eigenvalues λ_i are of the same sign, namely:

$$\lambda_1 = 1,764 \quad , \qquad \lambda_2 = 6,236 \quad . \tag{14a,b}$$

The critical value of the free term a of the equation (11) ($a \equiv a_c = 3,068$) is obtained by setting $\Delta_1 = 0$.

The elliptical QAE is visualized, in form of coaxial ellipses, which collapse, if the free term a is equal to the critical value a_c, which is located in the common center C of the coaxial ellipses, as given in (Fig. 1).

If $a < a_c$, the visualization of the elliptical QAE are coaxial ellipses, which all approach the critical point $a = a_c$, when the free term increases.

If $a \equiv a_c = 3,068$, the elliptical QAE collapse in this critical point (black point).

If $a > a_c$, elliptical QAE has no more real solutions and the spectral coefficients of the velocity's components are partially imaginary.

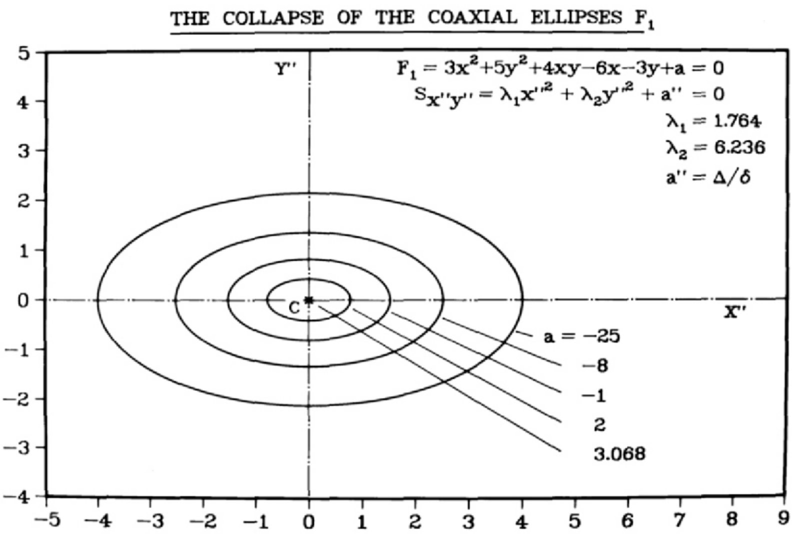

Fig. 1. Visualization of the Collapse of the Coaxial Ellipses.

Now the following hyperbolical QAE with two variables is taken, as exemplification, as in [5]:

$$F_2 \equiv 4x^2 + 7y^2 + 12xy - 4x - 5y + b = 0 \quad . \tag{15}$$

The canonical form of this equation, after the translation and rotation, is :

$$F_2 \equiv \lambda_1 x''^2 + \lambda_2 y''^2 + b'' = 0 \quad . \qquad (b'' = \Delta_2/\delta_2) \tag{16}$$

The discriminant δ_2 and great determinant Δ_2 of the QAE (15) are:

$$\delta_2 = -8 \quad , \qquad \Delta_2 = -8b + 7 \tag{17a,b}$$

and the eigenvalues λ_i are of opposite sign, i.e. :

$$\lambda_1 = -0,685 \quad , \qquad \lambda_2 = 11,685 \quad . \tag{18a,b}$$

The critical value of the free term b of the equation (15) ($b \equiv b_c = 0,875$) is obtained by setting $\Delta_2 = 0$.

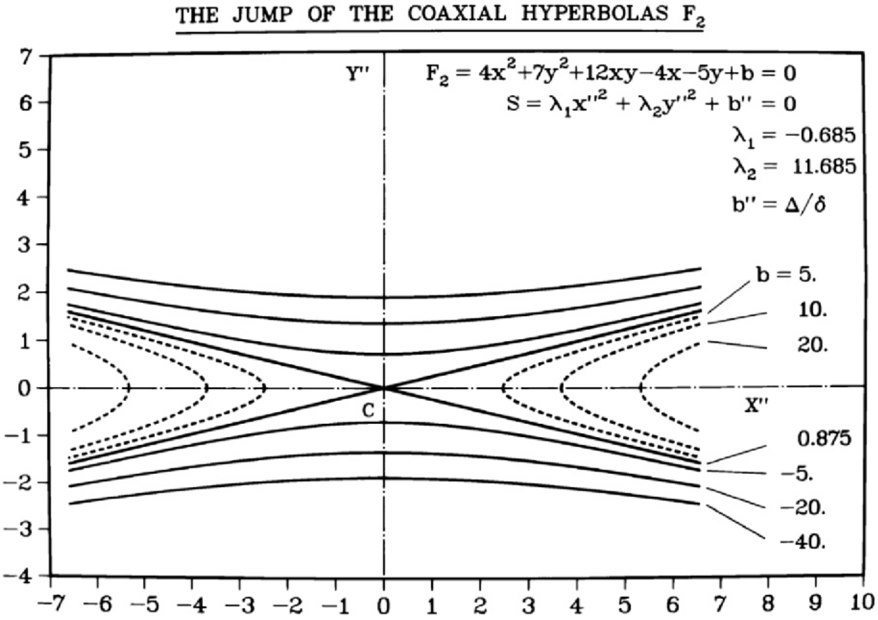

Fig. 2. The Visualization of the Jump of the Coaxial Hyperbolas with Two Sheets.

If the free term b in the equation (15) is systematically varied from $-\infty$ to $+\infty$ and:
- if $b < b_c$, the canonical equation $F_2 = 0$ is represented in form of coaxial hyperbolas with two sheets (Fig. 2), all centered in $C(x'' = 0,\ y'' = 0)$ and which approach their common, concurrent asymptotical lines, when b increases.
- if $b \equiv b_c = 0,875$, the corresponding hyperbola degenerates in its asymptotical lines;
- if $b > b_c$, the coaxial hyperbolas are jumping in the opposite angles of their concurrent asymptotical lines and are moving away from their asymptotical lines, when b increases. A saddle point occurs for $b = b_c$.

5. The Qualitative Analysis of Elliptical and Hyperbolical Quadratical Algebraic Equations with Variable Coefficients of Free and Linear Terms

Now, it is supposed that the free term $a_{33} = a$ and also one of the chosen coefficients of the linear term, namely $a_{13} = d$ of x, are variable :

$$F_1 \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2dx + 2a_{23}y + a = 0 \quad . \tag{19}$$

These QAEs have a critical parabolic line, which is obtained by cancellation of their great determinant, namely $\Delta = 0$. The implicit and the explicit forms of the equation of this critical parabola are the following:

$$a_{22}d^2 - 2a_{12}a_{23}d + \left(-a\delta + a_{11}a_{23}^2\right) = 0 \quad ,$$

$$a = \frac{1}{\delta}\left[a_{22}d^2 - 2a_{12}a_{23}d + a_{11}a_{23}^2\right] \quad . \tag{20a,b}$$

The implicit equation (20a) of the critical parabola, treated like a QAE in the variable d, furnishes two real critical values for d, namely d_{c_1} and d_{c_2} $(d_{c_1} < d_{c_2})$, as long as its determinant $\delta_p > 0$. This condition leads to an extremal value for a, namely $a = a_0$, obtained by cancelling the determinant of the QAE (20a), i.e. $\delta_p = 0$. The coordinates of the peak $P(a_0, d_0)$ of the critical parabola (20a,b) are the following:

$$a_0 = \frac{a_{23}^2}{a_{22}} \quad , \qquad d_0 = \frac{a_{12}a_{23}}{a_{22}} \quad . \tag{21a,b}$$

Now the following elliptical QAE is considered, for the visualization :

$$F_1 \equiv 3x^2 + 5y^2 + 4xy + 2dx - 3y + a = 0 \quad . \tag{22}$$

Its critical parabola is represented in the (Fig. 3). The equation (22) has a minimum critical value of their free term, namely $a \equiv a_0 = 0,44$, for $d \equiv d_0 = -0,6$ at the peak of the critical parabola.

For each given real value of $d = d_1$ of the elliptical QAE, there exists a critical value of the free term a, namely $a = a_c$, located on the critical parabola (20b), corresponding to this value of d.

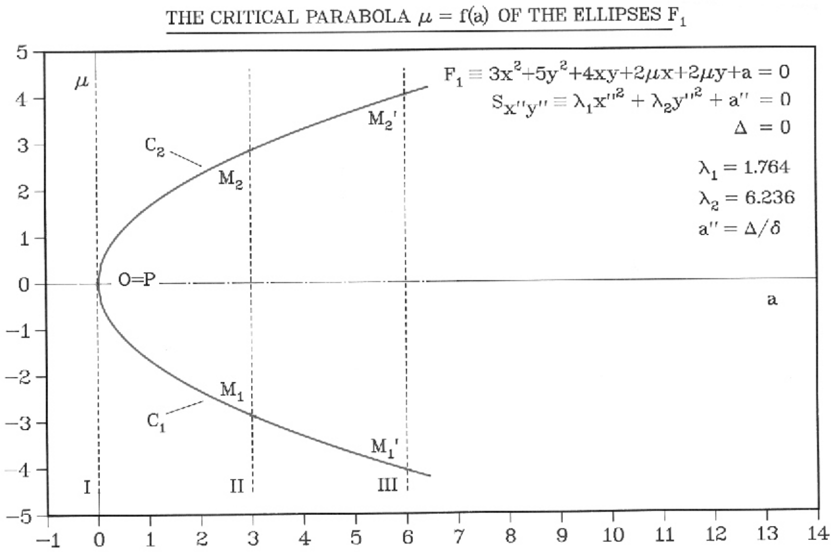

Fig. 3. The Visualization of the Critical Parabola of the Elliptical Equations $F_1 = 0$.

The coordinate $a \equiv a_0 = 0.44$ is the minimal critical value of the free term a. For a constant value of the free term a and
- if $a < a_0$, no critical point occurs and for each given a there exist no critical points;
- if $a = a_0$, the critical parabola has a double critical point for $d = d_0$, which is the peak P of the critical parabola. In this point the elliptical QAE collapse. The elliptical QAEs have real solutions, only if $d \leq d_0$;
- if $a > a_0$, there exist two critical values d_{c_1} and d_{c_2}, corresponding to the points M_1 and M_2 of the critical parabola, as in (Fig. 3). The elliptical QAE has real solutions for each $d \leq d_{c_1}$ or $d \geq d_{c_2}$. For $d_{c_1} < d < d_{c_2}$, there are no more real solutions for d. The inside of the critical parabola is a black hole for the elliptical QAE.

Now, for the visualization of the critical parabola, the following hyperbolical QAE with variable coefficients of its free term and of one of its linear terms is taken into consideration, as exemplification:

$$F_2 \equiv 4x^2 + 12xy + 7y^2 + 2dx - 5y + b = 0 \quad . \tag{23}$$

In the (Fig. 4) is represented the critical parabola $b = f(d)$, of this equation. The coordinates of the peak P of the critical parabola are $b_0 = 0.890$, $d_0 = -2.013$.

The value $b_0 = 0.890$ is now the maximal critical value of the free term b. For a constant value of the free term b and
- if $b > b_0$, no critical point occurs and the QAEs have real solutions;
- if $b = b_0$, there exists a double critical point $d = d_0$, located at the peak P of the critical parabola. In this point the hyperbolas degenerate in their asymptotes. By crossing the critical point P, the hyperbolas are jumping from one double angle of the asymptotes into their opposite double angle;
- if $b < b_0$, there exist two critical points d_{c_1} and d_{c_2}, corresponding to the points M_1 and M_2, located on the critical parabola, at the intersection with the transversal line $b = const.$, as in (Fig. 4). In these two points the hyperbolas degenerate in their asymptotes. By crossing the critical points M_1 and M_2, the hyperbolas are jumping from one double angle of the asymptotes into their opposite double angle, as before. The hyperbolic QAEs (for $M = 2$) degenerate in their asymptotes, in each point of their critical parabola. By crossing of this parabola, the hyperbolas approach their asymptotes in one of their double angles and, after jumping, they are going away from these asymptotes.

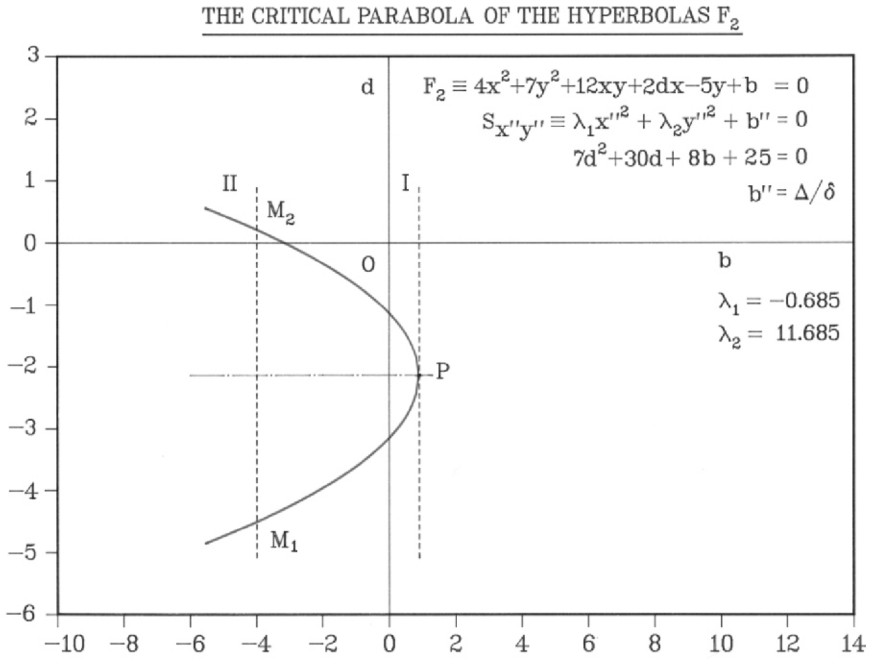

Fig. 4. The Visualization of the Critical Parabola of the Hyperbolical QAE $F_2 = 0$.

6. Conclusions

The stationary supersonic flow, governed by hyperbolical QAEs, degenerate along the critical lines and present two possible discontinuities, namely characteristic or shock surfaces. Bifurcations can also appear along the critical lines or surfaces. Hyperbolical solutions exist over all.

The stationary subsonic flow, governed by elliptical QAEs, tries to avoid the collapse and the black holes. Two possible changes of the flow can occur just before the black holes: to detach or to became instationary (governed by hyperbolical QAEs)!

The collapse points and lines of the elliptical QAE are useful for the determination of the position of detachment lines or for the beginning of transition. The saddle points, lines and surfaces of the hyperbolical QAE are useful for the determination of the position of characteristic or shock surfaces.

References

1. YOUNG, A.D., *Boundary Layers*, Blackwell Scientifical Publications, London, 1989, UK.
2. SCHLICHTING, H., *Boundary Layer Theory*, Mc. Graw-Hill, 1979, USA.
3. NASTASE, A., *New Zonal, Spectral Solutions for the Navier-Stokes Partial Differential Equations*, Proc. of the International Conference on Boundary and Interior Layers, ONERA Toulouse, 2004, France.
4. NASTASE, A., *Zonal, Spectral Solutions for Navier-Stokes Layer and Applications*, 4th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS), Jyväskylä, 2004, Finland.
5. NASTASE, A., *The Qualitative Analysis of the Asymptotical Behaviours of Quadratical Elliptical and Hyperbolical Partial Differential Equations and Its Applications to Aerodynamics*, Proc. of XXIII ARA Congress 1998 Rochester, New York, USA, Edit. Macarie, Bucharest, 1999, Romania.