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On Turbulent Marginal Separation: How the Logarithmic Law of the Wall is Superseded by the Half-Power Law*

B. Scheichl and A. Kluwick

Institute of Fluid Mechanics and Heat Transfer
Vienna University of Technology
Resselgasse 3/E322, A-1040 Vienna, Austria
bernhard.scheichl@tuwien.ac.at

1. Introduction and motivation

A novel rational theory of the incompressible nominally steady and two-dimensional turbulent boundary layer (TBL) exposed to an adverse pressure gradient, which is impressed by the prescribed external potential bulk flow, has been developed recently by the authors, see [1]. This asymptotic flow description exploits the Reynolds-averaged Navier–Stokes equations by taking the limit

$$Re = \tilde{U}\tilde{L}/\tilde{\nu} \ \rightarrow \infty \tag{1}$$

where the Reynolds number Re is formed by the global length and velocity scales $\tilde{L}$ and $\tilde{U}$, respectively, which are characteristic for the external inviscid and irrotational bulk flow, as well as the kinematic viscosity $\tilde{\nu}$. As the most important preliminary result of the approach presented in [1], in the limit (1) the slenderness of the boundary layer is seen to be independent of Re in formal limit $Re^{-1} = 0$. The rationale underlying this essential property can be cast in the following

Hypothesis (A) *Let all velocities be non-dimensional with the global reference value $\tilde{U}$. Furthermore, assume that in the limit (1) there exists a small non-dimensional local turbulent velocity scale U_t such that the vortical time-mean velocity variations are of $O(U_t)$. Then all components of the non-dimensional Reynolds stress tensor are quantities of $O(U_t^2)$.*

That is, the time-mean flow is presumed to be governed locally by a single velocity scale if the latter is sufficiently small compared to the free-stream velocity. Although this supposition is strongly suggested by physical intuition (and corroborated by an asymptotic analysis of any commonly used turbulence closures, see [2]), however, a rigorous justification will have to be based on an adequate investigation of the unsteady Navier–Stokes equations in the limit (1) — a challenging problem the authors intend to tackle yet. As a matter of course, the thereby expected affirmation of the hypothesis (A) must be regarded to be of fundamental impact on an asymptotic theory of turbulence.

The so-called classical theory, see for instance the pioneering paper [3], is capable of describing a strictly attached TBL only as it relies on the assumption of an asymptotically small streamwise velocity defect with respect to the external flow in the fully turbulent main part of the TBL. As demonstrated in [2], it can be derived solely by adopting (A) as the simplest example for an asymptotic description of a TBL, if in the main region U_t is identified with the skin friction velocity u_τ where u_τ^2 denotes the (non-dimensional) wall shear stress, in the following termed τ_w. Moreover, it is also elucidated in [2] that the method of matched asymptotic expansions allows for an extension of the classical approach which is capable of predicting flows having a velocity defect of $O(U_t)$ where U_t is still considered to be small but does not necessarily depend

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on Re. In a final step, by taking a streamwise velocity deficit of $O(1)$, which is a necessary characteristic of flows that may even undergo marginal separation, no asymptotically small turbulent velocity scale representative of the outer main flow regime is present there as the convective terms and, thus, the balanced shear stress gradient in the equations of motion must be quantities of $O(1)$ in this flow region. One then faces the remarkable consequence of (A), namely, that the slenderness of the boundary layer must be measured by an additional small parameter, in the following denoted by α, which is essentially independent of Re as $Re \to \infty$ and, in the current stage of development, to be determined experimentally. In the formal limit

$$\alpha \to 0, \quad Re^{-1} = 0 \tag{2}$$

then to be considered, the Reynolds shear stress must vanish at the surface. For finite values of Re, on the other hand, in an initially firmly attached TBL the shear stress in the flow regime which is located on top of the viscous wall layer adjacent to the surface assumes the value of τ_w in the overlap with that wall layer in leading order, in agreement with (A) as outlined in [4] and [2]. Therefore, in the former region, the so-called intermediate layer, U_t is set equal to u_τ and, in turn, seen to be of $O(1/\ln Re)$. The balance between convection and the Reynolds shear stress gradient requires the thickness of that intermediate layer to be of $O(U_t^2)$. But then the two-parameter expansion subject to (2) does not allow for a match of the velocity gradient normal to the surface with that holding in the outer main region having a thickness of $O(\alpha)$, cf. [1]. As a consequence, at the base of the latter a further layer has to be introduced the scaling of which is also seen to be independent of Re. That is, in the limit (2) the boundary layer comprises two layers, a so-called outer and inner wake layer, and, remarkably, is seen to closely resemble a turbulent free shear layer investigated asymptotically in [5].

The description of the outer main layer is addressed in [1]. Most important, it is demonstrated there analytically and numerically by adopting a local viscous/inviscid interaction strategy that in the primary limit (2) marginal separation is associated with the occurrence of closed reverse-flow regions where the surface slip velocity, which turns out to be a quantity of $O(1)$ in general, assumes negative values along a streamwise distance of $O(\alpha^{3/5})$. By taking into account the inner wake layer also, it is the objective of the present paper to demonstrate how the two near-wall flow regimes emerging for high but finite values of Re then singularly perturb the two-tiered wake flow referring to the limit (2). Particular emphasis is placed on the drastic variation of the universal flow behaviour in the overlap conjoining the viscous wall and the intermediate layer close to the locations of, respectively, separation and reattachment.

2. Weakly interacting flow in the limit of infinite Reynolds number

Problem formulation We consider a nominally steady and two-dimensional fully developed TBL driven by the prescribed incompressible and non-turbulent free-stream flow along a smooth and impermeable solid surface, being e.g. part of a diffuser duct. Let x, y, u, v, u', v', and p denote natural coordinates, respectively, along and perpendicular to the surface under consideration given by $y = 0$, the time-mean velocity components in x- and y-direction, the corresponding turbulent velocity fluctuations, and the time-mean fluid pressure. These quantities are non-dimensional with the reference values $\tilde{L}$ and $\tilde{U}$, respectively, introduced above, and the uniform fluid density. We furthermore define a stream function ψ by

$$\partial\psi/\partial y = u, \quad \partial\psi/\partial x = -h\, v, \quad h = 1 + k(x)\, y. \tag{3a}$$

Here $k(x) = O(1)$ is the accordingly non-dimensional surface curvature of the solid wall, which is defined as positive for a convex surface and assumed to be a quantity of $O(1)$. Adopting the usual notation for the turbulent stresses, the dimensionless time- or, equivalently, Reynolds-averaged

Navier–Stokes equations then read (cf. [6], p. 81)

$$h\Big(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\Big)\frac{\partial\psi}{\partial y}-k\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}=-h\frac{\partial p}{\partial x}-h\frac{\partial\langle u'^2\rangle}{\partial x}-\frac{\partial\, h^2\langle u'v'\rangle}{\partial y}+\frac{h^2}{Re}\frac{\partial\,\nabla^2\psi}{\partial y}, \tag{3b}$$

$$\Big(\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\Big)\Big(\frac{1}{h}\frac{\partial\psi}{\partial x}\Big)-k\Big(\frac{\partial\psi}{\partial y}\Big)^2=-h\frac{\partial p}{\partial y}-\frac{\partial\, h\langle v'^2\rangle}{\partial y}-\frac{\partial\langle u'v'\rangle}{\partial x}+k\langle u'^2\rangle-\frac{1}{Re}\frac{\partial\,\nabla^2\psi}{\partial x}, \tag{3c}$$

where $\nabla^2=h^{-1}[\partial/\partial x\,(h^{-1}\,\partial/\partial x)+\partial/\partial y\,(h\,\partial/\partial y)]$ is the Laplacian. The governing equations (3) are supplemented with the usual no-slip condition holding at the surface, i.e.

$$\psi=u=u'=v'=0 \quad \text{at} \quad y=0. \tag{4}$$

Since we exclude the presence of free-stream turbulence, the turbulent flow is essentially confined to the relatively thin boundary layer along the surface. As outlined in § 1., its thickness is measured by the quantity α, which is regarded as the primary perturbation parameter. For what follows, it is sufficient to represent the relatively distinct time-mean boundary layer edge by the sharp line $y=\delta(x;\alpha,Re)=O(\alpha)$. It is, furthermore, useful to express the Reynolds shear stress $-\langle u'v'\rangle$ by introducing the (positive) mixing length ℓ,

$$-\langle u'v'\rangle=\ell^2\frac{\partial u}{\partial y}\Big|\frac{\partial u}{\partial y}\Big|. \tag{5}$$

In addition, let $p_e(x;\beta)$, $u_e(x;\beta)$, and $u_s(x;\beta,\alpha)$ denote the pressure and the (streamwise) velocity component u, respectively, of the external potential main flow, evaluated at the surface $y=0$, as well as the slip velocity which denotes the, as will be seen, non-vanishing value of u at the surface in the limit (2). Here the control parameter $\beta=O(1)$ accounts for systematic variations in the external irrotational flow. For simplicity of notation, however, the explicit dependences on β will be omitted in the following.

Outer wake layer By considering the outer main layer first, inspection of (3) then suggests the following expansions there,

$$\begin{aligned}&\{\psi,\,-\langle u'v'\rangle,\,\delta\}=\alpha\{\mathit{\Psi}(x,Y),\,T(x,Y),\,\mathit{\Delta}(x)\}+O(\alpha^2),\\&\ell\sim\alpha^{3/2}L(x,Y)+\cdots,\quad p=p_e(x)+O(\alpha),\quad Y=y/\alpha.\end{aligned} \tag{6}$$

Herein the second-order terms represented by the Landau symbols account for, amongst others, the feedback of the induced external flow. In turn, the resulting leading-order shear layer approximation, allowing for a streamwise velocity deficit with respect to the imposed external flow of $O(1)$, reads

$$\frac{\partial\mathit{\Psi}}{\partial Y}\frac{\partial^2\mathit{\Psi}}{\partial Y\partial x}-\frac{\partial\mathit{\Psi}}{\partial x}\frac{\partial^2\mathit{\Psi}}{\partial Y^2}=-\frac{\mathrm{d}p_e}{\mathrm{d}x}+\frac{\partial T}{\partial Y},\quad -\frac{\mathrm{d}p_e}{\mathrm{d}x}=u_e\frac{\mathrm{d}u_e}{\mathrm{d}x},\quad T=L^2\frac{\partial^2\mathit{\Psi}}{\partial Y^2}\Big|\frac{\partial^2\mathit{\Psi}}{\partial Y^2}\Big|. \tag{7a}$$

These equations are subject to the wake-type boundary conditions

$$\mathit{\Psi}(x,0)=T(x,0)=0,\quad (\partial\mathit{\Psi}/\partial Y)\big(x,\mathit{\Delta}(x)\big)-u_e(x)=T\big(x,\mathit{\Delta}(x)\big)=0. \tag{7b}$$

The relationships for $Y=0$ in (7b) provide a match of the flow quantities with those in the inner wake region. A reasonable physical interpretation of the mathematical need of the latter is obtained from a consideration of the domain where surface effects play a dominant role and which remains finite even in the limit (2). One may then infer that in this limit under consideration effects on the mixing length which arise from the presence of the wall are restricted to that inner wake layer. As a finding substantiated by an asymptotic analysis of any commonly employed mixing length closures, see e.g. [6], this consideration implies that the rescaled mixing length L exhibits a finite value at the base of the outer main layer, say,

$$L(x,0) = L_0(x) = O(1), \tag{8}$$

see [1]. Most important, there it is also demonstrated numerically by adopting a specific closure for the Reynolds shear stress T which satisfies the requirement (8) and an (adverse) pressure distribution $p_e(x)$, that the boundary layer problem (7) indeed admits a "rotational" solution, i.e. one having $\partial^2\Psi/\partial Y^2 \not\equiv 0$ and $T \not\equiv 0$ in addition to the trivial, i.e. irrotational Eulerian, one given by $\partial\Psi/\partial Y - u_e(x) \equiv 0$, $T \equiv 0$. Then the associated streamwise velocity defect of $O(1)$ is conveniently measured by the slip velocity U_s and the related quantity λ,

$$U_s(x) = (\partial\Psi/\partial Y)(x, Y=0), \quad \lambda(x) = U_s\, \mathrm{d}U_s/\mathrm{d}x - u_e\, \mathrm{d}u_e/\mathrm{d}x > 0. \tag{9}$$

Note that for $U_s > 0$ and a given adverse pressure gradient, i.e. for $\mathrm{d}u_e/\mathrm{d}x < 0$, also $\mathrm{d}U_s/\mathrm{d}x < 0$ in general, whereas the sign of λ follows from the numerical solutions of (7), see [1]. Furthermore, from (7) one obtains the asymptotes

$$\Psi \sim U_s\, Y + 4/15\, (\lambda^{1/2}/L_0)\, Y^{5/2} + \cdots, \quad T = \lambda Y + O(Y^4), \quad Y \to 0. \tag{10}$$

Inner wake layer As a consequence of (6), (8), and (10), the inner wake regime has a thickness of $O(\alpha^{3/2})$ and is described by the expansions

$$\psi \sim \alpha^{3/2}\, U_s(x)\, \bar{Y} + \alpha^{9/4}\, \bar{\Psi}(x,\bar{Y}) + \cdots, \quad -\langle u'v' \rangle \sim \alpha^{3/2}\, \bar{T}(x,\bar{Y}) + \cdots,$$
$$\ell \sim \alpha^{3/2}\, \bar{L}(x,\bar{Y}) + \cdots, \quad p = p_e(x) + O(\alpha), \quad \bar{Y} = y/\alpha^{3/2}. \tag{11}$$

That is, the inner layer exhibits a streamwise velocity defect of $O(\alpha^{3/4})$ with respect to the slip velocity U_s, such that U_t is of $O(\alpha^{3/4})$. To leading order, there the equations of motion (3) reduce to

$$\lambda\bar{Y} = \bar{T} = \bar{L}^2 \frac{\partial^2\bar{\Psi}}{\partial\bar{Y}^2}\left|\frac{\partial^2\bar{\Psi}}{\partial\bar{Y}^2}\right|. \tag{12a}$$

Here the first of the boundary and matching conditions

$$\bar{T}(0,\bar{Y}) = \bar{\Psi}(0,\bar{Y}) = 0, \quad \partial\bar{\Psi}/\partial\bar{Y} - 2/3\, (\lambda^{1/2}/L_0)\, \bar{Y}^{3/2} \to 0 \quad \text{as} \quad \bar{Y} \to \infty, \tag{12b}$$

has been taken into account. The latter relationship in (12b) avoids the occurrence of an additional term of $O(\alpha^{3/4})$ in the expansion for ψ in (6). If $\bar{L} = L_0 + o(\bar{Y}^{-3/2})$ as $\bar{Y} \to \infty$ (which is the case for any known mixing length closure, see [1] and [6]), the solution of (12) for the streamwise velocity may be written as

$$\partial\bar{\Psi}/\partial\bar{Y} = 2/3\, (\lambda^{1/2}/L_0)\, \bar{Y}^{3/2} - \int_{\bar{Y}}^{\infty} (1/\bar{L} - 1/L_0)(\lambda\bar{Y})^{1/2}\, \mathrm{d}\bar{Y}. \tag{13}$$

By considering (A) in connection with the Re-dependent scaling of the intermediate layer, one readily infers from the linear behaviour of $\bar{T}$ given in (12a) that the $\bar{Y}$-dependent part of the streamwise velocity component varies with $\bar{Y}^{1/2}$ as $\bar{Y} \to 0$. This half-power law, known to hold on top of the viscous wall layer for a TBL on the verge of separation (for a further discussion see e.g. [6]) can be expressed as

$$\bar{L} \sim \chi(x)\, \bar{Y}, \quad \partial\bar{\Psi}/\partial\bar{Y} \sim \bar{U}_s(x) + [2/\chi(x)](\lambda\bar{Y})^{1/2}, \quad \bar{Y} \to 0. \tag{14}$$

The quantity $\chi(x)$ is regarded as part of a specific mixing length closure and, therefore, to be determined experimentally. As a result, the aforementioned overall slip velocity u_s takes on the form

$$u_s \sim U_s(x) + \alpha^{3/4}\, \bar{U}_s(x) + \cdots, \quad \bar{U}_s = -\int_0^{\infty} (1/\bar{L} - 1/L_0)(\lambda\bar{Y})^{1/2}\, \mathrm{d}\bar{Y} < 0. \tag{15}$$

It is anticipated in (15) that the integral exists. As will be shown in § 3., this is ensured by the behaviour of $\bar{L}$ for $\bar{Y} \to 0$ obtained from a match with the intermediate layer for high but finite values of Re. Then there is strong evidence that $0 \le \bar{L} < L_0$ for $0 \le \bar{Y} < \infty$, such that $\bar{U}_s < 0$.

Marginal separation in the triple-deck limit As a noteworthy cornerstone of the new asymptotic formulation of a TBL having a large streamwise velocity deficit, it is shown in [1] how triple-deck theory allows for a self-consistent asymptotic description of small reverse-flow regimes at the base of the outer wake layer: The locally weak interaction of the flow in that outermost layer, here denoted as main deck, with the induced potential flow in the so-called upper deck on top of the former is associated with the occurrence of small reverse-flow regimes at its base, in the so-called lower deck.

The analysis presented in [1] can be subsumed as follows: If the pressure gradient $\mathrm{d}p_e/\mathrm{d}x$ reaches a critical strength, described by a critical value β_c of the control parameter β introduced before, the solution of (7) exhibits the so-called marginal-separation singularity at some position, say, $x = x_c$, such that $U_s(x \neq x_c) > 0$ and $U_s(x_c) = 0$. Then an asymptotically correct treatment of (3) in the double limit (2) and $\beta \to \beta_c$ requires to generalise the boundary layer approximation (7) by taking into account turbulent/irrotational boundary layer interaction at a streamwise distance $\epsilon \propto -1/\ln|\beta - \beta_c|$ upstream of $x = x_c$ and over a small streamwise extent measured by $\sigma \propto (\alpha/\Lambda)^{3/5}$. Here the (positive) coupling parameter Λ characterises the triple-deck limit for it is proportional to $\alpha/\epsilon^{10/3}$ and kept fixed as $\alpha \to 0$, $\epsilon \to 0$, such that $\sigma = O(\epsilon^2)$. Then the lower deck is described by the following expansions by introducing rescaled variables

$$\psi/(L_{00}^{2/3}P_0) \sim \alpha\sigma^{5/6}\,\hat{\Psi}(X,\hat{Y}) + \cdots, \quad -\langle u'v'\rangle/(L_{00}^{2/3}P_0) \sim \alpha\sigma^{1/3}\,(\partial^2\hat{\Psi}/\partial\hat{Y}^2)^2 + \cdots,$$
$$p = p_e(x_c) + \sigma P(X;\Lambda) + O(\alpha), \quad X = (x - x_c + \epsilon)/\sigma, \quad \hat{Y} = Y/(L_0^{2/3}\sigma^{1/3});$$
$$L_{00} = L_0(x_c), \quad P_0 = (\mathrm{d}p_e/\mathrm{d}x)(x_c), \quad P(X;\Lambda) = P_0[X + \Lambda\,\hat{P}(X)]. \tag{16}$$

The numerical solutions of the triple-deck problem comprise both the canonical representations $\hat{\Psi}$ and $\hat{P}$ of the stream function and the induced pressure, respectively. Thus, the control parameter Λ accounts for the strength of the latter. As seen from (16), both the externally impressed and the superimposed induced pressure gradient are quantities of $O(1)$ in the triple-deck limit.

By taking into account the inner wake layer at the base of the lower deck, the expansion (15) for the slip velocity is locally replaced by

$$u_s = \sigma^{1/2}\,\hat{U}_s(X) + O(\alpha^{3/4}), \quad \hat{U}_s(X) = (\partial\hat{\Psi}/\partial\hat{Y})(X, \hat{Y} = 0). \tag{17}$$

Generally spoken, when $u_s = O(\alpha^{3/10})$ or even smaller the analysis of that flow region expressed by (9)–(15) remains valid if the formal substitutions

$$U_s \mapsto \sigma^{1/2}\,\hat{U}_s, \quad \lambda \mapsto \hat{U}_s\,\mathrm{d}\hat{U}_s/\mathrm{d}X + P_0 > 0, \quad p_e(x) \mapsto p_e(x_c) + P(X;\Lambda), \quad x \mapsto x_c \tag{18}$$

are applied. As indicated by (17), (15), and (14), a further breakdown of the asymptotic structure is encountered in the here most interesting case of triple-deck solutions which exhibit closed reverse-flow regimes such that $\hat{U}_s(X)$ changes sign at the positions of flow detachment and reattachment: For $u_s = O(\alpha^{3/4})$ a new inner wake region having a correspondingly smaller streamwise extent than the original one, and a sublayer closer to the wall where $y = O(u_s^2)$ when $u_s = o(\alpha^{3/4})$, have to be considered. However, as the inner wake layer analysis remains essentially unaltered there, it is disregarded. Both the asymptotic structure of the unperturbed TBL and its local splitting due to the interaction process in the limit (2) are sketched in Figure 1 (a).

3. Asymptotic structure of near-wall flow for high but finite Reynolds numbers

The analysis of the limit (2) outlined above serves as the starting point for the subsequent discussion of both the intermediate layer and the viscous wall region, which completes the flow description for large but finite values of Re and leads to the four-tiered TBL structure as depicted in Figure 1 (b) (here shown for flows sufficiently far from separation). As already stated in § 1., and in agreement with (A), in both layers the skin friction velocity

$$u_\tau = |\tau_w|^{1/2}, \quad \tau_w = Re^{-1}\,\partial u/\partial y \quad \text{at} \quad y = 0, \tag{19}$$

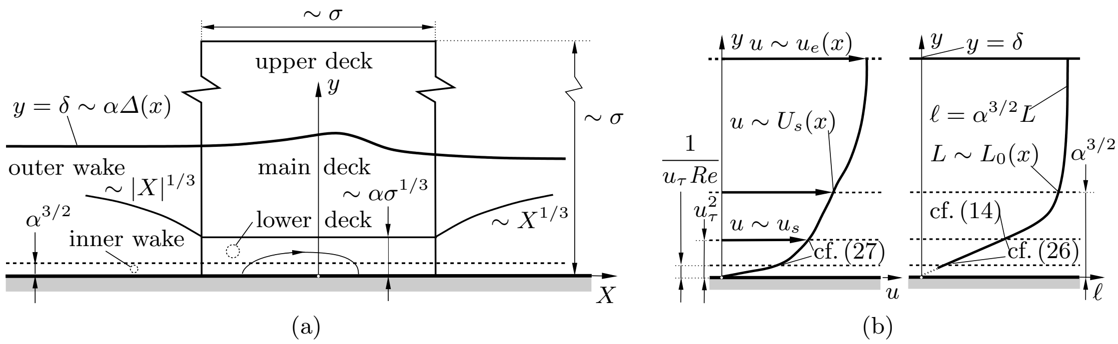

Figure 1: (a) Weakly interacting wake-type flow: separation bubble in lower deck indicated by detaching streamline, for onset/decay in local splitting for $|X| \to \infty$ see [1]; (b) four-tiered TBL.

serves as the appropriate velocity scale. We first briefly recall the essential properties of the viscous region close to the wall, which accounts for the no-slip condition expressed by (4) and where the Reynolds stresses are of the same magnitude as the viscous shear stress. A more comprehensive investigation of this flow regime is given in [3] and [2]. Let us also note a remarkable consequence of the hypothesis (A) revealed by the subsequent analysis, namely, that in the limit (1) the time-mean vorticity, asymptotically given by $\partial u/\partial y$, is found to be independent of Re outside the viscous wall layer.

We furthermore stress that the analysis presented in the following two paragraphs remains also valid in the triple-deck limit if the replacements (18) are taken into account.

Viscous wall layer There we expand in terms of the conventional scalings,

$$\frac{u}{u_\tau} \sim u^+(x,y^+) + \cdots, \quad -\frac{\langle u'v' \rangle}{u_\tau^2} \sim \tau^+(x,y^+) + \cdots, \quad p^+ = \frac{\mathrm{d}p_e/\mathrm{d}x}{u_\tau^3 Re}, \quad y^+ = y\, u_\tau Re, \tag{20}$$

In the equations of motion (3) the convective terms then are found to be negligibly small. Hence, they reduce to the balance of viscous and turbulent shear stress and the imposed (and induced) pressure gradient,

$$\partial u^+/\partial y^+ + \tau^+ \sim \operatorname{sgn}(\tau_w) + p^+ y^+, \quad Re \to \infty. \tag{21}$$

Note that $\partial u^+/\partial y^+ \to 0$ as $y^+ \to \infty$. Sufficiently far from separation (and also reattachment), the overall slip velocity u_s is a quantity of $O(1)$. Thus, in the intermediate layer the convective terms are linearised about $u \sim u_s$, and the Reynolds shear stress gradient approximately balances the sum of convection and the pressure gradient as expressed by the definition of λ given in (9) and entering also (10) and (18), such that the intermediate region has a thickness of $O(u_\tau^2)$. Consequently, in the limit $y^+ \to \infty$ the linear rise of τ^+ due to the pressure gradient is cancelled by the convective terms of higher order, see (21). As a result, in leading order the shear stress in the intermediate layer assumes the value of the wall shear stress τ_w in the overlap with the wall layer. This a posteriori justifies the choice of the velocity scale u_τ. Finally, matching of the quantity $y\, \partial u/\partial y$ then yields the logarithmic law of the wall,

$$u^+ \sim 1/A^+(x) \ln y^+ + B^+(x), \quad \operatorname{sgn}(A) = \operatorname{sgn}(\tau_w), \quad y^+ \to \infty. \tag{22}$$

Herein $A^+(x)$ and $B^+(x)$ are empirical functions. For $\tau_w > 0$, which characterises attached flows, and a perfectly smooth wall A^+ is the celebrated v. Kármán constant, commonly taken to be $\kappa \approx 0.421$, and $B^+ \approx 5.6$, cf. [2]. Note that for (mildly) separated flows having $\tau_w < 0$ an asymptotic behaviour akin to (22) was already proposed in [7] on semi-empirical grounds, together with explicit relationships for A^+ and B^+. Furthermore, the streamwise velocity components u in the intermediate and the wall layer match provided that the reduced skin friction velocity defined by $\gamma_s = u_\tau/u_s$ satisfies

$$\gamma_s = A^+(x)/\ln(u_\tau^3 Re)\,[1 + O(\ln^{-1} Re)], \tag{23}$$

cf. [4]. Here we note that $\mathrm{sgn}(\gamma_s) = \mathrm{sgn}(\tau_w)$. In (23) the higher-order terms abbreviated by the Landau symbol are seen to be closure-dependent. Inversion of the surface friction law (23) yields

$$\gamma_s/A^+(x) = \varepsilon[1 - 3\varepsilon \ln \varepsilon + O(\varepsilon)], \quad \mathrm{d}\gamma_s/\mathrm{d}x = O(\varepsilon^2), \quad \varepsilon = 1/\ln Re. \tag{24}$$

Then $p^+ = O(\ln^3 Re/Re)$, which suggests to drop the dependence on x of the leading-order quantities cited in (20), at least outside the separated region as indicated by the widely accepted empirical relationships for A^+ and B^+ given above.

Intermediate layer As pointed out in the preceeding paragraph, in the flow regime located on top of the viscous wall region the governing equations (3) enforce expansions of the form

$$\begin{aligned} &u/u_s(x;\alpha) \sim 1 + \gamma_s \hat{u}(x,\zeta) + \cdots, \quad -\langle u'v' \rangle/u_\tau^2 \sim 1 + \lambda(x)\,\zeta + \cdots, \\ &\ell \sim u_\tau^2\,\hat{l}(x,\zeta) + \cdots, \quad p = p_e(x) + O(\alpha), \quad \zeta = y/u_\tau^2, \end{aligned} \tag{25}$$

cf. [4]. Herein the new wall coordinate ζ is of $O(1)$. Then

$$\hat{l}\,\partial\hat{u}/\partial\zeta = [1 + \lambda(x)\,\zeta]^{1/2}, \quad \zeta \to 0: \ \hat{l} \sim |A^+(x)|\,\zeta, \quad \zeta \to \infty: \ \hat{l} \sim \chi(x)\,\zeta, \tag{26}$$

where the asymptotic behaviour of $\hat{l}$ in (26) reflects the relationships (22) and (14), respectively. As an important consequence, the match with the ambient layers reveals both the logarithmic as well as the half-power behaviour,

$$\zeta \to 0: \ \hat{u} = |A^+(x)|^{-1} \ln \zeta + O(1), \quad \zeta \to \infty: \ \hat{u} \sim 2/\chi(x)\,(\lambda(x)\,\zeta)^{1/2}. \tag{27}$$

Considering firmly attached flows, it is widely accepted to assume that the empirical function χ, introduced in (14), is independent of x and therefore equals the v. Kármán constant $\kappa = A^+$ (i.e. , if the flow is "locally in equilibrium", cf. [4] and [2]). In order to obtain a closed expression for the velocity variation $\hat{u}(\zeta)$ also for (mildly) separated flows, here we generalise this idea by setting $\hat{l}/\zeta = \chi(x) = |A^+(x)|$. Then integration of (26) gives rise to the so-called extended law of the wall,

$$\chi(x)\,\hat{u} = \ln \eta - 2\ln\big((1+\eta)^{1/2} + 1\big) + 2(1+\eta)^{1/2}, \quad \eta = \lambda\zeta. \tag{28a}$$

Without any loss of generality, the constant of integration in this relationship was chosen such that a contribution of $O(1)$ to its asymptotic expansion for $\eta \to \infty$ is discarded in order to avoid perturbations of the slip velocity $u_s(x,\alpha)$ arising from finite values of Re for the sake of simplicity. In turn,

$$\zeta \to 0: \quad \chi(x)\,\hat{u} = \ln \zeta + \ln(\lambda/4) + 2 + \lambda\zeta/2 + O(\zeta^2), \tag{28b}$$

$$\zeta \to \infty: \quad \chi(x)\,\hat{u} = 2(\lambda\zeta)^{1/2} - (\lambda\zeta)^{-1/2} + O(\zeta^{-3/2}). \tag{28c}$$

Note that the behaviour of $\hat{u}$ for $\zeta \to \infty$ in (27) or (28c) allows for a match of the streamwise velocity component in the inner wake region.

Gradual replacement of the logarithmic law in favour of the half-power law It is inferred from (25) and (24) that the expansion for the skin friction velocity u_τ given in (23) ceases to be valid if $\gamma_s = O(1)$. Stated equivalently, sufficiently close to locations of separation and reattachment, respectively, due to the breakdown of the expansion (17) mentioned at the end of § 2. both u_s and, as a consequence of (24), u_τ are quantities of $O(Re^{-1/3})$. Then the intermediate and the viscous wall layer merge at the base of the inner wake layer, such that in the resulting wall region having a thickness of $O(Re^{-2/3})$ the wall layer expansions (20) are expressed by adopting suitably redefined variables in the form

$$\begin{aligned} &u/u_p \sim u^\times(p^\times, y^\times) + \cdots, \quad -\langle u'v' \rangle/u_p^2 \sim \tau^\times(p^\times, y^\times) + \cdots, \\ &p^\times = \mathrm{sgn}(\tau_w)(u_p/u_\tau)^3, \quad u_p = (Re^{-1}\mathrm{d}P/\mathrm{d}X)^{1/3}, \quad y^\times = y\,u_p Re. \end{aligned} \tag{29}$$

Note that the function $P(X;\Lambda)$ is the overall pressure as defined in (16). Accordingly, the leading-order streamwise momentum balance (21) in that region is rewritten as

$$\partial u^\times/\partial y^\times + \tau^\times = (p^\times)^{-2/3} + y^\times. \tag{30}$$

Here the quantity $(p^\times)^{-2/3}$ denotes the rescaled wall shear stress which is generally considered to be of $O(1)$ but may change sign. In (30) the Reynolds shear stress $\tau^\times$ varies linearly due to the presence of the adverse pressure gradient $\mathrm{d}P/\mathrm{d}X$. Finally, by considering the asymptotic behaviour of the streamwise velocity for large values of $y^\times$, one recovers the square-root law (14) where any logarithmic variation in y is absent now,

$$u^\times \sim [2/\chi(x_c)](y^\times)^{1/2} + C^\times(p^\times), \quad B^\times \sim u_s/u_p = O(1), \quad y^\times \to \infty. \tag{31}$$

The function $C^\times(p^\times)$ has to be determined experimentally. Most important, the latter relationship in (31) follows from a match with the inner wake layer and represents the asymptotically correct continuation of the skin friction law (23) into the merged wall layer where flow reversal may take place.

4. Conclusions

In the present paper a fully self-consistent asymptotic theory of turbulent marginal separation, based on a minimum of physical assumptions regarding the nature of turbulence, is outlined in brief. Herein the non-dimensional boundary layer slenderness is measured by a small number, denoted by α, which is suggested to remain finite in the limit $Re \to \infty$ and is, therefore, taken as the principal perturbation parameter. The resulting two-tiered wake-type TBL flow is perturbed due to effects of high but finite values of Re. In particular, the rapid change of the structure of the near-wall flow as the skin friction changes sign is focussed upon. Open questions concern, amongst others, how to close the skin friction relationship $C^\times(p^+)$ in (31), which applies to a region of small streamwise extent up- and downstream of the positions of separation and reattachment, respectively, as $u_s = O(Re^{-1/3})$ there. Also, the fundamental assumption (A) requires further investigations, both experimentally and theoretically, i.e. on basis of the unsteady equations of motion.

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