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Grid Approximation of Parabolic Equations with Nonsmooth Initial Condition in the Presence of Boundary Layers of Different Types *

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1. Introduction

A Dirichlet problem for a singularly perturbed parabolic equation with a vector perturbation parameter $\overline{\varepsilon}$, $\overline{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, is considered on the semi-axis. The highest and first order derivatives with respect to x in the equation contain respectively the parameters ε_1 and ε_2 that take arbitrary values in the half-open interval $(0, 1]$ and in the interval $[-1, 1]$. Depending on the relationship between the parameters ε_1 and ε_2, the type of the equation may be reaction-diffusion or convection-diffusion. The first order derivative of the initial function has a jump discontinuity at some point x_0. For small values of the parameter ε_1, a boundary layer appears in a neighbourhood of the lateral boundary of the domain. The type of this layer depends on the relation between ε_1 and ε_2 and may be regular, parabolic, or hyperbolic (with their own characteristic length scales). In a neighbourhood of the set S^γ, that is, the characteristic of the reduced equation outgoing from the point $(x_0, 0)$, a parabolic interior (transient) layer arises.

Using the method of piecewise uniform meshes condensing in a neighbourhood of the boundary layer, we construct a special finite difference scheme that converges $\overline{\varepsilon}$-uniformly at the rate of the order 0.5. The method of additive splitting of a singularity (in the neighbourhood of the interior layer) applied on the basis of a domain decomposition method allows us to construct the improved scheme that converges $\overline{\varepsilon}$-uniformly with an order of convergence close to one.

2. Problem formulation. The aim for research

On the set $\overline{G}$, where

$$\overline{G} = G \cup S, \quad G = D \times (0, T], \quad D = (0, \infty), \tag{2.1}$$

we consider the boundary value problem for the singularly perturbed parabolic equation

$$L\, u(x,t) \equiv \left\{ \varepsilon_1 a(x,t) \frac{\partial^2}{\partial x^2} + \varepsilon_2 b(x,t) \frac{\partial}{\partial x} - c(x,t) - p(x,t) \frac{\partial}{\partial t} \right\} u(x,t) = f(x,t), \quad (x,t) \in G,$$
$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{2.2}$$

Here the parameters ε_1 and ε_2 are components of the vector parameter $\overline{\varepsilon}$ (or, in short, the parameter $\overline{\varepsilon}$) take arbitrary values in the half-open interval $(0, 1]$ and in the interval $[-1, 1]$, respectively. The coefficients $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$, and the right-hand side $f(x,t)$ are sufficiently smooth on $\overline{G}$, and also

$$a_0 \le a(x,t) \le a^0, \quad b_0 \le b(x,t) \le b^0, \quad c_0 \le c(x,t) \le c^0, \quad p_0 \le p(x,t) \le p^0, \tag{2.3a}$$

$$|\, f(x,t)\,| \le M, \quad (x,t) \in \overline{G}, \qquad a_0,\, b_0,\, p_0,\, c_0 > 0;$$

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the boundary function $\varphi(x,t)$ is continuous and bounded on S: *

$$|\,\varphi(x,t)\,| \le M, \quad (x,t) \in S, \tag{2.3b}$$

and it is sufficiently smooth on $\overline{S}^L$ and piecewise smooth on S_0. The derivative $(\partial/\partial x)\varphi(x,t)$ has a jump discontinuity (discontinuity of the first kind) on the set $S^{(*)} = \{(x,t)\colon\ x=d,\ t=0\}$. Here $S = S^L \cup S_0$, S_0 and S^L are the lower (for $t=0$) and lateral parts of the boundary S, $S_0 = \overline{S}_0$; $S^L = \Gamma \times (0,T]$, $\Gamma = \overline{D} \setminus D$; assume $S_0^* = S_0^l \cup S_0^r$, where $S_0^* = S_0 \setminus S^{(*)}$, $S_0^l = \{(x,t)\colon x \in [0,d),\ t=0\}$, $S_0^r = \{(x,t)\colon\ x \in (d,\infty),\ t=0\}$. For simplicity, compatibility conditions are assumed to be fulfilled on the set of "corner" points $S^c = \overline{S}^L \cap S_0$ (see, e.g., [6]), which ensure the necessary smoothness of the solution for each fixed $\overline{\varepsilon}$.

We are interested in finding an approximation to the solution $u(x,t)$, $(x,t) \in \overline{G}$. The derivative $(\partial/\partial x)u(x,t)$ is continuous on $\overline{G}^* = \overline{G} \setminus S^{(*)}$; for fixed $\overline{\varepsilon}$, it is bounded on $\overline{G}^*$ and discontinuous on the set $S^{(*)}$. Let us describe the behaviour of the solution and derivatives more precisely.

Let $S^\gamma = \{(x,t)\colon\ x=\gamma(t),\ (x,t) \in \overline{G}\}$, $x=\gamma(t)$, $t \ge 0$, be the characteristic of the reduced equation passing through the point $(d,\,0)$. For simplicity, we assume that the characteristic S^γ does not meet the boundary S^L. For $\varepsilon_1 \to 0$, boundary and interior layers with the typical length scales ε_0 and $\varepsilon_1^{1/2}$ appear in a neighbourhood of the sets S^L and S^γ, respectively, where

$$\varepsilon_0 = \varepsilon_0(\overline{\varepsilon}) = \begin{cases} \varepsilon_1^{1/2} & \text{for } \ |\varepsilon_2| \le \varepsilon_1^{1/2}, \\ \varepsilon_1\,\varepsilon_2^{-1} & \text{for } \ \varepsilon_2 > \varepsilon_1^{1/2}, \\ |\varepsilon_2| & \text{for } \ \varepsilon_2 < -\varepsilon_1^{1/2}; \end{cases} \tag{2.4}$$

$\varepsilon_1 \le \varepsilon_0 \le \varepsilon_1^{1/2} + |\varepsilon_2|$. The derivative $(\partial/\partial x)u(x,t)$ in the neighbourhood of the set S^L grows without limit as $\varepsilon_1 \to 0$ (see the second bound in (3.6) in Section 3). Note that the type of the boundary layer in the neighbourhood of S^L depends on the relation between the parameters ε_1 and ε_2. The boundary layer is regular for $\varepsilon_1^{1/2} \ll \varepsilon_2 \le 1$, parabolic for $|\varepsilon_2| \ll \varepsilon_1^{1/2}$, or hyperbolic for $\varepsilon_2 < 0$ and $\varepsilon_1^{1/2} \ll |\varepsilon_2| \ll 1$; no boundary layer appears for $\varepsilon_2 < 0$ and $|\varepsilon_2| \approx 1$. Unlike the boundary layer, the interior (parabolic) layer is weak (the first derivative of the interior-layer function in x is bounded $\overline{\varepsilon}$-uniformly; see bounds (3.4) in Section 3). In the nearest neighbourhood of $S^{(*)}$, we observe a discontinuity-singularity of the first derivative in x (see (3.4) for small values of ρ).

In the case of problem (2.2), (2.1) with sufficiently smooth initial data, for example, with $\left[\dfrac{\partial}{\partial x}\varphi(d,\,0)\right] = 0$, where $\left[\dfrac{\partial}{\partial x}\varphi(d,\,0)\right] = \dfrac{\partial}{\partial x}\varphi(d+0,\,0) - \dfrac{\partial}{\partial x}\varphi(d-0,\,0)$ is the jump of the derivative of the function $\varphi(x,t)$, the classical schemes do not converge $\overline{\varepsilon}$-uniformly in the neighbourhood of the set S^L. If the derivative of the initial function is discontinuous, then the rate of convergence of a finite difference scheme based on standard approximations of the problem with a scalar parameter ε is not higher than 0.5 (see, e.g., [4]).

Our aim is to construct a difference scheme for problem (2.2), (2.1) that converges $\overline{\varepsilon}$-uniformly with an order of convergence close to one.

Note that for a model heat transfer problem in the case of flow past a flat plate with suction of the boundary layer [1], we have $\varepsilon_1 = Pe^{-1}$ and $\varepsilon_2 = Re^{-1/2} + v_0$, where Re and Pe are the Reynolds and Péclet numbers, and $v_0 \ge 0$ is the intensity of the suction. The temperature of the flowing fluid is piecewise smooth in some cross-section of the flow, and the first derivative of the temperature in this cross-section has a discontinuity at some distance from the plate.

* Here and below we denote by M, M_i (by m) sufficiently large (small) positive constants that are independent of $\overline{\varepsilon}$ and the discretization parameters. The notation $L_{(j.k)}$ $(M_{(j.k)},\ G_{h(j.k)})$ means that this operator (constant, set) is defined in formula $(j.k)$.

3. A priori bounds on the solution of the boundary value problem

Let us present some bounds on the solution of problem (2.2) and its derivatives. These bounds are derived using the technique from [2, 7, 8]. The functions $f(x,t)$ and $\varphi(x,t)$ satisfying (2.3) are assumed to be sufficiently smooth on the sets $\overline{G}$ and $\overline{S}^{\,L}$, $\overline{S}_0^{\,l}$, $\overline{S}_0^{\,r}$, respectively.

We represent the domain $\overline{G}$ as the sum of overlapping sets

$$\overline{G} = \bigcup_j \overline{G}^{\,j}, \quad j = 1,\, 2,\, 3, \tag{3.1}$$

where

$$\begin{aligned}
G^1 &= G^1(m^1) = \{(x,t):\ |x-\gamma(t)| < m^1,\ \ t\in(0,T]\},\\
G^2 &= G^2(m^2) = \{(x,t):\ x\in(0, 0+m^2),\ \ t\in(0,T]\},\\
G^3 &= G^3(m^3) = G\setminus\left\{G^1(m^3)\bigcup G^2(m^3)\right\}, \quad m^3 < m^1,\, m^2,
\end{aligned}$$

G^1 and G^2 are the vicinities of the interior and the boundary layers, respectively; let $\overline{G}^{\,1}\cap\overline{G}^{\,2} = \emptyset$.

Using the results obtained in [2, 7, 8], we find the bound

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, u(x,t)\right| \le M, \quad (x,t)\in\overline{G}^{\,3}, \quad k+2k_0 \le K; \tag{3.2}$$

the value of K is determined by the data of the problem.

The function $u(x,t)$, $(x,t)\in\overline{G}^{\,1}$, can be represented as the sum of functions

$$u(x,t) = U^{\,1}(x,t) + W^{\,1}(x,t), \quad (x,t)\in\overline{G}^{\,1}, \tag{3.3}$$

where $U^{\,1}(x,t)$ and $W^{\,1}(x,t)$ are the regular and singular components of the solution; namely, $W^{\,1}(x,t)$ is the transient layer.

For the components in representation (3.3), we have the bounds

$$\begin{aligned}
&\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U^1(x,t)\right| \le M\left[1+\varepsilon_1^{(i+1-k-k_0)/2}\,\rho^{\,i+1-k-k_0} + \varepsilon_1^{(i+1-k)/2}\,\rho^{\,i+1-k-2k_0}\right],\ (x,t)\in\overline{G}^{\,1},\\
&\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, W^1(x,t)\right| \le M\left[1+\varepsilon_1^{(1-k-k_0)/2}\,\rho^{\,1-k-k_0} + \varepsilon_1^{(1-k)/2}\,\rho^{\,1-k-2k_0}\right], \quad (x,t)\in\overline{G};\\
&\left|\frac{\partial^{k+k_0}}{\partial \xi^k \partial t^{k_0}}\widehat{U}^1(\xi,t)\right| \le M\left[1+\varepsilon_1^{(i+1-k)/2}\,\widehat{\rho}^{\ i+1-k-2\,k_0}\right], \quad (\xi,t)\in\overline{\widehat{G}}^{\,1},\\
&\left|\frac{\partial^{k+k_0}}{\partial \xi^k \partial t^{k_0}}\widehat{W}^1(\xi,t)\right| \le M\left[1+\varepsilon_1^{(1-k)/2}\,\widehat{\rho}^{\ 1-k-2\,k_0}\right]\exp(-m\,\varepsilon_1^{-1/2}\,|\xi|\,), \quad (\xi,t)\in\overline{\widehat{G}},\\
&\qquad\qquad k+2k_0 \le K, \quad i = 1,\, 2,
\end{aligned} \tag{3.4}$$

where $\xi = x-\gamma(t)$, $\rho = \rho(x,t;\,\varepsilon_1) = \varepsilon_1^{-1/2}\,\bigl|x-\gamma(t)\bigr| + t^{1/2}$, $\widehat{\rho} = \widehat{\rho}(\xi,t;\,\varepsilon_1) = \varepsilon_1^{-1/2}\,\bigl|\xi\bigr| + t^{1/2}$, and m is an arbitrary number.

We represent the solution of the boundary value problem (2.2), (2.1) on the set $\overline{G}^{\,2}$ for all admissible $\overline{\varepsilon}$ as the sum of functions

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t)\in\overline{G}^{\,2}, \tag{3.5}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular components of the solution; namely, $V(x,t)$ is the boundary layer.

The components in representation (3.5) satisfy the bounds written in compact form

$$\begin{aligned}
&\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x,t)\right| \le M,\\
&\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} V(x,t)\right| \le M\lambda^{-k}\exp(-m_1\lambda^{-1}x), \quad (x,t)\in\overline{G}^{\,2},\\
&\qquad k+k_0 \le K, \quad k_0 \le K_0, \quad K = 3, \quad K_0 = 2.
\end{aligned} \tag{3.6}$$

Here $\lambda = \lambda(\varepsilon_1, \varepsilon_2)$, $\lambda = \lambda_1 \equiv \varepsilon_1(\varepsilon_1 + m^{(1)}\varepsilon_2^2)^{-1/2}$ for $\varepsilon_2 \geq 0$, and $\lambda = \lambda_2 \equiv 2(\varepsilon_1 + M^{(1)}\varepsilon_2^2)^{1/2}$ for $\varepsilon_2 < 0$, $m^{(1)} = 4^{-1} \min_{\overline{G}} \left[a^{-1}(x,t)\, b^2(x,t)\, c^{-1}(x,t)\right]$, $M^{(1)} = 4^{-1} \max_{\overline{G}} \left[a^{-1}(x,t)\, b^2(x,t)\, c^{-1}(x,t)\right]$, m_1 is an arbitrary number in the interval $(0, m_1^0)$, $m_1^0 = \min_{\overline{G}} \left[a^{-1/2}(x,t)\, c^{1/2}(x,t)\right]$.

To derive the *a priori* bounds, we assumed that the data of the boundary value problem satisfy the condition

$$a,\ b,\ c,\ p,\ f \in C^{l_1+\alpha,\, l_0+\alpha}(\overline{G}),\ \varphi \in C^{l_0+\alpha}(\overline{S}^L) \cap C^{l_1+\alpha}(\overline{S}_0^l) \cap C^{l_1+\alpha}(\overline{S}_0^r),\ l_0 \geq 4,\ l_1 \geq 7,\ \alpha > 0, \tag{3.7}$$

moreover, compatibility conditions [6] are satisfied on the set S^c to guarantee the inclusion

$$u \in C^{3+\alpha,\, 2+\alpha}(\overline{G}^*). \tag{3.8}$$

Theorem 3.1 *Let the data of the boundary value problem* (2.2), (2.1) *satisfy conditions* (2.3) *and* (3.7)*, and let the solution satisfy condition* (3.8)*. Then the solution $u(x,t)$ and its components in representations* (3.3) *and* (3.5) *satisfy bounds* (3.2)*,* (3.4)*, and* (3.5)*; in* (3.2) *and* (3.4)*, $K = 4$.*

4. Classical difference scheme

For problem (2.2), (2.1), we consider a classical finite difference scheme and give conditions for its convergence. On the set $\overline{G}$, we define the grid

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{4.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are meshes on the set $\overline{D}$ and on $[0, T]$, respectively; $\overline{\omega}$ and $\overline{\omega}_0$ are meshes with arbitrarily distributed mesh points satisfying only the condition $h \leq MN^{-1}$, $h_t \leq MN_0^{-1}$, where $h = \max_i h^i$, $h^i = x^{i+1} - x^i$, $x^i, x^{i+1} \in \overline{\omega}$, $h_t = \max_j h_t^j$, $h_t^j = t^{j+1} - t^j$, $t^j, t^{j+1} \in \overline{\omega}_0$. Here $N + 1$ and $N_0 + 1$ are the minimal number of points on a unit interval in the mesh on $\overline{D}$ and the number of points in the mesh $\overline{\omega}_0$, respectively. Also, it is of interest to consider schemes on the simplest grids

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{4.2}$$

which are uniform with respect to both x and t, with the step-sizes $h = N^{-1}$ and $h_t = TN_0^{-1}$. Problem (2.2), (2.1) is approximated by the implicit finite difference scheme [9]

$$\begin{aligned} \Lambda z(x,t) \equiv \left\{\varepsilon_1 a(x,t)\delta_{\overline{x}\widehat{x}} + \varepsilon_2^+ b(x,t)\delta_x + \varepsilon_2^- b(x,t)\delta_{\overline{x}} - c(x,t) - p(x,t)\delta_{\overline{t}}\right\} z(x,t) = \\ = f(x,t), \quad (x,t) \in G_h, \\ z(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{4.3}$$

Here $\delta_{\overline{x}\widehat{x}}\, z(x,t)$ and $\delta_x\, z(x,t)$, $\delta_{\overline{x}}\, z(x,t)$, $\delta_{\overline{t}}\, z(x,t)$ are the second and the first difference derivatives; $\delta_{\overline{x}\widehat{x}}\, z(x,t) = 2\left(h^i + h^{i-1}\right)^{-1} \{\delta_x - \delta_{\overline{x}}\}\, z(x,t)$, $x = x^i$; $\varepsilon_2^+ = 2^{-1}(\varepsilon_2 + |\varepsilon_2|)$, $\varepsilon_2^- = 2^{-1}(\varepsilon_2 - |\varepsilon_2|)$.

The maximum principle holds for the difference scheme (4.3), (4.1) [9].

Taking into account the *a priori* bounds on the solution of problem (2.2), (2.1), we find the following error estimate for solutions of the difference scheme (4.3), (4.2):

$$\begin{aligned} &|\, u(x,t) - z(x,t)\,| \leq \\ &\leq M \left\{ \begin{array}{ll} \left[(\varepsilon_1^{1/2} + N^{-1})^{-1}N^{-1} + N^{-1/2} + N_0^{-1/2}\right] & \text{for } |\varepsilon_2| \leq M_0\, \varepsilon_1^{1/2}, \\ \left[(\varepsilon_1\varepsilon_2^{-1} + N^{-1})^{-1}N^{-1} + N^{-1/2} + N_0^{-1/2}\right] & \text{for } \varepsilon_2 > M_0\varepsilon_1^{1/2}, \\ \left[(|\varepsilon_2| + N^{-1})^{-1}N^{-1} + N^{-1/2} + N_0^{-1/2}\right] & \text{for } \varepsilon_2 < -M_0\varepsilon_1^{1/2} \end{array} \right\}, \quad (x,t) \in \overline{G}_h, \end{aligned} \tag{4.4}$$

where M_0 is an arbitrary (sufficiently large) constant.

Thus, the condition ($h \ll \varepsilon_0$)

$$\varepsilon_0^{-1} = o(N), \quad \varepsilon_0 = \varepsilon_{0(2.4)}(\overline{\varepsilon}), \quad N \to \infty \tag{4.5}$$

is necessary and sufficient for scheme (4.3), (4.2) to be convergent; scheme (4.3), (4.2) does not converge $\overline{\varepsilon}$-uniformly.

Theorem 4.1 *Let the solution of the boundary value problem* (2.2), (2.1) *and its components in representations* (3.3), (3.5) *satisfy the a priori bounds of Theorem* 3.1. *Then, in the case of the difference scheme* (4.3) *on the mesh* (4.2), *condition* (4.5) *is necessary and sufficient for its convergence. The numerical solutions satisfy estimate* (4.4).

Remark 1 If the condition $|\varepsilon_2| = \mathcal{O}\big(\varepsilon_1^{1/2}\big)$ holds, the solution of the problem has a parabolic-layer-type singularity. Using the technique given in [2, 3, 10], it is possible to show that, under this condition, there are no schemes based on the fitted operator method that converge $\overline{\varepsilon}$-uniformly.

5. Special finite difference scheme

To construct an $\overline{\varepsilon}$-uniformly convergent scheme, we use meshes condensing in a neighbourhood of the boundary layer.

On the set $\overline{G}$, we introduce the grid

$$\overline{G}_h = \overline{\omega}^* \times \overline{\omega}_0, \tag{5.1a}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(4.2)}$, and $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a piecewise uniform mesh on $\overline{D}$. The mesh sizes of $\overline{\omega}^*$ are constant on the intervals $[0,\sigma]$ and $[\sigma,\infty)$ and are equal to $h^{(1)} = 2\,\sigma\,N^{-1}$ and $h^{(2)} = 2(1-\sigma)N^{-1}$, respectively. The value σ, $\sigma = \sigma(\varepsilon_1,\varepsilon_2,N)$, is chosen so as to satisfy the condition

$$\sigma = \begin{cases} \min\,[2^{-1},\ m_1^{-1}\,\varepsilon_1^{1/2}\,\ln N] & \text{for} \quad |\varepsilon_2| \le M^0\,\varepsilon_1^{1/2}, \\ \min\,[2^{-1},\ m_2^{-1}\,\varepsilon_1\,\varepsilon_2^{-1}\,\ln N] & \text{for} \quad \varepsilon_2 > M^0\,\varepsilon_1^{1/2}, \\ \min\,[2^{-1},\ m_3^{-1}\,|\varepsilon_2|\,\ln N] & \text{for} \quad \varepsilon_2 < -M^0\,\varepsilon_1^{1/2}. \end{cases} \tag{5.1b}$$

Here M^0 is an arbitrary constant, m_1 is an arbitrary number from the interval $(0,m_1^0)$, $m_1^0 = m_1^0\left(M^0\right) = \min\left\{2^{-1}(M^0)^{-1}\min\limits_{\overline{G}}\left[b^{-1}(x,t)\,c(x,t)\right],\ 2^{-1/2}\min\limits_{\overline{G}}{}^{1/2}\left[a^{-1}(x,t)\,c(x,t)\right]\right\}$; m_2 is an arbitrary constant from $(0,m_0^2)$, $m_0^2 = \min\limits_{\overline{G}}\left[a^{-1}(x,t)\,b(x,t)\right]$; m_3 is an arbitrary number from $(0,m_3^0)$, $m_3^0 = m_3^0\left(M^0\right) = \min\left\{2^{-1/2}\,M^0\,\min\limits_{\overline{G}}{}^{1/2}\left[a^{-1}(x,t)c(x,t)\right],\ 2^{-1}\min\limits_{\overline{G}}\left[b^{-1}(x,t)c(x,t)\right]\right\}$.

On the grid $\overline{G}_{h(5.1)}$ constructed in such a way, the discrete solution approximates the singular component $V(x,t)$ $\overline{\varepsilon}$-uniformly with the accuracy $\mathcal{O}\big(N^{-1}\,\ln N + N_0^{-1}\big)$.

Using the majorant function technique from [2, 3] and taking into account the *a priori* bounds on the solution of problem (2.2), (2.1), we obtain the error estimate for the solutions of scheme (4.3), (5.1):

$$|\,u(x,t) - z(x,t)\,| \le M\big[\,N^{-1/2} + N_0^{-1/2}\,\big], \quad (x,t) \in \overline{G}_h. \tag{5.2}$$

The estimate is unimprovable with respect to N and N_0 for $\varepsilon_1 = \varepsilon_2 = 1$.

Thus, scheme (4.3), (5.1) converges $\overline{\varepsilon}$-uniformly.

Provided that

$$\left[\frac{\partial}{\partial x}\,\varphi(x,t)\right] = 0, \quad (x,t) \in S^{(*)}, \tag{5.3}$$

we obtain the error estimate

$$
|\, u(x,t) - z(x,t)\,| \le M N^{-1} \left\{ \begin{array}{ll} \min\,[\varepsilon_1^{-1/2},\, \ln N] & \text{for} \ \ |\varepsilon_2| \le M^0\, \varepsilon_1^{1/2}, \\ \min\,[\varepsilon_1^{-1}\varepsilon_2,\, \ln N] & \text{for} \ \ \varepsilon_2 > M^0 \varepsilon_1^{1/2}, \\ \min\,[|\varepsilon_2|^{-1},\, \ln N] & \text{for} \ \ \varepsilon_2 < -M^0 \varepsilon_1^{1/2} \end{array} \right\} + M N_0^{-1+\alpha_0}, \qquad (5.4)
$$

$$
(x,t) \in \overline{G}_h,
$$

where α_0 is an arbitrary constant from the interval (0,1). We also have the $\overline{\varepsilon}$-uniform estimate

$$
|\, u(x,t) - z(x,t)\,| \ \le \ M \left[N^{-1} \ln N + N_0^{-1+\alpha_0} \right], \quad (x,t) \in \overline{G}_h. \qquad (5.5)
$$

Theorem 5.1 *Let the assumptions of Theorem* 4.1 *hold. Then the solution of the difference scheme* (4.3), (5.1) *converges* $\overline{\varepsilon}$*-uniformly. The numerical solutions satisfy estimate* (5.2) *and, under condition* (5.3), *estimates* (5.4) *and* (5.5).

6. Domain decomposition method for problem (2.2), (2.1)

We now construct a scheme based on the domain decomposition method for problem (2.2), (2.1), where the set $\overline{G}$ is decomposed into subdomains containing no more than a single singularity.

6.1. First, we present a continual domain decomposition method. We represent the set $\overline{G}$ as the sum of overlapping subsets

$$
\overline{G} = \bigcup\nolimits_k \overline{G}^{(k)}, \quad k = 1,2,3, \qquad (6.1)
$$

where $\overline{G}^{(k)} = G^{(k)} \cup S^{(k)}$, $G^{(k)} = D^{(k)} \times (0,T]$, $k = 1,2,3$, $D^{(i)} = \left(d^{iL},\, d^{iR}\right)$, $i = 1,2$, and $D^{(3)} = (d^{3L},\, \infty)$; the set $\overline{G}^{(2)}$ contains the set S^γ together with its $m^{(2)}$-neighbourhood, and also S^γ has no common points with the m^2-neighbourhood of the set $\overline{G}^{(1)} \cup \overline{G}^{(3)}$. By δ, we denote the minimal overlap of the subdomains from (6.1); generally, δ may depend on the parameter $\overline{\varepsilon}$.

Let us describe the modified Schwarz method for problem (2.2), (2.1) (see, e.g., [5]). Let $\overline{\omega}_0$ be a uniform mesh on [0,T] with step-size τ; $\overline{\omega}_0 = \overline{\omega}_{0(4.2)}$. By $G(t_1)$, we denote the strip

$$
G(t_1) = G \ \cap \ \{\, t_1 < t \le t_1 + \tau \,\}, \quad t_1,\, t_1 + \tau \in \overline{\omega}_0.
$$

Assume $\overline{G}(t_1) = G(t_1) \cup S(t_1)$, and let the function $v(x,t) = v(x,t;t_1)$ be defined on $S(t_1)$. We denote the extension of the function $v(x,t)$ onto the whole set $\overline{G}(t_1)$ by $\overline{v}(x,t;t_1)$. The function $\overline{v}(x,t;t_1)$ is assumed to satisfy the Lipschitz condition with respect to t. We subdivide the strip $G(t_1)$ into the sections $\overline{G}^{(k)}(t_1) = G^{(k)}(t_1) \cup S^{(k)}(t_1)$, where $G^{(k)}(t_1) = G^{(k)} \cap G(t_1)$, $k = 1,2,3$.

Suppose that the function $u(x,t)$, $(x,t) \in \overline{G}$, for $t^n \in \overline{\omega}_0$, $t \le t^n < T$, $n = 0,1,\ldots, N_0 - 1$, has already been constructed. We find the function $u(x,t)$ for $t \le t^{n+1}$, constructing $u(x,t)$ on the strip $\overline{G}(t^n)$ in the following way. We first find the functions $u^{(k)}(x,t)$ on the sets $\overline{G}^{(k)}(t^n)$ by solving the boundary value problems

$$
\left.
\begin{array}{ll}
L_{(6.2)}(u^{(k)}(x,t)) = 0, & (x,t) \in G^{(k)}(t^n), \\
u^{(k)}(x,t) = \left\{ \begin{array}{ll} \overline{u}(x,t;t^n), & k = 1, \\ u^{(k-1)}(x,t), & k \ge 2 \end{array} \right\}, & (x,t) \in S^{(k)}(t^n)
\end{array}
\right\} \ \text{for} \ \ (x,t) \in \overline{G}^{(k)}(t^n), \qquad (6.2\text{a})
$$

$$
k = 1,2,3; \quad t^n \in \overline{\omega}_0, \quad n \le N_0 - 1.
$$

Here $L_{(6.2)}(u(x,t)) \equiv L_{(2.2)}u(x,t) - f(x,t)$, $(x,t) \in G$. We find the functions $u^{(k)}(x,t)$ on $\overline{G}^{(k)}(t^n)$; further, we extend these functions for each k onto the whole strip $\overline{G}(t^n)$ as follows:

$$
u^{(k)}(x,t) = \left\{
\begin{array}{ll}
u^{(k)}(x,t), & (x,t) \in \overline{G}^{(k)}(t^n), \\
\left. \begin{array}{ll} \overline{u}(x,t;t^n), & k = 1, \\ u^{(k-1)}(x,t), & k \ge 2 \end{array} \right\}, & (x,t) \in \overline{G}(t^n) \setminus \overline{G}^{(k)}(t^n)
\end{array}
\right\} \ \text{for} \ (x,t) \in \overline{G}(t^n), \qquad (6.2\text{b})
$$

$$
k = 1,2,3, \quad t^n \in \overline{\omega}_0.
$$

Finding the function $u^{(k)}(x,t)$, $(x,t)\in\overline{G}(t^n)$ for $k=3$, we define the function $u(x,t)$ on the whole strip $\overline{G}(t^n)$ so that

$$u(x,t)=u^{(3)}(x,t),\quad (x,t)\in\overline{G}(t^n),\quad t^n\in\overline{\omega}_0. \tag{6.2c}$$

Thus, the function $u(x,t)$ has been constructed on the domain $\overline{G}$ for $t\in[0,t^{n+1}]$.

In the relations (6.2a) and (6.2b), $\overline{u}(x,t;t^n)=\overline{v}(x,t;t^n)$, $(x,t)\in\overline{G}(t^n)$. The function $v(x,t;t^n)$ is determined by

$$v(x,t;t^n)=\begin{cases}\varphi(x,t),\quad (x,t)\in S(t^n), & t^n=t^0=0,\\ \left.\begin{array}{ll}\varphi(x,t), & (x,t)\in S(t^n)\cap S,\quad t\ge t^n,\\ u(x,t), & (x,t)\in S(t^n)\setminus S,\quad t=t^n\end{array}\right\}, & t^n>0,\quad (x,t)\in S(t^n),\end{cases} \tag{6.2d}$$
$$n=0,1,\ldots,N_0-1.$$

Problem (6.2), (6.1) is the continual modified Schwarz method that corresponds to problem (2.2), (2.1). Note that the method does not iterate in the strict sense. The boundary value problems are solved only once at those points of $\overline{G}$ that do not belong to the intersection of the subdomains $\overline{G}^{(k)}$, and they are solved twice only on the intersection of these subdomains.

Under the condition

$$\delta=\delta_{(6.1)}(\overline{\varepsilon})>0,\quad \inf_{\overline{\varepsilon}}\left[\varepsilon_0^{-1}\delta_{(6.1)}(\overline{\varepsilon})\right]>0, \tag{6.3}$$

equivalent to the condition $\delta=\delta_{(6.1)}(\overline{\varepsilon})\ge m_{(6.3)}\varepsilon_0$, where $\varepsilon_0=\varepsilon_{0(2.4)}(\overline{\varepsilon})$, the function $u_{(6.2)}(x,t)$, i.e., the solution of problem (6.2), (6.1), converges $\overline{\varepsilon}$-uniformly as $N_0\to\infty$:

$$|\,u(x,t)-u_{(6.2)}(x,t)\,|\le MN_0^{-1},\quad (x,t)\in\overline{G}.$$

If condition (6.3) is violated, then the function $u_{(6.2)}(x,t)$ does not converge $\overline{\varepsilon}$-uniformly.

The following theorem similar to that given in [5, 11] is valid.

Theorem 6.1 *Condition* (6.3) *is necessary and sufficient for the $\overline{\varepsilon}$-uniform convergence (as $N_0\to\infty$) of the solution $u_{(6.2)}(x,t)$ of problem* (6.2), (6.1) *to the solution of problem* (2.2), (2.1).

6.2. On the set $\overline{G}^{(2)}(t^n)$, the function $u^{(2)}(x,t)$, i.e., the solution of the subproblem in (6.2a)

$$\begin{aligned}L_{(6.2)}\big(u^{(2)}(x,t)\big)&=0, && (x,t)\in G^{(2)}(t^n),\\ u^{(2)}(x,t)&=u^{(1)}(x,t), && (x,t)\in S^{(2)}(t^n),\end{aligned} \tag{6.4a}$$

can be represented as the sum of functions

$$u^{(2)}(x,t)=u_1^{(2)}(x,t)+u_2^{(2)}(x,t),\quad (x,t)\in\overline{G}^{(2)}(t^n), \tag{6.4b}$$

where $u_1^{(2)}(x,t)$ and $u_2^{(2)}(x,t)$ are the regular and singular (its main term) components of the solution of problem (6.4a). Here

$$u_2^{(2)}(x,t)=2^{-1}\left[\frac{\partial}{\partial x}\varphi(d,0)\right]\Big\{(x-\gamma(t))\,v\big(2^{-1}\varepsilon_1^{-1/2}(x-\gamma(t))\,\vartheta^{-1/2}(t)\big)+$$
$$+2\,\pi^{-1/2}\,\varepsilon_1^{1/2}\,\vartheta^{1/2}(t)\exp\big(-4^{-1}\varepsilon_1^{-1}(x-\gamma(t))^2\,\vartheta^{-1}(t)\big)\Big\}\exp\big(-\alpha(t)\big),\quad (x,t)\in\overline{G}^{(2)}, \tag{6.5}$$

where

$$v(\xi)=\operatorname{erf}(\xi)=2\,\pi^{-1/2}\int_0^{\xi}\exp(-\alpha^2)\,d\alpha,\quad \xi\in I\!R,$$

$$\vartheta(t)=\int_0^t a(\gamma(t_1),t_1)\,p^{-1}(\gamma(t_1),t_1)\,dt_1,\quad \alpha(t)=\int_0^t c(\gamma(t_1),t_1)\,p^{-1}(\gamma(t_1),t_1)\,dt_1.$$

The function $u_1^{(2)}(x,t)$ is the solution of the problem

$$
\begin{aligned}
L_{(6.4)}\big(u_1^{(2)}(x,t)\big) &= 0, && (x,t)\in G^{(2)}(t^n),\\
u_1^{(2)}(x,t) &= u^{(1)}(x,t)-u_2^{(2)}(x,t), && (x,t)\in S^{(2)}(t^n),
\end{aligned}
\tag{6.4c}
$$

where
$$
\begin{aligned}
L_{(6.4)}\big(u_1^{(2)}(x,t)\big) &\equiv L_{(2.2)}u_1^{(2)}(x,t)-f^{(2)}(x,t), && (x,t)\in G^{(2)}(t^n),\\
f^{(2)}(x,t) &\equiv f(x,t)-L_{(2.2)}u_2^{(2)}(x,t), && (x,t)\in \overline{G}^{(2)}\setminus S^{(*)}.
\end{aligned}
$$

The domain decomposition method (6.2), (6.4), (6.1), in which the solution $u^{(2)}(x,t)$ of subproblem (6.4a) is found via the function $u_1^{(2)}(x,t)$ from representation (6.4b), i.e., the solution of subproblem (6.4c), is the domain decomposition method with the additive splitting of the interior-layer-type singularity (its main term) in the solution on the subdomain $\overline{G}^{(2)}$ (or, in short, the domain decomposition method with the additive splitting of the singularity).

The discretization of problem (6.2), (6.4), (6.1), similar to that made in [5, 11], leads to a difference scheme of the decomposition method with the additive splitting of the singularity that converges $\overline{\varepsilon}$-uniformly (as $N, N_0 \to \infty$) at the rate $\mathcal{O}\big(N^{-1}\ln N + N_0^{-1+\alpha_0}\big)$, where $\alpha_0 = \alpha_{0(5.4)}$, i.e., with the order of convergence close to one.

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