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A posteriori Adapted Meshes in the Approximation of Singularly Perturbed Quasilinear Parabolic Convection-diffusion Equations *

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1. Introduction

A Dirichlet problem on a interval for a quasilinear parabolic convection-diffusion equation with a small parameter ε multiplying the highest derivative is considered. For this problem, a solution of the classical finite difference scheme on a uniform mesh converges only under the condition $h \ll \varepsilon$, where h is the step-size of the space mesh; moreover, the order of convergence in x is $\mathcal{O}\left(\varepsilon N^{-1}\right)$, where $N+1$ is the number of nodes in the uniform mesh with respect to x.

Methods for the construction of ε-uniformly convergent schemes on special meshes, *a priori* adapted in boundary and transient layers (see, e.g., [1]–[5]) are sufficiently well developed. It is of interest another approach to construct adaptive methods, where finite difference schemes on *a posteriori* condensing meshes are applied; see, e.g., [5, 6]. In these monographs, finite difference schemes were considered on meshes that are refined by some a way in subdomains in which computed solutions turn out to be unsatisfactory accurate. In a number of methods, the process of the local mesh refinement is defined by the *gradients of solutions* to intermediate grid problems [7]–[9]. However, such schemes for nonlinear equations earlier were not considered.

In the present paper for the boundary value problem, using nonlinear and linearized base schemes, finite difference schemes on *a posteriori adapted meshes* are constructed. Subdomains, where a refinement of solutions is required, are defined by the gradient of the grid solution; the improvement of the solutions is performed only locally. Under this process, *uniform meshes* are used; in the adaptation procedure *piecewise uniform meshes* are generated that condense in a neighbourhood of the boundary layer. On the adapted meshes based on the *solution gradient*, rather simple finite difference schemes are constructed for which errors of solutions weakly depend on the parameter ε. The scheme on a posteriori adapted meshes converges *"almost ε-uniformly"*, i.e. under the condition $N^{-1} \ll \varepsilon^{\nu}$, the value ν can be chosen arbitrary from the half-open interval $(0,1]$.

2. Problem formulation. The aim of research

On the set $\overline{G}$

$$\overline{G} = G \bigcup S, \quad G = D \times (0,T], \tag{2.1}$$

where $D=(0,d)$, we consider the boundary value problem for the quasilinear singularly perturbed parabolic equation

$$L\left(u(x,t)\right) \equiv L^2\, u(x,t) - f\big(x,t,\, u(x,t)\big) = 0, \qquad (x,t) \in G, \tag{2.2}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S.$$

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Here

$$L^2 = \varepsilon\, a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} - c(x,t) - p(x,t)\frac{\partial}{\partial t}, \quad (x,t) \in G,$$

the functions $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t,\,u)$ and $\varphi(x,t)$ are assumed to be sufficiently smooth on $\overline{G}$, $\overline{G}\times R$ and S respectively, moreover,[1]

$$a_0 \le a(x,t) \le a^0, \quad b_0 \le b(x,t) \le b^0, \quad |c(x,t)| \le c^0, \quad p_0 \le p(x,t) \le p^0, \quad (x,t) \in \overline{G}; \tag{2.3}$$

$$|f(x,t,\,u)| \le M, \;\; c_1 \le c(x,t) + \frac{\partial}{\partial u} f(x,t,\,u) \le c^1, \;\; (x,t,\,u) \in \overline{G}\times R; \;\; |\varphi(x,t)| \le M, \;\; x \in S;$$

$a_0,\, b_0,\, c_1,\, p_0 > 0$; the parameter ε takes arbitrary values in the half-open interval $(0,1]$.

For small values of the parameter ε, a regular boundary layer appears in a neighbourhood of the set $S_1^L = \{(x,t):\; x=0,\; 0<t\le T\}$. Here S_1^L and S_2^L are the left and right parts of the lateral boundary; $S = S^L \bigcup S_0$, $S^L = S_1^L \bigcup S_2^L$, $S_0 = \overline{S}_0$ is the lower part of the boundary.

From estimate (3.4) for the error of the discrete solution in section 3. it follows that the solution of classical difference scheme (3.2) on uniform mesh (3.3) converges under very restrictive condition ($h \ll \varepsilon$) $\varepsilon^{-1} = o(N)$, where $N+1$ is the number of nodes in the uniform mesh in x. If this condition is violated, e.g., for $\varepsilon^{-1} = \mathcal{O}(N)$, then, in general, the solution of difference scheme (3.2), (3.3) for $N,\, N_0 \to \infty$ does not converge to the solution of problem (2.2), (2.1); N_0+1 is the number of nodes in the mesh in t.

Let us give definitions. Let $z(x,t)$, $(x,t) \in \overline{G}_h$ be the solution of a difference scheme and let for the function $z(x,t)$ the following estimate is satisfied

$$|u(x,t) - z(x,t)| \le M\lambda(\varepsilon^{-\nu}N^{-1}, N_0^{-1}), \quad (x,t) \in \overline{G}_h, \tag{2.4}$$

where $\lambda(\xi_1,\,\xi_2) \to 0$ for $\xi_1,\,\xi_2 \to \infty$ uniformly with respect to the parameter ε, $\nu \ge 0$. By definition, the solution of this difference scheme converges on the set $\overline{G}_h$ *uniformly with respect to the parameter* ε (or ε*-uniformly*), if in the estimate (2.4) $\nu = 0$; in that case we also will say that the scheme converges ε*-uniformly*. For $\nu > 0$ we will say that the scheme converges with *defect* ν. In that case when the value ν can be chosen arbitrary small, moreover, for the solution of the difference scheme the estimate (2.4) is satisfied, we will say that the scheme converges *almost* ε*-uniformly with the defect* ν (or, briefly, *almost* ε*-uniformly*).

The defect of scheme (3.2), (3.3) is equal to one.

Our **aim** is for the boundary value problem (2.2), (2.1) to construct a difference scheme on *a posteriori* adapted meshes whose solution converges almost ε-uniformly.

3. Base scheme for problem (2.2), (2.1)

On the set $\overline{G}$ we introduce the rectangular mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are arbitrary, in general, nonuniform meshes on the intervals $[0,d]$ and $[0,T]$ respectively. Let $h^i = x^{i+1} - x^i$, $x^i,\, x^{i+1} \in \overline{\omega}$, $h = \max_i h^i$, and $h_t^k = t^{k+1} - t^k$, $t^k,\, t^{k+1} \in \overline{\omega}_0$, $h_t = \max_k h_t^k$. Assume that the condition $h \le M\,N^{-1}$, $h_t \le M\,N_0^{-1}$ is fulfilled, where $N+1$ and N_0+1 are the number of nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$ respectively.

Problem (2.2), (2.1) is approximated by the finite difference scheme [10]

$$\begin{aligned} \Lambda\,(z(x,t)) &\equiv \Lambda^2\, z(x,t) - f(x,t,\, z(x,t)) = 0, \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{3.2}$$

[1] Throughout this paper, M, M_i (or m) denote sufficiently large (small) positive constants that do not depend on ε and on the discretization parameters.

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$; $\Lambda^2 \equiv \varepsilon\, a(x,t)\, \delta_{\overline{x}\widehat{x}} + b(x,t)\, \delta_x - c(x,t) - p(x,t)\, \delta_{\overline{t}}$, $(x,t) \in G_h$, $\delta_{\overline{x}\widehat{x}}\, z(x,t)$ is the central difference derivative on the nonuniform mesh, $\delta_{\overline{x}\widehat{x}}\, z(x,t) = 2(h^i + h^{i-1})^{-1}[\delta_x\, z(x,t) - \delta_{\overline{x}}\, z(x,t)]$, $(x,t) = (x^i,t) \in G_h$; $\delta_x\, z(x,t)$ and $\delta_{\overline{x}}\, z(x,t)$, $\delta_{\overline{t}}\, z(x,t)$ are the first (forward and backward) derivatives.

Nonlinear base scheme (3.2), (3.1) is monotone [10] ε-uniformly.

In the case of meshes, uniform in both variables,

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.3}$$

using the maximum principle, we obtain the estimate

$$|u(x,t) - z(x,t)| \le M \left[\left(\varepsilon + N^{-1}\right)^{-1} N^{-1} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h. \tag{3.4}$$

Now we construct the base scheme convergent ε-uniformly (see, e.g., [2, 3]). On the set $\overline{G}$ we introduce the mesh

$$\overline{G}_h = \overline{\omega}^* \times \overline{\omega}_0, \tag{3.5}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(3.3)}$, $\overline{\omega}^*$ is a *piecewise* uniform mesh which is constructed as follows. The interval $[0,d]$ is divided into two parts $[0,\sigma]$, $[\sigma,d]$, step-sizes in the intervals $[0,\sigma]$ and $[\sigma,d]$ are constant and equal to $h^{(1)} = 2\,\sigma\, N^{-1}$ and $h^{(2)} = 2(d-\sigma)N^{-1}$ respectively. The parameter σ is defined by the relation $\sigma = \sigma(\varepsilon,\, N) = \min\left[2^{-1}\, d,\; m^{-1}\,\varepsilon\, \ln N\right]$, where m is an arbitrary number in the interval $(0,m_0)$, $m_0 = \min_{\overline{G}}[a^{-1}(x,t)\, b(x,t)]$.

For solutions of the difference scheme (3.2), (3.5) we obtain the ε-uniform estimate

$$|u(x,t) - z(x,t)| \le M \left[N^{-1} \ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h. \tag{3.6}$$

Theorem 1 *Let for the solution $u(x,t)$ of the problem* (2.2), (2.1) *the estimates of Theorem 2 be satisfied. Then the base difference scheme* (3.2), (3.5) *(scheme* (3.2), (3.3)*) converges ε-uniformly (under the condition $\varepsilon^{-1} = o(N)$). For discrete solutions the estimates* (3.4), (3.6) *are valid.*

4. Grid approximations on locally refined meshes

We describe a formal iterative algorithm for construction of approximate solutions for the boundary value problem (2.2), (2.1). On the set $\overline{G}$ we introduce the coarse (start) mesh

$$\overline{G}_{1h} = \overline{\omega}_1 \times \overline{\omega}_0, \tag{4.1a}$$

where $\overline{\omega}_1$ and $\overline{\omega}_0$ are uniform meshes, $\overline{\omega}_0 = \overline{\omega}_{0(3.3)}$; the step-size $\overline{\omega}_1$ is $h_1 = d\, N^{-1}$. We denote by $z_1(x,t)$, $(x,t) \in \overline{G}_{1h}$, where $\overline{G}_{1h} = \overline{G}_{1h(4.1)} = \overline{G}_{h(3.3)}$, the solution of problem (3.2), (4.1a).

Let the value $d_1 \in \overline{\omega}_1$ be found in some a way so that for $x \ge d_1$ the discrete solution $z_1(x,t)$, $(x,t) \in \overline{G}_{1h}$ well approximates the solution of the problem (2.2), (2.1), moreover,

$$|u(x,t) - z_1(x,t)| \le M\, \delta, \quad (x,t) \in \overline{G}_{1h}, \quad x \ge d_1, \tag{4.2a}$$

where $\delta > 0$ is an arbitrary sufficiently small number, the constant M is independent of δ; $d_1 \in [0,d)$.

If it turns out that $d_1 > 0$, we refine the solution.

Let for $k \ge 2$, the grid set $\overline{G}_{k-1,h}$ and the grid function $z_{k-1}(x,t)$ on this set already be constructed. Further, let the value $d_{k-1} \in \omega_{k-1}$ be found in some a way so that for $x \ge d_{k-1}$ the discrete solution $z_{k-1}(x,t)$, $(x,t) \in \overline{G}_{k-1,h}$ well approximates the solution of the problem (2.2), (2.1), moreover,

$$|u(x,t) - z_{k-1}(x,t)| \le M\, \delta, \quad (x,t) \in \overline{G}_{k-1,h}, \quad x \ge d_{k-1}; \tag{4.2b}$$

$M_{(4.2\mathrm{b})} = M^*_{(4.2\mathrm{b})}(k-1)$, where M^* is independent of k. Here $\overline{G}_{k-1,h} = \overline{\omega}_{k-1} \times \overline{\omega}_0$, $\overline{\omega}_{k-1}$ is a mesh generating the mesh $\overline{G}_{k-1,h}$; N_k+1 is the number of nodes in the mesh $\overline{\omega}_k$, $k \geq 2$; $N_1 = N$.

If it turns out that $d_{k-1} > 0$, we define the subdomain

$$\overline{G}_{(k)} = G_{(k)} \cup S_{(k)}, \quad G_{(k)} = G_{(k)}(d_{k-1}), \quad G_{(k)} = D_{(k)} \times (0,T], \quad D_{(k)} = (0, d_{k-1}). \tag{4.1c}$$

On the set $\overline{G}_{(k)}$ we introduce the mesh

$$\overline{G}_{(k)h} = \overline{\omega}_{(k)} \times \overline{\omega}_0, \tag{4.1d}$$

where $\overline{\omega}_{(k)}$ is the uniform mesh with the number of nodes $N+1$; $h_{(k)}$ is the step-size in the mesh $\overline{\omega}_{(k)}$. Let $z_{(k)}(x,t)$, $(x,t) \in \overline{G}_{(k)h}$ be the solution of the grid problem

$$\begin{aligned} \Lambda_{(3.2)}\big(z_{(k)}(x,t)\big) &= 0, \qquad (x,t) \in G_{(k)h}, \\ z_{(k)}(x,t) &= \begin{cases} z_{k-1}(x,t), & (x,t) \in S_{(k)h} \setminus S, \\ \varphi(x,t), & (x,t) \in S_{(k)h} \bigcap S. \end{cases} \end{aligned} \tag{4.1e}$$

Set

$$\overline{G}_{kh} = \overline{G}_{(k)h} \bigcup \left\{ \overline{G}_{k-1,h} \setminus \overline{G}_{(k)} \right\}, \quad z_k(x,t) = \begin{cases} z_{(k)}(x,t), & (x,t) \in \overline{G}_{(k)h}, \\ z_{k-1}(x,t), & (x,t) \in \overline{G}_{k-1,h} \setminus \overline{G}_{(k)}. \end{cases}$$

If for some value $k = K_0$ it turns out that $d_{K_0} = 0$, then we set $d_k = 0$ for $k \geq K_0$. For $k \geq K_0 + 1$, we assume that the sets $\overline{G}_{(k)}$ are empty, and we do not compute the functions $z_{(k)}(x,t)$. For example, for $k \geq K_0$ we have $z_k(x,t) = z_{K_0}(x,t)$, $\overline{G}_{kh} = \overline{G}_{K_0 h}$.

For $k = K$, where K is a given fixed number, $K \geq 1$, we set

$$\overline{G}^K_h = \overline{G}_{Kh} \equiv \overline{G}_h, \quad z^K(x,t) = z_K(x,t) \equiv z(x,t). \tag{4.1f}$$

Let the value $d^K \in \overline{\omega}_K$, $d^K = d_K$, be found so that for $x \geq d_K$ the solution $z_K(x,t)$ approximates the solution of the problem (2.2), (2.1); in that case we have

$$|u(x,t) - z^K(x,t)| \leq M\,\delta, \quad (x,t) \in \overline{G}^K_h, \quad x \geq d^K. \tag{4.2c}$$

We call the function $z_{(4.1)}(x,t)$, $(x,t) \in \overline{G}_{h(4.1)}$ the solution of the scheme (3.2), (4.1), and the functions $z_k(x,t)$, $(x,t) \in \overline{G}_{kh}$, $k = 1, ..., K$ are called by the components of the solution of the difference scheme.

The given algorithm (we call it $A_{(4.1)}$) allows us to construct the solution of problem (3.2), (4.1) on the basis of the sequence of the values d_k, $k = 1, ..., K$. The value $N_K + 1$ is the number of nodes in the mesh $\overline{\omega}^K = \overline{\omega}_K$, used for the construction of the function $z^K(x,t)$. For the value N_K we have the estimate $N_K \leq K\,(N-1) + 1 \leq K\,N$.

5. Adaptive scheme based on a bound of the solution gradient

5.1. As a preliminary, we give estimates of the solution of the boundary value problem and its derivatives. We represent the solution of problem (2.2) as the sum of functions

$$u(x,t) = U(x,t) + V(x,t), \qquad (x,t) \in \overline{G}, \tag{5.1}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular parts of the solution.

For the functions $U(x,t)$, $V(x,t)$ the following estimates are valid

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, U(x,t) \right| \leq M\left[1 + \varepsilon^{2-k}\right], \quad \left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}\, V(x,t) \right| \leq M\,\varepsilon^{-k} \exp\big(-m\varepsilon^{-1} r(x,\Gamma_1)\big), \tag{5.2}$$

$$(x,t) \in \overline{G}, \quad k + 2\,k_0 \leq 4, \quad k \leq 3,$$

where m is an arbitrary number in the interval $(0, m_0)$, $m_0 = \min_{\overline{G}} \left[a^{-1}(x,t)\, b(x,t)\right]$; $r(x, \Gamma_1)$ is the distance from the point x to the left boundary Γ_1 of the set D.

Theorem 2 *Let the data of the boundary value problem* (2.2), (2.1) *satisfy the condition* (2.3), *the condition* $a,\, b,\, c,\, p \in C^{6+\alpha}(\overline{G})$, $f \in C^{6+\alpha}(\overline{G} \times R)$, $\varphi \in C^{6+\alpha}(S)$, $\alpha > 0$, *and also the condition*

$$\varphi(x,t) = 0, \quad (x,t) \in S_0; \quad \frac{\partial^{k_0}}{\partial t^{k_0}}\, \varphi(x,t) = 0, \quad \frac{\partial^{k+k_0+k_u}}{\partial x^k\, \partial t^{k_0}\, \partial u^{k_u}}\, f(x,t,\, u) = 0, \quad (x,t) \in S^*, \quad u = 0,$$

where $k,\, k_0,\, k_u \le 6$, $S^* = \overline{S}^L \bigcap S_0$. *Then the solution of the boundary value problem and its components in representation* (5.1) *satisfy the estimates* (5.2).

5.2. Let us give the construction of the indicator on the basis of bounds for the solution gradient.

5.2.1. Let us define the width of the boundary layer in the case of the boundary value problem (2.2), (2.1). Let the component $U(x,t)$ in representation (5.1) satisfy the estimate

$$\left| \frac{\partial}{\partial x}\, U(x,t) \right| \le M_1, \qquad (x,t) \in \overline{G}. \tag{5.3a}$$

Assume that values of the parameter ε are sufficiently small, $\varepsilon \le \varepsilon_0$. We say that $\sigma_0^c = \sigma_0^c(M_0;\ D,\, \varepsilon)$, where M_0 is an arbitrary sufficiently large number, $M_0 > M_1$, is the width of the boundary layer in a neighbourhood of the side S_1^L (defined by the gradient of the boundary layer function), if σ_0^c is the minimum of the value σ, for which the following estimate holds

$$\left| \frac{\partial}{\partial x}\, V(x,t) \right| \le M_0, \qquad (x,t) \in \overline{G}, \quad r(x, \Gamma_1) \ge \sigma. \tag{5.3b}$$

5.2.2. Let us define the width of the boundary layer in the case of difference scheme (3.2), (3.1). We denote by $z_v(x,t)$, $(x,t) \in \overline{G}$ the solution of the difference problem

$$\Lambda_{(3.2)} z(x,t) = L_{(2.2)}\, v(x,t), \quad (x,t) \in G_h, \quad z(x,t) = v(x,t), \quad (x,t) \in S_h,$$

where $v(x,t)$ is an arbitrary sufficiently smooth function, $v \in C^{2,1}(G) \bigcap C(\overline{G})$. We represent the solution of problem (3.2), (3.1) as the sum of functions

$$z(x,t) = z_U(x,t) + z_V(x,t), \quad (x,t) \in \overline{G}_h, \tag{5.4}$$

where $z_U(x,t)$ and $z_V(x,t)$ are grid functions which approximate the components $U(x,t)$ and $V(x,t)$ in representation (5.1). Let the component $z_U(x,t)$ satisfies the estimate

$$|\, \delta_x\, z_U(x,t)\, | \le M_1, \qquad (x,t) \in \overline{G}. \tag{5.5a}$$

We say that $\sigma_0 = \sigma_0(M_0) = \sigma_0(M_0;\, D, \varepsilon, h)$, where $\varepsilon \in (0, \varepsilon_0]$, M_0 and ε_0 are sufficiently large and small constants, $M_0 > M_1$, $\varepsilon_0 = \varepsilon_0(M_0)$, is the width of the discrete boundary layer in a neighbourhood of the side S_1^L (defined by the gradient of the singular component $z_{V(5.4)}(x,t)$), if σ_0 is the minimum of the value σ, for which the estimate holds

$$|\delta_{\overline{x}} z_V(x,t)| \le M_0, \quad (x,t) \in \overline{G}, \quad r(x, \Gamma_1) \ge \sigma. \tag{5.5b}$$

Thus, the function $\sigma_0(M_0) = \sigma_{0(5.5)}(M_0;\, D, \varepsilon, h)$ is constructed.

5.3. In order that a formal grid construction (3.2), (4.1) would be constructive, it is required to give the values K and d_k, $k = 1, 2, \ldots, K$.

Let $K \geq 1$. Define the values $d_{k(4.1)}$. Set

$$d_1 = \sigma_1, \tag{5.6a}$$

where $\sigma_1 = \sigma_{0(5.5)}(M_0; D, \varepsilon, h)$, $D = D_{(2.1)}$, $h = h_{(3.3)}$, M_0 is a sufficiently large number. Let the value d_{k-1} already be found. Further, we find the value

$$\sigma_k = \sigma_0(k\, M_0;\ D_{(k)},\, \varepsilon,\, h_{(k)}), \quad k \geq 2, \tag{5.6b}$$

where $\sigma_{0(5.6)}(M;\ D,\, \varepsilon,\, h) = \sigma_{0(5.5)}(M;\ D,\, \varepsilon,\, h)$, $D_{(k)} = D_{(k)(4.1c)}$, $h_{(k)} = h_{(k)(4.1)}$. If the relation $\sigma_k \leq m_0\, \sigma_{k-1}$ is valid, then we set

$$d_k = \sigma_k; \tag{5.6c}$$

here m_0 is a sufficiently small number. But if for some value $k = k_0$, it turned out that $\sigma_{k_0} > m_0\, \sigma_{k_0-1}$, then we set $d_k = d_{k_0}$ for $k \geq k_0$.

The iterative process is stopped at $k = K$.

The difference scheme (3.2), (4.1), (5.6) is the scheme on adapted meshes that are constructed on the basis of a bound of the solution gradient.

6. Investigation of scheme (3.2), (4.1), (5.6)

6.1. For the solution of difference scheme (3.2), (3.3) the following estimates are fulfilled

$$|u(x,t) - z(x,t)| \leq M \left[(\varepsilon + N^{-1})^{-1} N^{-1} + N_0^{-1} \right], \tag{6.1a}$$

$$|u(x,t) - z(x,t)| \leq M \left[q^{-r_1 h^{-1}} + N^{-1} + N_0^{-1} \right], \qquad (x,t) \in \overline{G}, \tag{6.1b}$$

where $r_1 = r(x, \Gamma_1)$, $q = 1 + a_{(1)}^{-1} b_{(1)}\, \varepsilon^{-1} h$, $a_{(1)} = \max_{\overline{G}} a(x,t)$, $b_{(1)} = \min_{\overline{G}} b(x,t)$.

From these estimates it follows that the scheme converges on $\overline{G}_h$ under the condition $\varepsilon^{-1} = o(N)$, and also it converges ε-uniformly outside of σ_0–neighbourhood of the set S_1^L:

$$|u(x,t) - z(x,t)| \leq M \left[N^{-1/2} + N_0^{-1} \right], \qquad (x,t) \in \overline{G}, \quad \textit{for} \quad r(x, \Gamma_1) \geq \sigma_0, \tag{6.2a}$$

$$\sigma_0 \leq M \left[\varepsilon \ln \varepsilon^{-1} + N^{-1} \ln N \right], \quad \textit{where} \quad \sigma_0 = \sigma_{0(5.5)}\,(M_0;\ D,\, \varepsilon,\, h)\,. \tag{6.2b}$$

The neighbourhood outside of which the estimate (6.2a) is valid, shrinks as $\varepsilon \to 0$, $N \to \infty$.

Theorem 3 *Let for the solution of boundary value problem* (2.2), (2.1) *the estimates of Theorem* 2 *be fulfilled. Then the solution of the difference scheme* (3.2), (3.3) *converges on $\overline{G}$ to the solution of the boundary value problem under the condition $\varepsilon^{-1} = o(N)$, and also it converges ε-uniformly (at the rate $\mathcal{O}\left(N^{-1/2} + N_0^{-1}\right)$) outside of σ_0–neighbourhood of the set S_1^L. For the discrete solution the error estimates* (6.1), (6.2) *are valid.*

6.2. Let us consider the difference scheme (3.2), (4.1), (5.6).

The component $z_1(x,t) = z(x,t)$ of the solution of this scheme satisfies the estimates (6.1). Taking account of the estimate for the function $z_V(x)$, we find the estimate for the value σ_1, i.e. the width of the boundary layer:

$$\sigma_1 \leq M \left[\varepsilon \ln \varepsilon^{-1} + N^{-1} \ln N \right], \quad \varepsilon \in (0, \varepsilon_0], \quad h \leq h_0.$$

For the value h_2, i.e. the step-size of the mesh $\overline{\omega}_{(2)}$, we have the estimate

$$h_2 \leq M\, N^{-1} \left[\varepsilon\, \ln\, \varepsilon^{-1} + N^{-1}\, \ln N \right].$$

Taking into account the bound (6.1b), we estimate $u(x,t) - z_1(x,t)$ on the boundary of the set $\overline{G}_{(2)h}$, and also $u(x,t) - z_{(2)}(x,t)$ on the set $\overline{G}_{(2)h}$ itself. For $u(x,t) - z_2(x,t)$ we obtain the estimate

$$|u(x,t) - z_2(x,t)| \le M\left[N^{-1/2} + \varepsilon^{-1}N^{-2}\ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_{2h},$$

and outside of σ_2–neighbourhood of the set S_1^L we have

$$|u(x,t) - z_2(x,t)| \le M\left[N^{-1/2} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_{2h}, \quad r(x,\Gamma_1) \ge \sigma_2.$$

For the value σ_2, we have the estimate

$$\sigma_2 \le M\,N^{-1}\ln N.$$

By a similar way we find the estimates

$$|u(x,t) - z(x,t)| \le M\left[N^{-1/2} + \varepsilon^{-1}N^{-K}\ln^{K-1}N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h;$$

$$|u(x,t) - z(x,t)| \le M\left[N^{-1/2} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h, \quad r(x,\Gamma_1) \ge \sigma_K;$$

$$\sigma_K \le \left\{ \begin{array}{ll} M[\varepsilon\ln\varepsilon^{-1} + N^{-1}\ln N], & K = 1, \\ MN^{-K+1}\ln^{K-1}N, & K \ge 2 \end{array} \right\}; \tag{6.3}$$

where $z(x,t) = z_{(4.1)}(x,t)$, $\overline{G} = \overline{G}_{h(4.1)}$.

The function $z(x,t)$ for N, $N_0 \to \infty$ converges (to the solution of boundary value problem (2.2), (2.1)) ε-uniformly outside of σ_K–neighbourhood of the set S_1^L, and also on the set $\overline{G}$ for sufficiently small (but not too small) values of the parameter ε, namely, under the condition

$$\varepsilon \ge \varepsilon_0(N), \quad \varepsilon_0^{-1}(N) = o(N^K\ln^{-K+1}N). \tag{6.4}$$

Thus, difference scheme (3.2), (4.1), (5.6), i.e. the scheme on adapted meshes, converges almost ε-uniformly. In order to provide the convergence defect of the function $z(x,t)$ not higher than the value $\nu_{(2.4)}$, it is required to choose the value K satisfying the condition

$$K > K(\nu), \quad K(\nu) = \nu^{-1}. \tag{6.5}$$

Theorem 4 *Let hypothesis of Theorem* 2 *be fulfilled. Then the function* $z(x,t)$, $(x,t) \in \overline{G}$, *i.e. the solution of the difference scheme* (3.2), (4.1), (5.6) *converges on* $\overline{G}$ *to the solution of the boundary value problem* (2.2), (2.1) *under the condition* (6.4), *and also it converges* ε*-uniformly (at the rate* $\mathcal{O}\left(N^{-1/2} + N_0^{-1}\right)$*) outside of* σ_K*–neighbourhood of the set* S_1^L*; the solution of the difference scheme* (3.2), (4.1), (5.6), (6.5) *converges to the solution of the boundary value problem almost* ε*-uniformly with the defect* ν*. For the discrete solutions the estimates* (6.3) *are valid.*

7. Linearized scheme on adapted meshes

7.1. On mesh (3.1) we consider the linearized (see [10]) difference scheme

$$\begin{aligned} \Lambda_{(5.1)}\left(z(x,\,t)\right) &\equiv \Lambda^2_{(3.2)}z(x,\,t) - f\left(x,t,\,\check{z}(x,\,t)\right) = 0, \quad (x,\,t) \in G_h, \\ z(x,\,t) &= \varphi(x,\,t), \quad (x,\,t) \in S_h. \end{aligned} \tag{7.1}$$

Here $\check{z}(x,\,t) = z(x,t-h_t)$, $(x,t) \in \overline{G}_h$, $t > 0$.

The difference scheme (7.1), (3.1) is monotone under the condition

$$f_u(x,t,\,u) \le c(x,t), \quad (x,t,\,u) \in \overline{G} \times R. \tag{7.2}$$

For simplicity, we assume that the condition (7.2) is satisfied.

Taking into account estimates of the solution to problem (2.2), (2.1) for the linearized difference scheme (7.1) on the special mesh (3.5), we obtain the estimate (similar to estimate (3.6))

$$|u(x,t) - z(x,t)| \leq M \left[N^{-1} \ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h. \tag{7.3}$$

Theorem 5 *Let hypothesis of Theorem* 2 *and condition* (7.2) *be fulfilled. Then the solution of the linearized difference scheme* (7.1), (3.5) *converges to the solution of the boundary value problem* (2.2), (2.1) *ε-uniformly; for the discrete solutions the estimate* (7.3) *holds.*

7.2. To the boundary value problem (2.2), (2.1) corresponds difference scheme (7.1), (4.1), (5.6), i.e. the linearized difference scheme on *a posteriori* adapted meshes.

For solutions of the difference scheme (7.1), (4.1), (5.6), statements about convergence hold that are similar to those for the scheme (3.2), (4.1), (5.6).

Theorem 6 *Let hypothesis of Theorem* 5 *be satisfied. Then the function $z(x,t)$, $(x,t) \in \overline{G}_h$, i.e. the solution of the difference scheme* (7.1), (4.1), (5.6) *converges on $\overline{G}$ to the solution of the boundary value problem* (2.2), (2.1) *under the condition* (6.4), *and also it converges ε-uniformly (at the rate $\mathcal{O}\left(N^{-1/2} + N_0^{-1}\right)$) outside of σ_K–neighbourhood of the set S_1^L; the solution of the difference scheme* (7.1), (4.1), (5.6), (6.5) *converges to the solution of the boundary value problem almost ε-uniformly with the defect ν. For the discrete solutions the estimates* (6.3) *are valid.*

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