Int. Conference on Boundary and Interior Layers
BAIL 2006
G. Lube, G. Rapin (Eds)

An adaptive method for the numerical solution of an elliptic convection diffusion problem*

S. Sikwila[1], A.F. Hegarty[1] and G.I. Shishkin[2]

[1] Department of Mathematics and Statistics, University of Limerick, Ireland.
stephen.sikwila@ul.ie, alan.hegarty@ul.ie,

[2] Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences
16 S. Kovalevskaya Street, Ekaterinburg 620219, Russia
shishkin@imm.uran.ru

Abstract

We consider the two dimensional linear steady state convection diffusion problem

$$\frac{1}{Re}\triangle u(x,y) + \mathbf{a}.\nabla u(x,y) = f(x,y)$$

where Re represents the Reynolds number, i.e., $1/Re$ is a small parameter, and $\mathbf{a} = (a_1, a_2)$. Numerical methods for solving such problems, based on the use of piecewise uniform meshes appropriately condensed in the layer regions, were shown to be uniformly convergent with respect to the perturbation parameter, e.g., in [3], [4]. These methods relied on *a priori* knowledge of the location of the layer region. It is of interest to examine whether similarly robust results can be obtained without this *a priori* knowledge. To this end, a new adaptive technique is presented, again based on piecewise uniform meshes, where the location of the cut-off points between the coarse and fine meshes are moved iteratively. Algorithms for one-dimensional cases were described in [5] and [6]. Numerical experiments indicate that the computed solutions are robust with respect to Re.

1. Formulation of the Algorithms

We will consider problems where an interior layer is induced from the inflow boundary data. This interior layer will propagate across Ω_2 as shown in figure 1. The basic algorithm produces a

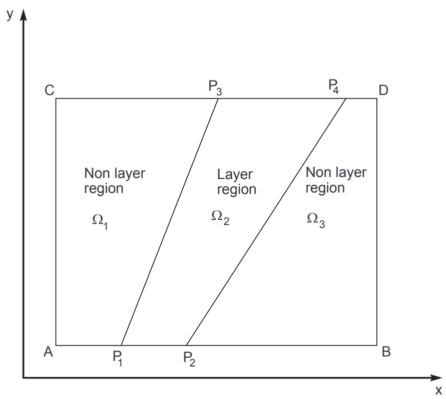


Figure 1: Graphical representation of the domain Ω showing the location of the layer region.

mesh comprising piecewise-uniform submeshes which are more and more refined within the layer region. The mesh is uniform in the y-direction and defined by "cut-off points" in the x-direction. The following algorithm to find the cut-off points along every fixed y value for the problem P_{Re}:

$$P_{Re} \;:\; \begin{cases} \frac{1}{Re}\triangle u(x,y) + \mathbf{a}.\nabla u(x,y) = f(x,y), & (x,y) \in \Omega; \\ u_\varepsilon(x,y) = g(x,y), & (x,y) \in \Gamma, \end{cases}$$

*This work was supported by the Russian Foundation for Basic Research under grant No. 04-01-00578.

For the algorithm $diff_{i,j} = |u_{i,j} - u_{i-1,j}|$ and pos represents position. $PASS$ will represent the number of passes which are set by the user.
Algorithm

- $k = 0$.
- While $k < PASS$
  - Given $x^{(k)}$ and $y^{(k)}$ solve P_{Re}.
  - $j = 0$.
  - While $j < N$
    - $q_j = \max_i \{diff_{i,j} \mid 1 \le i \le N\}$.
    - $w = pos(q_j) - 1$
    - While $diff_{w,j} > tol$
      - $w := w - 1$.
    - End while.
    - $\tau_{1,j} = x_w^{(k)}$
    - $e = pos(q_j) + 1$
    - While $diff_{e,j} > tol$
      - $e := e + 1$.
    - End while.
    - $\tau_{2,j} = x_e^{(k)}$
    - $j := j + 1$.
  - End while
  - $\mathrm{L} = \{\tau_{1,j}\}_{j=0}^{N}$ and $\mathrm{R} = \{\tau_{2,j}\}_{j=0}^{N})$.
  - $\tilde{\mathrm{L}}_j = \dfrac{y_j - y_0}{y_N - y_0}(\tau_{1,N} - \tau_{1,1}) + \tau_{1,1}$ and $\tilde{\mathrm{R}}_j = \dfrac{y_j - y_0}{y_N - y_0}(\tau_{2,N} - \tau_{2,1}) + \tau_{2,1}$. New cut off points will be given by

$$\hat{\tau}_{1,j} = \tilde{\mathrm{L}}_j + \min\{0, \min_j\{\tau_{1,j} - \tilde{\mathrm{L}}_j\}\}$$
$$\hat{\tau}_{2,j} = \tilde{\mathrm{R}}_j + |\min\{0, \min_j\{\tilde{\mathrm{R}}_j - \tau_{2,j}\}\}|.$$

  - Form Ω_2 in figure 2 by defining

$$x_l(y) = \frac{y_j - y_0}{y_N - y_0}(\hat{\tau}_{1,N} - \hat{\tau}_{1,1}) + \hat{\tau}_{1,1}$$
$$x_u(y) = \frac{y_j - y_0}{y_N - y_0}(\hat{\tau}_{2,N} - \hat{\tau}_{2,1}) + \hat{\tau}_{2,1}$$
$$y_l(x) = 0 \quad \text{and} \quad y_u(x) = 1.$$

  - Use 1 to transform Ω_2 to Ω_2' (figure 2) and P_{Re} to P_{Re}'.
  - $k := k + 1$. Solve P_{Re}' on Ω_2'.
- End while

The number of mesh-points in a mesh produced by this algorithm depends on the number of passes used. If it is required to maintain a fixed number of mesh-points, the following can be added to the algorithm:

- Form $\Omega^N = \Omega_A^N \times \Omega_B^N$ where $\Omega_A^N = \{x_{i,j} \mid 1 \le i, j \le N-1\}$, $\Omega_B^N = \{y_j \mid 1 \le j \le N-1\}$, $y_j = j/N$ and

$$x_{i,j} = \begin{cases} 4i\hat{\tau}_{1,j}, & 0 \le i \le N/4; \\ \hat{\tau}_{1,j} + 2(j - N/4)(\hat{\tau}_{2,j} - \hat{\tau}_{1,j})/N, & N/4 < j \le 3N/4; \\ \hat{\tau}_{2,j} + 4(j - 3N/4)(1 - \hat{\tau}_{2,j})/N, & 3N/4 < j \le N. \end{cases}$$

- Solve P_{Re} on Ω^N using the transformation $\xi = x + (1-y)x^*$ or $\xi = x - yx^*$ and $\eta = y$ where $x^* = x_{i,N} - x_{i,1}$ to transform the grid to an orthogonal grid.

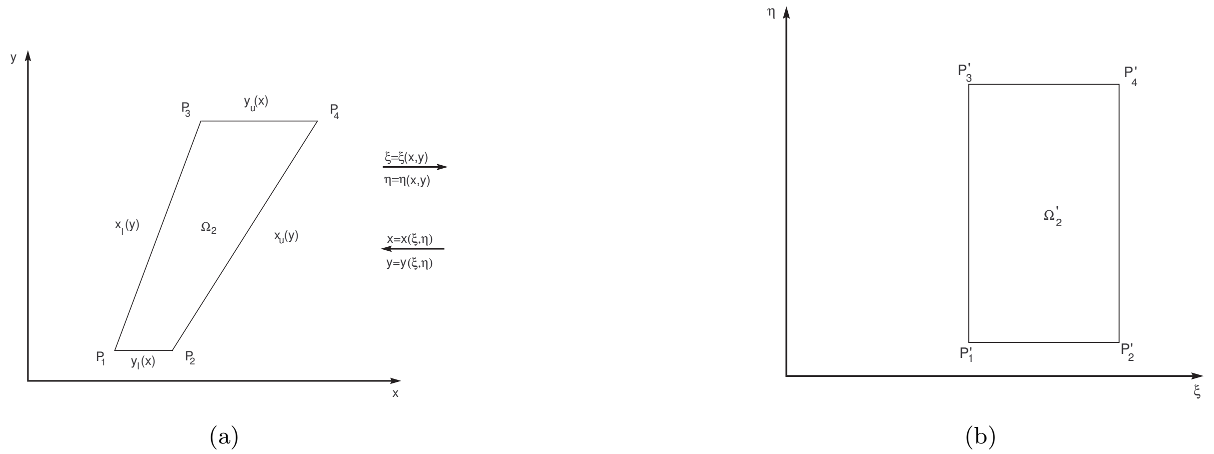


Figure 2: Transformation to a rectangular mesh

We use the following transformations

$$\xi = \frac{x - x_l(y)}{x_u(y) - x_l(y)} \tag{1a}$$

$$\eta = \frac{y - y_l(x)}{y_u(x) - y_l(x)} \tag{1b}$$

from [2] where x_l, x_u, y_l and y_u are the four sides in figure 2(a). This transformation will be used to transform the non rectangular domain Ω_2 to a rectangular domain Ω_2' and P_{Re} to P'_{Re}

$$P'_\varepsilon \;:\; \begin{cases} A'\frac{\partial^2 u_\varepsilon}{\partial \xi^2} + B'\frac{\partial^2 u_\varepsilon}{\partial \xi \partial \eta} + C'\frac{\partial^2 u_\varepsilon}{\partial \eta^2} + D'\frac{\partial u_\varepsilon}{\partial \xi} + E'\frac{\partial u_\varepsilon}{\partial \eta} + F'u_\varepsilon = f(\xi, \eta), & (\xi, \eta) \in \Omega_2'; \\ u_\varepsilon = g', & (\xi, \eta) \in \Gamma_2, \end{cases}$$

where $\Gamma_2 = \bar{\Omega}_2' \backslash \Omega_2'$,

$$\begin{aligned} A' &= \xi_x^2 + \xi_y^2 \\ B' &= 2(\xi_x \eta_x + \xi_y \eta_y) \\ C' &= \eta_x^2 + \eta_y^2 \\ D' &= \xi_{xx} + \xi_{yy} + a_1 \xi_x + a_2 \eta_x \\ E' &= \eta_{xx} + \eta_{yy} + a_1 \xi_y + a_2 \eta_y \\ F' &= d(\xi, \eta) \end{aligned}$$

2. Numerical Experiments: Test Problem

We will use the transformation $\psi = x - yx^*$, $\eta = y$.
Consider the problem

$$\frac{1}{Re}\triangle u(x,y) + \mathbf{a}.\nabla u(x,y) = f(x,y), \quad ,(x,y) \in \Omega \tag{2}$$

where $\Omega = (-1,1) \times (0,1)$, Re is the Reynolds number, $\mathbf{a} = (a_1, a_2)$

$$a_1(x,y) = \frac{3}{4} - \frac{1}{4(1+\exp((-4x+4y-2)Re/32))}$$

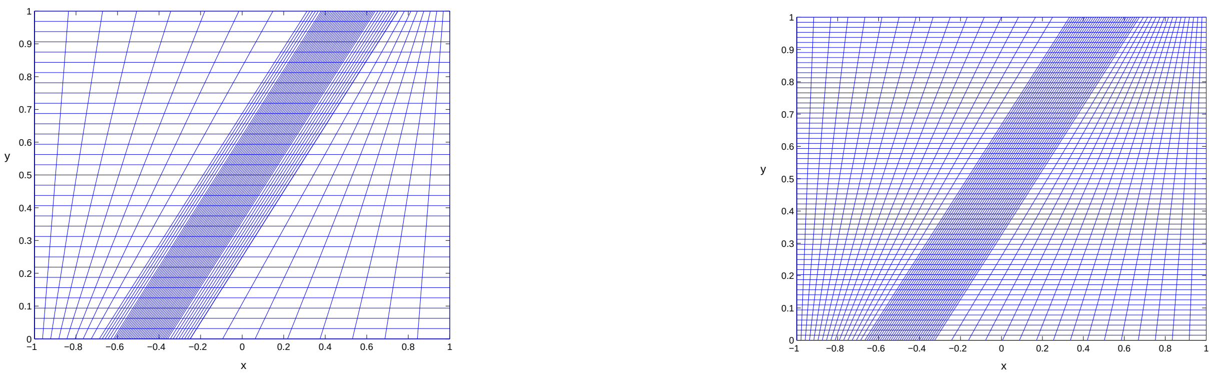
and

$$a_2(x,y) = \frac{3}{4} + \frac{1}{4(1+\exp((-4x+4y-2)Re/32))}.$$

The function $f(x,y)$ is chosen such that

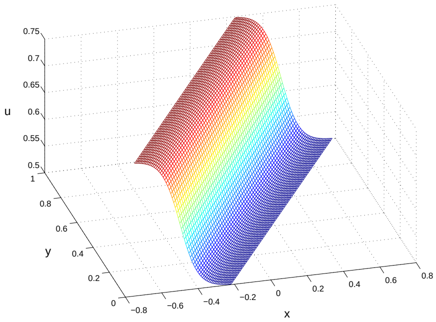
$$u^{exact} = \frac{3}{4} - \frac{1}{4(1+\exp((-4x+4y-2)Re/32))}$$

is the exact solution and the boundary conditions are calculated using the exact solution. This test problem was also used as an example by Bahdir [1]. The problem will be solved using the tolerance $tol = 0.05N^{-1}$.

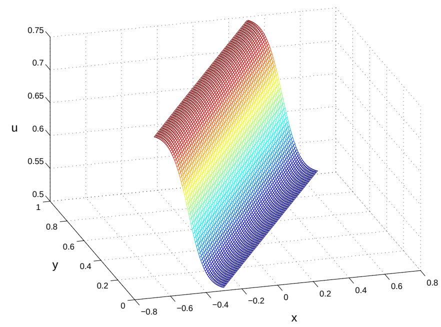


(a) Initial $N = 32$ (b) N=64

Figure 3: Grid Structure for (2) with $Re = 160$.

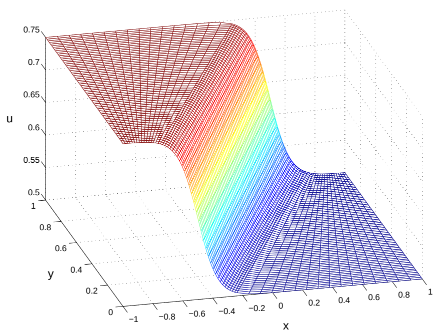


(a) First Iteration

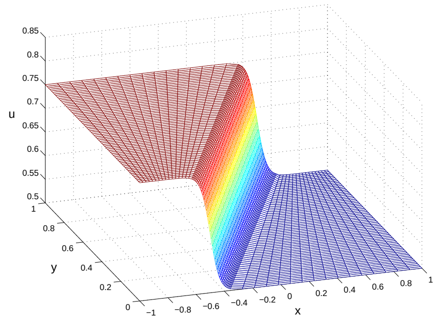


(b) Second Iteration

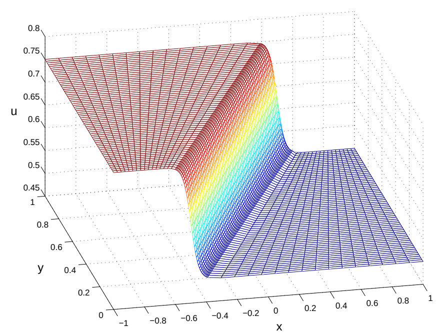
Figure 4: Iterations for (2) with $N = 64$ and $Re = 160$.



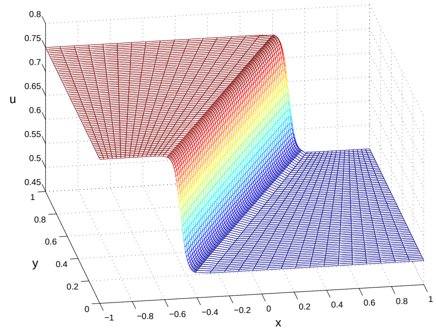
(a) $Re = 120$



(b) $Re = 240$



(c) $Re = 320$



(d) $Re = 400$

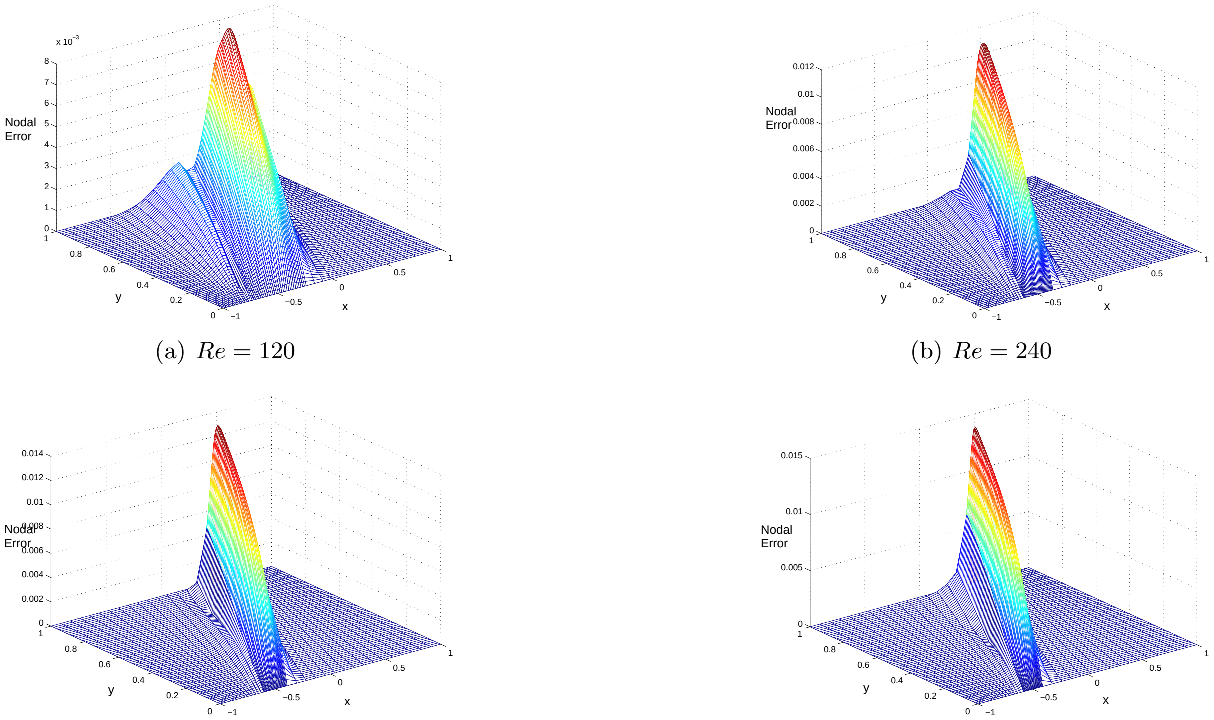
Figure 5: Numerical solutions of (2) for different Re values with $N = 64$.

Table 1: Maximum pointwise errors for the numerical solution of (2).

Iterations / Re	Number of intervals N: 32, 2	32, 3	64, 2	64, 3	128, 2	128, 3
100	0.13D-01	0.13D-01	0.71D-02	0.77D-02	0.40D-02	0.43D-02
120	0.14D-01	0.15D-01	0.79D-02	0.84D-02	0.44D-02	0.47D-02
140	0.16D-01	0.17D-01	0.86D-02	0.90D-02	0.47D-02	0.50D-02
160	0.17D-01	0.17D-01	0.91D-02	0.96D-02	0.50D-02	0.52D-02
180	0.18D-01	0.19D-01	0.96D-02	0.10D-01	0.52D-02	0.55D-02
200	0.19D-01	0.20D-01	0.10D-01	0.18D-01	0.54D-02	0.57D-02
220	0.21D-01	0.21D-01	0.11D-01	0.11D-01	0.57D-02	0.59D-02
240	0.21D-01	0.22D-01	0.11D-01	0.12D-01	0.58D-02	0.60D-02
260	0.23D-01	0.23D-01	0.12D-01	0.12D-01	0.60D-02	0.62D-02
280	0.23D-01	0.23D-01	0.12D-01	0.12D-01	0.62D-02	0.63D-02
300	0.24D-01	0.24D-01	0.12D-01	0.12D-01	0.63D-02	0.64D-02
320	0.25D-01	0.25D-01	0.13D-01	0.13D-01	0.65D-02	0.65D-02
340	0.25D-01	0.26D-01	0.13D-01	0.13D-01	0.66D-02	0.66D-02
360	0.27D-01	0.27D-01	0.13D-01	0.13D-01	0.67D-02	0.67D-02
380	0.27D-01	0.27D-01	0.14D-01	0.13D-01	0.68D-02	0.68D-02
400	0.27D-01	0.28D-01	0.14D-01	0.14D-01	0.69D-02	0.69D-02
420	0.28D-01	0.28D-01	0.14D-01	0.14D-01	0.69D-02	0.69D-02
440	0.28D-01	0.28D-01	0.14D-01	0.14D-01	0.70D-02	0.69D-02
E^N_{max}	0.28D-01	0.28D-01	0.14D-01	0.014D-01	0.70D-02	0.69D-02

Table 2: Convergence rates for the numerical solution of (2).

Iterations / Re	Number of intervals N: 32, 2	32, 3	64, 2	64, 3
100	0.81	0.74	0.85	0.85
120	0.83	0.83	0.86	0.84
140	0.87	0.90	0.88	0.85
160	0.90	0.88	0.88	0.88
180	0.94	0.92	0.89	0.90
200	0.88	0.89	0.93	0.92
220	0.96	0.92	0.92	0.91
240	0.92	0.93	0.94	0.94
260	0.97	0.96	0.95	0.92
280	0.95	0.95	0.96	0.94
300	0.98	0.99	0.97	0.95
320	0.98	0.98	0.98	0.97
340	0.97	1.01	0.99	0.95
360	1.03	1.01	0.99	0.97
380	0.97	1.02	1.00	0.97
400	0.94	1.04	1.03	0.98
420	0.99	1.03	1.01	0.99
440	0.98	1.04	1.01	0.99
R^N_{max}	0.98	1.04	1.01	0.99



(a) $Re = 120$ (b) $Re = 240$

(c) $Re = 320$ (d) $Re = 400$

Figure 6: Nodal errors for the numerical solution of (2) for different Re values with $N = 64$.

3. Conclusion

The adaptive method resolves the internal layer well for large values of Re and is computationally efficient since one needs at most two iterations to fully resolve the interior layer.

References

[1] A. Refik Bahadir, *A Fully Implicit Finite-Difference Scheme For Two Dimensional Burgers' Equations*, Applied Mathematics and Computation, (**137**) 131–137 (2003).

[2] J. Crank, *Free and Moving Boundary Problems*, Clarendon Press, Oxford (1984).

[3] J.J.H. Miller, E. O'Riordan and G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific, London (1996).

[4] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan and G.I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Chapman and Hall/CRC, Boca Raton (2000).

[5] S. Sikwila, A.F. Hegarty and G.I. Shishkin, *Numerical solution of problems with boundary and interior layers using an adaptive mesh method*, Proceedings of 3rd annual workshop on Numerical Methods for Problems with Layer Phenomena, University of Limerick, February 2004.

[6] S. Sikwila, A.F. Hegarty and G.I. Shishkin, Novel robust adaptive techniques for the numerical solution of convection diffusion problem, Proceedings of BAIL 2004 Conference, Toulouse, July 2004, ONERA, 2004.