Full Asymptotic Analysis of the Navier-Stokes Equations in the Problems of Gas Flows over Bodies with Large Reynolds Number

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1. Introduction

Traditionally, a gas flow in aerodynamics is assumed to be incompressible in both, outside and inside the boundary layer if the Mach number in the free stream $M_\infty < 0.3$ and the characteristic temperature difference is small enough, otherwise the model for compressible gas flow is used. At the same time in many practical problems of gas flows over bodies, for instance in the induced convection in heat exchangers, we have hyposonic gas flows in which the gas temperature across the boundary layer on the body surface can be varied significantly that leads to considerable changes in the gas density. As a result, the gas flow outside the boundary layer can be considered as incompressible one, whereas inside the layer, as essentially compressible one. In the last case, researchers use usually the common model of the compressible gas both inside the boundary layer and in the external inviscid flow area. However, such approach implies solving the compressible Euler equations in the external area that leads to significant computational difficulties if the Mach number is small. There are many papers devoted to the problem of the asymptotic analysis of the Navier-Stokes equations with small Mach number (as in [1]-[2]). In this paper the asymptotic analysis with another main small parameter $\varepsilon = 1/\sqrt{Re_\infty}$ is considered.

2. Formulation of the problem

The main aim of this paper is to carry out the strict asymptotic analysis based on the method of matched asymptotic expansions of the complete Navier-Stokes equations in all possible situations:

1) flows with the Mach number M_∞ greater than 0.3 and the large temperature difference across the boundary layer,

2) flows with $M_\infty > 0.3$ and the small temperature difference across the boundary layer,

3) flows with $M_\infty < 0.3$ and the large temperature difference across the boundary layer,

4) flows with $M_\infty < 0.3$ and the small temperature difference across the boundary layer,

and as a result to suggest the setting of the problem for the Euler equations and the boundary layer equations including the procedure of matching the appropriate gas parameters.

Cases 1) and 4) are seen to give us the well-known models for the compressible and incompressible gas flow in both areas respectively. Because of this, they have not been the major subject of investigation in the present study. At the same time cases 2) and 3) give us some intermediate situations which are of great interest. This problem is not trivial for example for case 3) because the application of the Euler equations for incompressible flow outside the boundary layer and the classical equations of the compressible boundary layer near the body surface makes impossible the agreement between both equation systems.

Investigation of the gas flows over bodies with large Reynolds number in most cases can be significantly simplified when the flow area is divided into two parts: the external inviscid one and the narrow area near the body surface known as the viscous boundary layer. As this takes place, the Navier-Stokes equations describing such flows are splitted into the Eulier equations

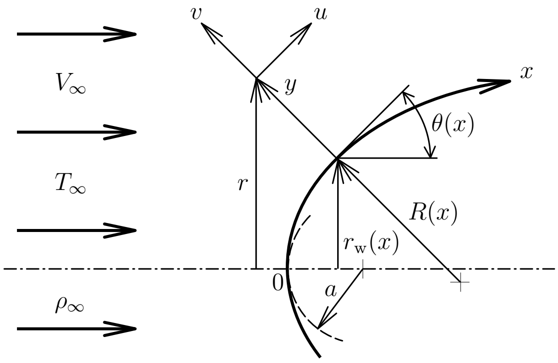

Figure 1: The boundary layer coordinate system.

for the external flow area and the Prandtl equations for a boundary layer.

We shall describe the idea of the analysis by an example of plane and axisymmetric flow around a blunt body. Consider both the sub- and supersonic gas flows that means that the Mach number in the undisturbed flow M_∞ is assumed to be arbitrary. Gas is assumed to be perfect with constant specific heats. The viscosity–temperature relation is taken in the form of a power function.

Consider a steady-state two-dimensional (plane) or axisymmetric flow. In the last case we assume that the azimutal gas velocity can be non-zero that corresponds to the flow over a rotating body. Introduce the boundary layer coordinate system (x, y) in the plane of the flow for the 2D flow and the coordinate system (x, y, ϑ) for the axisymmetric flow. Here the axis x is directed along the body contour, the axis y is normal to x, ϑ is azimuthal angle (see Figure 1). This coordinate system is orthogonal curvilinear. We consider the case $\partial/\partial\vartheta = 0$.

Take the Navier-Stokes equations as the governing equations describing the gas flow over a body. In the boundary layer coordinates they takes the form [3]:

$$\frac{\partial}{\partial x^*}\left(\rho^* u^* r^{*j}\right) + \frac{\partial}{\partial y^*}\left(\rho^* v^* r^{*j}\ \frac{R^*+y^*}{R^*}\right) = 0, \tag{1}$$

$$\rho^*\left[\frac{R^*u^*}{R^*+y^*}\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + \frac{u^*v^*}{R^*+y^*} - \frac{R^*w^{*2}}{R^*+y^*}\frac{\partial \ln r^{*j}}{\partial x^*}\right] = -\frac{R^*}{R^*+y^*}\frac{\partial p^*}{\partial x^*} + \rho^* g^*\cos\theta + \Phi_1^*, \tag{2}$$

$$\rho^*\left(\frac{R^*u^*}{R^*+y^*}\ \frac{\partial v^*}{\partial x^*} + v^*\ \frac{\partial v^*}{\partial y^*} - \frac{u^{*2}}{R^*+y^*} - w^{*2}\ \frac{\partial \ln r^{*j}}{\partial y^*}\right) = -\frac{\partial p^*}{\partial y^*} - \rho^* g^*\sin\theta + \Phi_2^*, \tag{3}$$

$$j\rho^*\left(\frac{R^*u^*}{R^*+y^*}\ \frac{\partial w^*}{\partial x^*} + v^*\ \frac{\partial w^*}{\partial y^*} + \frac{R^*}{R^*+y^*}\ u^*w^*\ \frac{\partial \ln r^{*j}}{\partial x^*} + v^*w^*\ \frac{\partial \ln r^{*j}}{\partial y^*}\right) = j\Phi_3^*, \tag{4}$$

$$\rho^* c_{\rm p}^*\left(\frac{R^*u^*}{R^*+y^*}\ \frac{\partial T^*}{\partial x^*} + v^*\ \frac{\partial T^*}{\partial y^*}\right) = \frac{R^*u^*}{R^*+y^*}\ \frac{\partial p^*}{\partial x^*} + v^*\ \frac{\partial p^*}{\partial y^*} + \frac{\partial}{\partial x^*}\left(\lambda^*\ \frac{R^*}{R^*+y^*}\ \frac{\partial T^*}{\partial x^*}\right) +$$

$$\frac{\partial}{\partial y^*}\left(\lambda^*\ \frac{\partial T^*}{\partial y^*}\right) + \lambda^*\left(\frac{R^*}{R^*+y^*}\right)^2\frac{\partial \ln r^{*j}}{\partial x^*}\ \frac{\partial T^*}{\partial x^*} + \lambda^*\ \left(\frac{1}{R^*+y^*} + \frac{\partial \ln r^{*j}}{\partial y^*}\right)\ \frac{\partial T^*}{\partial y^*} + \Phi_4^*, \tag{5}$$

$$p^* = \rho^*\Re T^*, \tag{6}$$

$$\frac{\mu^*}{\mu_0^*} = \left(\frac{T^*}{T_0^*}\right)^\omega. \tag{7}$$

Here $j=0$ is for a plane flow, $j=1$ for an axisymmetric one; the superscript $*$ relates to the dimensional parameters, the subscript 0 to the stagnation point parameters; t is the time, R the curvature radius of a body contour, r the distance from the axis of the symmetry, u and v the $x-$ and $y-$ component of the gas velocity, w the azimuthal velocity, ρ, T, p, μ, λ the density, the temperature, the pressure, the viscosity and the thermal conductivity of the gas, $\Re$ the gas constant, c_p the gas specific heat at constant pressure, ω the exponent in the viscosity–temperature relation, g the acceleration of gravity. $\Phi_i^* = \Phi_i(X^*)$, $i = 1, \ldots, 4$ are viscosity-dependent terms ($X = (x, y, R, r, u, v, w, \mu)^\mathrm{T}$):

$$\Phi_1^* = \frac{R^*}{R^*+y^*}\,\frac{\partial}{\partial x^*}\left[\frac{4}{3}\,\mu^*\left(\frac{R^*}{R^*+y^*}\,\frac{\partial u^*}{\partial x^*}+\frac{v^*}{R^*+y^*}\right)-\frac{2}{3}\,\mu^*\left(\frac{\partial v^*}{\partial y^*}+\frac{R^*u^*}{R^*+y^*}\,\frac{\partial \ln r^{*j}}{\partial x^*}+\right.\right.$$
$$\left.\left.v^*\cdot\frac{\partial \ln r^{*j}}{\partial y^*}\right)\right]+\frac{\partial}{\partial y^*}\left[\mu^*\left(\frac{\partial u^*}{\partial y^*}+\frac{R^*}{R^*+y^*}\,\frac{\partial v^*}{\partial x^*}-\frac{u^*}{R^*+y^*}\right)\right]+2\mu^*\,\frac{R^*}{R^*+y^*}\,\frac{\partial \ln r^{*j}}{\partial x^*}\,\cdot$$
$$\left(\frac{R^*}{R^*+y^*}\,\frac{\partial u^*}{\partial x^*}+\frac{v^*}{R^*+y^*}-\frac{R^*u^*}{R^*+y^*}\,\frac{\partial \ln r^{*j}}{\partial x^*}-v^*\frac{\partial \ln r^j}{\partial y^*}\right)+\mu^*\left(\frac{2}{R^*+y^*}+\frac{\partial \ln r^{*j}}{\partial y^*}\right)\cdot$$
$$\left(\frac{\partial u^*}{\partial y^*}+\frac{R^*}{R^*+y^*}\,\frac{\partial v^*}{\partial x^*}-\frac{u^*}{R^*+y^*}\right), \tag{8}$$

$$\Phi_2^* = \frac{R^*}{R^*+y^*}\,\frac{\partial}{\partial x^*}\left[\mu^*\left(\frac{\partial u^*}{\partial y^*}+\frac{R^*}{R^*+y^*}\,\frac{\partial v^*}{\partial x^*}-\frac{u^*}{R^*+y^*}\right)\right]+\frac{\partial}{\partial y^*}\left[\frac{4}{3}\,\mu^*\frac{\partial v^*}{\partial y^*}-\frac{2}{3}\,\mu^*\left(\frac{\partial u^*}{\partial x^*}\,\cdot\right.\right.$$
$$\left.\left.\frac{R^*}{R^*+y^*}+v^*\left(\frac{1}{R^*+y^*}+\frac{\partial \ln r^{*j}}{\partial y^*}\right)+\frac{R^*u^*}{R^*+y^*}\,\frac{\partial \ln r^{*j}}{\partial x^*}\right)\right]+2\mu^*\,\frac{1}{R^*+y^*}\,\left(\frac{\partial v^*}{\partial y^*}-\frac{\partial u^*}{\partial x^*}\,\cdot\right.$$
$$\left.\frac{R^*}{R^*+y^*}-\frac{v^*}{R^*+y^*}\right)+2\mu^*\,\frac{\partial \ln r^{*j}}{\partial y^*}\left(\frac{\partial v^*}{\partial y^*}-\frac{R^*u^*}{R^*+y^*}\,\frac{\partial \ln r^{*j}}{\partial x^*}-v^*\,\frac{\partial \ln r^{*j}}{\partial y^*}\right)+$$
$$\frac{R^*}{R^*+y^*}\cdot\frac{\partial \ln r^{*j}}{\partial x^*}\,\mu^*\left(\frac{\partial u^*}{\partial y^*}+\frac{R^*}{R^*+y^*}\,\frac{\partial v^*}{\partial x^*}-\frac{u^*}{R^*+y^*}\right), \tag{9}$$

$$\Phi_3^* = \frac{R^*}{R^*+y^*}\,\frac{\partial}{\partial x^*}\left[\frac{R^*}{R^*+y^*}\cdot\mu^*\,\left(\frac{\partial w^*}{\partial x^*}-w^*\frac{\partial \ln r^{*j}}{\partial x^*}\right)\right]+\frac{\partial}{\partial y^*}\,\left[\mu^*\left(\frac{\partial w^*}{\partial y^*}-w^*\frac{\partial \ln r^{*j}}{\partial y^*}\right)\right]+$$
$$\mu^*\left(\frac{1}{R^*+y^*}+2\,\frac{\partial \ln r^{*j}}{\partial y^*}\right)\left(\frac{\partial w^*}{\partial y^*}-w^*\frac{\partial \ln r^{*j}}{\partial y^*}\right)+2\mu^*\,\left(\frac{R^*}{R^*+y^*}\right)^2\frac{\partial \ln r^{*j}}{\partial x^*}\,\cdot$$
$$\left(\frac{\partial w^*}{\partial x^*}-w^*\frac{\partial \ln r^{*j}}{\partial x^*}\right), \tag{10}$$

$$\Phi_4^* = \mu^*\left\{2\left[\left(\frac{R^*}{R^*+y^*}\,\frac{\partial u^*}{\partial x^*}+\frac{v^*}{R^*+y^*}\right)^2+\left(\frac{\partial v^*}{\partial y^*}\right)^2+\left(\frac{R^*u^*}{R^*+y^*}\,\frac{\partial \ln r^{*j}}{\partial x^*}+v^*\,\frac{\partial \ln r^{*j}}{\partial y^*}\right)^2\right]+\right.$$
$$\left(\frac{\partial u^*}{\partial y^*}+\frac{R^*}{R^*+y^*}\,\frac{\partial v^*}{\partial x^*}-\frac{u^*}{R^*+y^*}\right)^2-\frac{2}{3}\,\left(\frac{\partial v^*}{\partial y^*}+\frac{R^*u^*}{R^*+y^*}\,\frac{\partial \ln r^{*j}}{\partial x^*}+v^*\,\frac{\partial \ln r^{*j}}{\partial y^*}+\right.$$
$$\left.\left.\frac{R^*}{R^*+y^*}\,\frac{\partial u^*}{\partial x^*}+\frac{v^*}{R^*+y^*}\right)^2+\left(\frac{\partial w^*}{\partial y^*}-w^*\,\frac{\partial \ln r^{*j}}{\partial y^*}\right)^2\right\}. \tag{11}$$

Boundary conditions for the Navier-Stokes equations are the following:

$$y^* = 0: \qquad u^* = v^* = 0, \quad w^* = w_\mathrm{w}^*, \quad T^* = T_\mathrm{w}^*, \tag{12}$$
$$y^* \to \infty: \qquad u^* \to V_\infty^*\cos\theta, \quad v^* \to -V_\infty^*\sin\theta, \quad w^* \to 0, \quad \rho^* \to \rho_\infty^*, \quad T^* \to T_\infty^*, \tag{13}$$

where the subscript ∞ relates to the free stream parameters.

3. Nondimensionalization of the governing equations

3.1. Traditionally known dimensionless variables

Some flow parameters can vary in wide ranges, whereas the others in narrow ones. For mathematical analysis and numerical solving it is much more convenient to have the Navier-Stokes equations written in terms of such dimensionless variables which vary in the ranges from 0 to $O(1)$. Two systems of dimensionless variables are widely known here: one for flows with $M_\infty \sim 1$ (see, for example, [4] and another for $M_\infty \ll 1$ (see [2]). They differ from each other by the way of nondimensionalization of the pressure and the temperature.

First, introduce the dimensionless variables except the pressure and the temperature

$$x = \frac{x^*}{a^*}, \quad y = \frac{y^*}{a^*}, \quad r = \frac{r^*}{a^*}, \quad R = \frac{R^*}{a^*}, \quad u = \frac{u^*}{V^*_\infty}, \quad v = \frac{v^*}{V^*_\infty}, \quad w = \frac{w^*}{V^*_\infty},$$

$$\rho = \frac{\rho^*}{\rho^*_\infty}, \quad \mu = \frac{\mu^*}{\mu^*_\infty}, \quad \lambda = \frac{\lambda^*}{\lambda^*_\infty}, \quad (\mu = \lambda), \tag{14}$$

where a^* is the body contour radius at the stagnation point (for example, $R = 1$ for a sphere).

Consider flows with the Mach number $M_\infty \sim 1$. In this case the following nondimensionalization of the pressure and the temperature are used [4]

$$p = \frac{p^*}{\rho^*_\infty V^{*2}_\infty}, \quad T = \frac{T^*}{V^{*2}_\infty / c^*_\mathrm{p}}, \quad h = \frac{h^*}{V^{*2}_\infty}, \quad (h = T), \tag{15}$$

where h is the gas enthalpy. The characteristic range of the pressure and the temperature is bounded by the values in the free stream area and at the stagnation point. Then the bounds of the dimensionless pressure and temperature are the following

$$p_\infty = \frac{p^*_\infty}{\rho^*_\infty V^{*2}_\infty} = \frac{\rho^*_\infty \Re T^*_\infty}{\rho^*_\infty V^{*2}_\infty} = \frac{\gamma \Re T^*_\infty}{\gamma V^{*2}_\infty} = \frac{c^{*2}_\infty}{\gamma V^{*2}_\infty} = \frac{1}{\gamma M^2_\infty},$$

$$T_\infty = \frac{T^*_\infty}{V^{*2}_\infty / c^*_\mathrm{p}} = \frac{\gamma \Re T^*_\infty}{\gamma \Re V^{*2}_\infty / c^*_\mathrm{p}} = \frac{a^{*2}_\infty}{\gamma V^{*2}_\infty (c^*_\mathrm{p} - c^*_\mathrm{V}) / c^*_\mathrm{p}} = \frac{1}{\gamma(\gamma - 1) M^2_\infty},$$

$$p_0 = \frac{p^*_0}{\rho^*_\infty V^{*2}_\infty} = \frac{p^*_0}{p^*_\infty} \cdot \frac{p^*_\infty}{\rho^*_\infty V^{*2}_\infty} = \left(1 + \frac{\gamma - 1}{2} M^2_\infty\right)^{\frac{\gamma}{\gamma - 1}} \cdot \frac{1}{\gamma M^2_\infty},$$

$$T_0 = \frac{T^*_0}{V^{*2}_\infty / c^*_\mathrm{p}} = \frac{T^*_0}{T^*_\infty} \cdot \frac{T^*_\infty}{V^{*2}_\infty / c^*_\mathrm{p}} = \left(1 + \frac{\gamma - 1}{2} M^2_\infty\right) \cdot \frac{1}{\gamma(\gamma - 1) M^2_\infty},$$

where $p^*_\infty = \rho^*_\infty \Re T^*_\infty$, $c^*_\infty = \sqrt{\gamma \Re T^*_\infty}$ is the sound speed in the free stream, γ the isentropic exponent, c^*_V the gas specific heat at constant volume, $M_\infty = V^*_\infty / c^*_\infty$. To estimate the suitability of the nondimensionalization method calculate the parameter ranges for the air ($\gamma = 1.4$). We have $[0.179, 1.397]$ and $[0.446, 0.804]$ as the ranges for p and T, respectively, for $M_\infty = 2$, and $[7.937, 8.448]$ and $[19.841, 20.198]$ for $M_\infty = 0.3$. It is obvious, that for $M_\infty < 0.3$ we obtain more narrow ranges which are far from 0 (for example, $[71.429, 71.930]$ and $[178.571, 178.929]$ for $M_\infty = 0.1$). This fact leads to the necessity to take another nondimensionalization in the case of hyposonic flow. Usually, the following dimensionless variables are used here [2]

$$p = \frac{p^* - p^*_\infty}{\rho^*_\infty V^{*2}_\infty}, \quad T = \frac{T^*}{T^*_\infty}, \quad h = \frac{h^*}{c^*_\mathrm{p} T^*_\infty}, \quad (h = T). \tag{16}$$

Then the characteristic bounds of the variables are

$$p_\infty = \frac{p^*_\infty - p^*_\infty}{\rho^*_\infty V^{*2}_\infty} = 0, \qquad T_\infty = \frac{T^*_\infty}{T^*_\infty} = 1,$$

$$p_0 = \frac{p^*_0 - p^*_\infty}{\rho^*_\infty V^{*2}_\infty} = \left(\frac{p^*_0}{p^*_\infty} - 1\right) \cdot \frac{p^*_\infty}{\rho^*_\infty V^{*2}_\infty} = \left(\left(1 + \frac{\gamma - 1}{2} M^2_\infty\right)^{\frac{\gamma}{\gamma - 1}} - 1\right) \cdot \frac{1}{\gamma M^2_\infty} \overset{(M_\infty \ll 1)}{\approx}$$

$$\left(\left(1 + \frac{\gamma}{\gamma - 1} \cdot \frac{\gamma - 1}{2} M^2_\infty + \frac{\gamma}{\gamma - 1} \cdot \left(\frac{\gamma}{\gamma - 1} - 1\right) \cdot \left(\frac{\gamma - 1}{2} M^2_\infty\right)^2 + \ldots\right) - 1\right) \cdot \frac{1}{\gamma M^2_\infty} \approx \frac{1}{2}$$

$$T_0 = \frac{T^*_0}{T^*_\infty} = 1 + \frac{\gamma - 1}{2} M^2_\infty.$$

In this case the ranges for p and T are $[0, 0.5]$ and $[1, 1.018]$ for $M_\infty = 0.3$. At the same time the last formulae being applied to the flow with $M_\infty = 2$ give us $[0, 1.219]$ and $[1, 1.8]$.

As is seen, the last system of nondimensionalization is preferable (especially for p) and it is taken in the paper as a basis.

3.2. The Navier-Stokes equations in the dimensionless form

Rewrite the Navier-Stokes equations in the chosen system of the dimensionless variables

$$\frac{\partial}{\partial x}\left(\rho u r^j\right)+\frac{\partial}{\partial y}\left(\rho v r^j\ \frac{R+y}{R}\right)=0, \tag{17}$$

$$\rho\left(\frac{Ru}{R+y}\ \frac{\partial u}{\partial x}+v\ \frac{\partial u}{\partial y}+\frac{uv}{R+y}-\frac{Rw^2}{R+y}\ \frac{\partial \ln r^j}{\partial x}\right)=-\frac{R}{R+y}\ \frac{\partial p}{\partial x}+\frac{1}{Fr}\ \rho\cos\theta+\frac{1}{Re_\infty}\ \Phi_1, \tag{18}$$

$$\rho\left(\frac{Ru}{R+y}\ \frac{\partial v}{\partial x}+v\ \frac{\partial v}{\partial y}-\frac{u^2}{R+y}-w^2\ \frac{\partial \ln r^j}{\partial y}\right)=-\frac{\partial p}{\partial y}-\frac{1}{Fr}\rho\sin\theta+\frac{1}{Re_\infty}\ \Phi_2, \tag{19}$$

$$j\rho\left(\frac{Ru}{R+y}\ \frac{\partial w}{\partial x}+v\ \frac{\partial w}{\partial y}+\frac{R}{R+y}\ uw\ \frac{\partial \ln r^j}{\partial x}+vw\ \frac{\partial \ln r^j}{\partial y}\right)=j\ \frac{1}{Re_\infty}\ \Phi_3, \tag{20}$$

$$\rho\left[\frac{Ru}{R+y}\ \frac{\partial h}{\partial x}+v\ \frac{\partial h}{\partial y}\right]=(\gamma-1)M_\infty^2\left[\frac{Ru}{R+y}\ \frac{\partial p}{\partial x}+v\ \frac{\partial p}{\partial y}\right]+\frac{1}{Pr}\frac{1}{Re_\infty}\left[\frac{\partial}{\partial x}\left(\mu\frac{R}{R+y}\frac{\partial h}{\partial x}\right)+\right.$$
$$\left.\frac{\partial}{\partial y}\left(\mu\ \frac{\partial h}{\partial y}\right)+\mu\left(\frac{R}{R+y}\right)^2\frac{\partial \ln r^j}{\partial x}\ \frac{\partial h}{\partial x}++\mu\ \left(\frac{1}{R+y}+\frac{\partial \ln r^j}{\partial y}\right)\ \frac{\partial h}{\partial y}\right]+(\gamma-1)\frac{M_\infty^2}{Re_\infty}\ \Phi_4, \tag{21}$$

$$\rho h=1+\gamma M_\infty^2 p, \tag{22}$$

$$\mu=h^\omega, \tag{23}$$

where $Re_\infty=\rho_\infty^* V_\infty^* a^*/\mu_\infty^*$, $Pr=\mu_\infty^* c_{\rm p}^*/\lambda_\infty^*$, $Fr=V_\infty^{*2}/g^*a^*$ are the Reynolds number, the Prandtl number and the Frud number in the free stream. The relations for Φ_1-Φ_4 have the same form as (8-11) if the asterisk is omitted.

The dimensionless boundary conditions take the form

$$y=0:\qquad u=v=0,\quad w=w_{\rm w},\quad h=h_{\rm w}, \tag{24}$$

$$y\to\infty:\qquad u\to\cos\theta,\quad v\to-\sin\theta,\quad w\to 0,\quad \rho\to 1,\quad h\to 1. \tag{25}$$

3.3. New dimensionless variables

As indicated above the main idea of nondimensionalization is to obtain parameters which vary from 0 to $O(1)$. In the chosen system (14, 16) this condition does not assert for T and, as a result, for the gas parameters connected with the temperature, such as the density and the viscosity. Try to satisfy the requirements and introduce new dimensionless variables, differing from each others in the mentioned areas, namely

- in the inviscid external area

$$\vartheta^o=\frac{T^*-T_\infty^*}{T_0^*-T_\infty^*}=\frac{T-1}{T_0-1},\qquad \hat{\rho^o}=\frac{\rho^*-\rho_\infty^*}{\rho_0^*-\rho_\infty^*}=\frac{\rho-1}{\rho_0-1},\qquad \hat{\mu^o}=\frac{\mu^*-\mu_\infty^*}{\mu_0^*-\mu_\infty^*}=\frac{\mu-1}{\mu_0-1}, \tag{26}$$

- inside the boundary layer

$$\vartheta^i=\frac{T^*-T_{\rm w}^*}{T_0^*-T_{\rm w}^*}=\frac{T-T_{\rm w}}{T_0-T_{\rm w}},\qquad \hat{\rho^i}=\frac{\rho^*-\rho_{\rm w}^*}{\rho_0^*-\rho_{\rm w}^*}=\frac{\rho-\rho_{\rm w}}{\rho_0-\rho_{\rm w}},\qquad \hat{\mu^i}=\frac{\mu^*-\mu_{\rm w}^*}{\mu_0^*-\mu_{\rm w}^*}=\frac{\mu-\mu_{\rm w}}{\mu_0-\mu_{\rm w}}. \tag{27}$$

As a result, all governing parameters that can be both small and not small are indicated explicitly as the coefficients in the equation system. Among these parameters are M_∞^2 and $\Delta T/T_0$ (ΔT is the temperature difference and T_0 is the characteristic value of the temperature in the area).

4. Method of the matched asymptotic expansions

We consider the gas flows over bodies with the large Reynolds number. In this case we can divide the flow area into two parts: the external inviscid one and the narrow area near the body surface known as a viscous boundary layer. Mathematically, it means that the Navier-Stokes equations are splitted into the Eulier equations for the external inviscid flow area and the Prandtl boundary layer equations. Such splitting of the problem is obtained on the basis of the matched asymptotic expansions method proposed by Van Dike for hypersonic flow [4]. All gas parameters are written in the form of power series in the small parameter $\varepsilon = 1/\sqrt{Re_\infty}$:

- for the outer expansion

$$\psi^o = \psi_1^o + \varepsilon\psi_1^o + \varepsilon^2\psi_2^o + \cdots, \qquad \text{where } \psi^o = \{u, v, w, p, \rho, T(\text{and } h), \mu(\text{and } \lambda)\}, \tag{28}$$

- for the inner expansion (we introduce the so-called extended variable $\eta = y\sqrt{Re_\infty} = y/\varepsilon$)

$$\psi^i = \psi_1^i + \varepsilon\psi_1^i + \varepsilon^2\psi_2^i + \cdots, \qquad \text{where } \psi^o = \{u, w, p, \rho, T(\text{and } h), \mu(\text{and } \lambda)\}$$
$$\text{and } v^i = \varepsilon v_1^i + \varepsilon^2 v_2^i + \varepsilon^3 v_3^i + \cdots. \tag{29}$$

As the result the system of the Navier-Stokes equations reduces to the sequences of partial differential equation systems in the external flow area and the internal flow area. In particular, the first approximation gives us the Euler equations (in the external flow area) and the Prandtl boundary layer equations.

5. The equations for the external flow area

5.1. Relations between the dimensionless variables

Derive the relationship between variables (16) and (26) accepted for the nondimensionalization in the external inviscid area

$$h = T = 1 + \vartheta^o(T_0 - 1) = 1 + \vartheta^o \cdot \left(1 + \frac{\gamma - 1}{2} M_\infty^2 - 1\right) = 1 + \frac{\gamma - 1}{2} M_\infty^2 \cdot \vartheta^o, \tag{30}$$

$$\rho = 1 + \hat{\rho}^o(\rho_0 - 1) = 1 + \hat{\rho}^o \cdot \left(\left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{\frac{1}{\gamma - 1}} - 1\right) = 1 + A_\rho \cdot \hat{\rho}^o, \tag{31}$$

$$\mu = 1 + \hat{\mu}^o(\mu_0 - 1) = 1 + \hat{\mu}^o \cdot \left(\left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{\omega} - 1\right) = 1 + A_\mu \cdot \hat{\mu}^o. \tag{32}$$

These formulae are valid for the different arbitrary Mach number, but in the case of the hyposonic flows they can be simplified as follows

$$A_\rho = \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{\frac{1}{\gamma - 1}} - 1 \overset{(M_\infty \ll 1)}{\approx} 1 + \frac{1}{\gamma - 1} \cdot \frac{\gamma - 1}{2} M_\infty^2 + \cdots - 1 = \frac{1}{2} M_\infty^2 + \ldots, \tag{33}$$

$$A_\mu = \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{\omega} - 1 \overset{(M_\infty \ll 1)}{\approx} 1 + \omega \cdot \frac{\gamma - 1}{2} M_\infty^2 + \cdots - 1 = \omega \cdot \frac{\gamma - 1}{2} M_\infty^2. \tag{34}$$

In other words, the coefficients A_ρ and A_μ are the small parameters in the hyposonic flow.

5.2. The asymptotic analysis in the inviscid area

The Navier-Stokes equations in terms of the new variables contain three parameters $\varepsilon^2 = 1/Re_\infty$, M_∞ and $1/Fr$. It should be noticed that in the hyposonic flows the gravity force can affect significantly the gas flow (in studies of free and induced convection). The full asymptotic analysis is carried out on the basis of comparison of M_∞^2 and $1/Fr$ with the small parameter ε in order of magnitude [4].

The results for the first approximation give us the Euler equation for the cases of incompressible ($M_\infty \ll 1$) and compressible ($M_\infty \sim 1$) flow. These equations, well-known, but written in the new variables, are excluded from the text of the paper due to the paper length restrictions.

6. The boundary layer equations

6.1. Relations between the dimensionless variables

Derive the relationship between variables (16) and (27)

$$h = T = T_{\rm w} + \vartheta^i (T_0 - T_{\rm w}) = T_0 \cdot \left[\frac{T_{\rm w}}{T_0} + \vartheta^i \cdot \left(1 - \frac{T_{\rm w}}{T_0} \right) \right] =$$
$$\left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right) \cdot [1 - \varepsilon_{\rm w} + \vartheta^i \varepsilon_{\rm w}] = B_{h1} + B_{h2} \cdot \vartheta^i, \tag{35}$$

where $\varepsilon_{\rm w}$ is the parameter which can be both small and not small

$$\varepsilon_{\rm w} = 1 - T_{\rm w}/T_0; \tag{36}$$

$$\rho = \rho_{\rm w} + \hat{\rho^i}(\rho_0 - \rho_{\rm w}) = \rho_0 \cdot \left[\frac{\rho_{\rm w}}{\rho_0} + \hat{\rho^i} \cdot \left(1 - \frac{\rho_{\rm w}}{\rho_0} \right) \right] = \rho_0 \cdot \left[\frac{p_{\rm w}}{p_0} \cdot \frac{T_0}{T_{\rm w}} + \hat{\rho^i} \left(1 - \frac{p_{\rm w}}{p_0} \cdot \frac{T_0}{T_{\rm w}} \right) \right] =$$
$$\left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right)^{\frac{1}{\gamma - 1}} \cdot \left[\frac{p_{\rm w}}{p_0} \cdot \frac{1}{1 - \varepsilon_{\rm w}} + \hat{\rho^i} \left(1 - \frac{p_{\rm w}}{p_0} \cdot \frac{1}{1 - \varepsilon_{\rm w}} \right) \right] = B_{\rho 1} + B_{\rho 2} \cdot \hat{\rho^i}, \tag{37}$$

$$\mu = \mu_{\rm w} + \hat{\mu^i}(\mu_0 - \mu_{\rm w}) = \mu_0 \cdot \left[\frac{\mu_{\rm w}}{\mu_0} + \hat{\mu^i} \cdot \left(1 - \frac{\mu_{\rm w}}{\mu_0} \right) \right] = T_0^\omega \cdot \left[\left(\frac{T_{\rm w}}{T_0} \right)^\omega + \hat{\mu^i} \ \left(1 - \left(\frac{T_{\rm w}}{T_0} \right)^\omega \right) \right] =$$
$$\left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right)^\omega \ \left[(1 - \varepsilon_{\rm w})^\omega + \hat{\mu^i} \ (1 - (1 - \varepsilon_{\rm w})^\omega) \right] = B_{\mu 1} + B_{\mu 2} \cdot \hat{\mu^i}. \tag{38}$$

These formulae are valid for the different arbitrary Mach number and $\varepsilon_{\rm w}$, but they can be simplified in the following cases

$$h = B_{h1} + B_{h2} \ \vartheta^i = \begin{cases} \left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right), & \text{if } M_\infty^2 \gg \varepsilon, \ \varepsilon_{\rm w} = O(\varepsilon), \\ 1 - \varepsilon_{\rm w} + \varepsilon_{\rm w} \vartheta^i, & \text{if } M_\infty^2 = O(\varepsilon), \ \varepsilon_{\rm w} \gg \varepsilon, \\ 1, & \text{if } M_\infty^2 = O(\varepsilon), \ \varepsilon_{\rm w} = O(\varepsilon), \end{cases} \tag{39}$$

$$\rho^i = B_{\rho 1} + B_{\rho 2} \ \hat{\rho^i} = \begin{cases} \left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right)^{\frac{1}{\gamma - 1}} \left[\frac{p_{\rm w}}{p_0} + \left(1 - \frac{p_{\rm w}}{p_0} \right) \hat{\rho^i} \right], & \text{if } M_\infty^2 \gg \varepsilon, \ \varepsilon_{\rm w} = O(\varepsilon), \\ \frac{p_{\rm w}}{p_0} \frac{1}{1 - \varepsilon_{\rm w}} + \left(1 - \frac{p_{\rm w}}{p_0} \frac{1}{1 - \varepsilon_{\rm w}} \right) \ \hat{\rho^i}, & \text{if } M_\infty^2 = O(\varepsilon), \ \varepsilon_{\rm w} \gg \varepsilon, \\ \frac{p_{\rm w}}{p_0} + \left(1 - \frac{p_{\rm w}}{p_0} \right) \ \hat{\rho^i}, & \text{if } M_\infty^2 = O(\varepsilon), \varepsilon_{\rm w} = O(\varepsilon), \end{cases} \tag{40}$$

$$\mu^i = B_{\mu 1} + B_{\mu 2} \ \hat{\mu^i} = \begin{cases} \left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right)^\omega, & \text{if } M_\infty^2 \gg \varepsilon, \ \varepsilon_{\rm w} = O(\varepsilon), \\ (1 - \varepsilon_{\rm w})^\omega + (1 - (1 - \varepsilon_{\rm w})^\omega) \ \hat{\mu^i}, & \text{if } M_\infty^2 = O(\varepsilon), \ \varepsilon_{\rm w} \gg \varepsilon, \\ 1, & \text{if } M_\infty^2 = O(\varepsilon), \ \varepsilon_{\rm w} = O(\varepsilon). \end{cases} \tag{41}$$

6.2. The Navier-Stokes equations in terms of the new variables

Rewrite the Navier-Stokes equations in terms of the dimensionless variables (16) and (27)

$$\frac{\partial}{\partial x} \left((B_{\rho 1} + B_{\rho 2} \ \hat{\rho^i}) u^i r^j \right) + \frac{\partial}{\partial y} \left((B_{\rho 1} + B_{\rho 2} \ \hat{\rho^i}) v^i r^j \ \frac{R + y}{R} \right) = 0, \tag{42}$$

$$(B_{\rho 1} + B_{\rho 2} \ \hat{\rho^i}) \left(\frac{R u^i}{R + y} \ \frac{\partial u^i}{\partial x} + v^i \ \frac{\partial u^i}{\partial y} + \frac{u^i v^i}{R + y} - \frac{R w^{i2}}{R + y} \ \frac{\partial \ln r^j}{\partial x} \right) = - \frac{R}{R + y} \ \frac{\partial p^i}{\partial x} +$$
$$\frac{1}{Fr} \ (B_{\rho 1} + B_{\rho 2} \ \hat{\rho^i}) \cos \theta + \frac{1}{Re_\infty} \ \Phi_1, \tag{43}$$

$$(B_{\rho 1} + B_{\rho 2} \ \hat{\rho^i}) \left(\frac{R u^i}{R + y} \ \frac{\partial v^i}{\partial x} + v^i \ \frac{\partial v^i}{\partial y} - \frac{u^{i2}}{R + y} - w^{i2} \ \frac{\partial \ln r^j}{\partial y} \right) = - \frac{\partial p}{\partial y} -$$
$$\frac{1}{Fr} (B_{\rho 1} + B_{\rho 2} \ \hat{\rho^i}) \sin \theta + \frac{1}{Re_\infty} \ \Phi_2, \tag{44}$$

$$j (B_{\rho 1} + B_{\rho 2} \ \hat{\rho^i}) \left(\frac{R u^i}{R + y} \ \frac{\partial w^i}{\partial x} + v^i \ \frac{\partial w^i}{\partial y} + \frac{R}{R + y} \ u^i w^i \ \frac{\partial \ln r^j}{\partial x} + v^i w^i \ \frac{\partial \ln r^j}{\partial y} \right) = j \frac{1}{Re_\infty} \ \Phi_3, \tag{45}$$

$$(B_{\rho 1}+B_{\rho 2}\;\hat{\rho^i})B_{h2}\cdot\left(\frac{Ru^i}{R+y}\;\frac{\partial\vartheta^i}{\partial x}+v^i\;\frac{\partial\vartheta^i}{\partial y}\right)=(\gamma-1)M_\infty^2\left(\frac{Ru^i}{R+y}\;\frac{\partial p^i}{\partial x}+v^i\;\frac{\partial p^i}{\partial y}\right)+\frac{B_{h2}}{Pr}\cdot$$
$$\frac{1}{Re_\infty}\;\left[\frac{\partial}{\partial x}\left((B_{\mu 1}+B_{\mu 2}\;\hat{\mu^i})\;\frac{R}{R+y}\;\frac{\partial\vartheta^i}{\partial x}\right)+\frac{\partial}{\partial y}\left((B_{\mu 1}+B_{\mu 2}\;\hat{\mu^i})\;\frac{\partial\vartheta^i}{\partial y}\right)+(B_{\mu 1}+B_{\mu 2}\;\hat{\mu^i})\cdot\right.$$
$$\left.\left(\frac{R}{R+y}\right)^2\frac{\partial\ln r^j}{\partial x}\;\frac{\partial\vartheta^i}{\partial x}+(B_{\mu 1}+B_{\mu 2}\;\hat{\mu^i})\;\left(\frac{1}{R+y}+\frac{\partial\ln r^j}{\partial y}\right)\;\frac{\partial\vartheta^i}{\partial y}\right]+(\gamma-1)\frac{M_\infty^2}{Re_\infty}\;\Phi_4,\quad(46)$$

$$(B_{\rho 1}+B_{\rho 2}\;\hat{\rho^i})\cdot\left(B_{h1}+B_{h2}\;\vartheta^i\right)=1+(\gamma-1)M_\infty^2 p^i,\qquad(47)$$

$$(B_{\mu 1}+B_{\mu 2}\;\hat{\mu^i})=\left(B_{h1}+B_{h2}\;\vartheta^i\right)^\omega.\qquad(48)$$

and the boundary conditions

$$y=0:\qquad u^i=v^i=0,\quad w^i=w_{\rm w},\quad\vartheta^i=1,\qquad(49)$$

$$y\to\infty:\qquad u^i\to\cos\theta,\quad v^i\to-\sin\theta,\quad w^i\to 0,\quad\rho^i\to\rho^i_\infty,\quad\vartheta^i\to\vartheta^i_\infty.\qquad(50)$$

6.3. The asymptotic analysis in the boundary layer

The equations (42-48) contain four parameters $\varepsilon^2=1/Re_\infty$, M_∞, $\varepsilon_{\rm w}$ and $1/Fr$. The full asymptotic analysis is carried out on the base of comparison of all parameters in the boundary layer (M_∞^2, $\varepsilon_{\rm w}$, $1/Fr$) with the standard small parameter ε in order of magnitude [4].

Substitute the power series in ε for all gas parameters into the system (42-48). All coefficients in all items of Φ_1 - Φ_4 are of orders of unit, $1/\varepsilon$ and $1/\varepsilon^2$. It means that it is necessary to take them into account in all expansions.

To derive the equations inside the boundary layer, take into account the following expansions

$$r(x,y)=r_{\rm w}(x)+y\cos\theta=r_{\rm w}(x)+\varepsilon\eta\cos\theta(x)\qquad(51)$$

$$\frac{R}{R+y}=\frac{R}{R+\varepsilon\eta}=\frac{1}{1+\varepsilon\cdot\frac{\eta}{R}}\overset{(\varepsilon\ll 1)}{\approx}1-\varepsilon\;\frac{\eta}{R}+\cdots,\qquad(52)$$

$$\frac{\partial\ln r^j}{\partial x}=\frac{j}{r}\cdot\frac{\partial r}{\partial x}=j\;\frac{1}{r_{\rm w}(x)+\varepsilon\eta\cos\theta(x)}\;(r'_{\rm w}-\varepsilon\sin\theta\;\theta')\overset{(\varepsilon\ll 1)}{\approx}$$
$$\frac{j}{r_{\rm w}}\;\left(1-\varepsilon\frac{\eta}{r_{\rm w}}\cos\theta+\dots\right)\;(r'_{\rm w}-\varepsilon\sin\theta\;\theta')=j\;\frac{r'_{\rm w}}{r_{\rm w}}+o(\varepsilon),\qquad(53)$$

$$\frac{\partial\ln r^j}{\partial y}=\frac{j}{r}\cdot\frac{\partial r}{\partial y}=\frac{j}{r}\cos\theta=\frac{j}{r_{\rm w}(x)+\varepsilon\eta\cos\theta(x)}\;\cos\theta\overset{(\varepsilon\ll 1)}{\approx}$$
$$\frac{j}{r_{\rm w}}\;\left(1-\varepsilon\frac{\eta}{r_{\rm w}}\cos\theta+\dots\right)\;\cos\theta=\frac{j\cos\theta}{r_{\rm w}}+o(\varepsilon).\qquad(54)$$

Let's first consider the eq. (44) in the case when $M_\infty^2\gg\varepsilon$ and $\varepsilon_{\rm w}\gg\varepsilon$. The equation of the first approximation can be obtained after the equalization of the coefficients at $1/\varepsilon$. If we assume that the coefficient $B_{\rho 1}$ and $B_{\rho 2}$ are not the values of order of $1/\varepsilon$ (in other words, that $\frac{p_{\rm w}}{p_0}\neq O\left(\frac{1}{\varepsilon}\right)$) the equation (44) can be transferred to the form $0=-\frac{\partial p_1^i}{\partial\eta}$, that leads to the fact that p_1^i is the constant value across the boundary layer. Therefore hereinafter in the the paper we can put $p_{\rm w}=p_0$ and rewrite the relation (40) in the form

$$\rho^i=B_{\rho 1}+B_{\rho 2}\;\hat{\rho^i}=\begin{cases}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}}\;\left(\frac{1}{1-\varepsilon_{\rm w}}-\frac{\varepsilon_{\rm w}}{1-\varepsilon_{\rm w}}\;\hat{\rho^i}\right), & \text{if } M_\infty^2\gg\varepsilon,\ \varepsilon_{\rm w}\gg\varepsilon,\\ \left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}}, & \text{if } M_\infty^2\gg\varepsilon,\ \varepsilon_{\rm w}=O(\varepsilon),\\ \dfrac{1}{1-\varepsilon_{\rm w}}-\dfrac{\varepsilon_{\rm w}}{1-\varepsilon_{\rm w}}\;\hat{\rho^i}, & \text{if } M_\infty^2=O(\varepsilon),\ \varepsilon_{\rm w}\gg\varepsilon,\\ 1, & \text{if } M_\infty^2=O(\varepsilon),\varepsilon_{\rm w}=O(\varepsilon),\end{cases}\qquad(55)$$

Write the equations system in four different cases:

For the non-hyposonic and non-isothermal boundary layer (when $M_\infty^2 \gg \varepsilon$ and $\varepsilon_\mathrm{w} \gg \varepsilon$)

$$\frac{\partial}{\partial x}\left[(1-\varepsilon_\mathrm{w}\rho_1^i)u_1^i r_\mathrm{w}^j\right] + \frac{\partial}{\partial \eta}\left[(1-\varepsilon_\mathrm{w}\rho_1^i)v_1^i r_\mathrm{w}^j\right] = 0, \tag{56}$$

$$\begin{gathered}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}} \left\{\frac{1}{1-\varepsilon_\mathrm{w}} - \frac{\varepsilon_\mathrm{w}}{1-\varepsilon_\mathrm{w}}\ \rho_1^i\right\}\left[u_1^i\ \frac{\partial u_1^i}{\partial x} + u_1^i\ \frac{\partial u_1^i}{\partial x} + v_1^i \frac{\partial u_1^i}{\partial \eta} - \right. \\ \left. j\frac{r'_\mathrm{w}}{r_\mathrm{w}} w_1^{i2}\right] = -\frac{\partial p_1^i}{\partial x} + \left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\omega} \frac{\partial}{\partial \eta}\left[\left[(1-\varepsilon_\mathrm{w})^\omega + (1-(1-\varepsilon_\mathrm{w})^\omega)\,\mu_1^i\right]\cdot\frac{\partial u_1^i}{\partial \eta}\right] + \\ \begin{cases} \frac{1}{Fr}\ \left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}}\ \left\{\frac{1}{1-\varepsilon_\mathrm{w}} - \frac{\varepsilon_\mathrm{w}}{1-\varepsilon_\mathrm{w}}\ \rho_1^i\right\}\cos\theta, & \text{if } 1/Fr \gg \varepsilon, \\ 0, & \text{if } 1/Fr = O(\varepsilon), \end{cases}\end{gathered} \tag{57}$$

$$0 = -\ \frac{\partial p_1^i}{\partial \eta}, \tag{58}$$

$$\begin{gathered} j\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}-\omega} \left\{\frac{1}{1-\varepsilon_\mathrm{w}} - \frac{\varepsilon_\mathrm{w}}{1-\varepsilon_\mathrm{w}}\ \rho_1^i\right\}\left[u_1^i\ \frac{\partial w_1^i}{\partial x} + v_1^i\ \frac{\partial w_1^i}{\partial \eta} + \right. \\ \left. u_1^i w_1^i\ j\frac{r'_\mathrm{w}}{r_\mathrm{w}}\right] = j\ \frac{\partial}{\partial \eta}\left[\left[(1-\varepsilon_\mathrm{w})^\omega + (1-(1-\varepsilon_\mathrm{w})^\omega)\,\mu_1^i\right]\ \frac{\partial w_1^i}{\partial \eta}\right], \end{gathered} \tag{59}$$

$$\begin{gathered}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{\gamma}{\gamma-1}} \left\{\frac{1}{1-\varepsilon_\mathrm{w}} - \frac{\varepsilon_\mathrm{w}}{1-\varepsilon_\mathrm{w}}\ \rho_1^i\right\}\varepsilon_\mathrm{w}\left[u_1^i\ \frac{\partial \vartheta_1^i}{\partial x} + v_1^i \frac{\partial \vartheta_1^i}{\partial \eta}\ \right] = (\gamma-1)M_\infty^2\left(u_1^i\ \frac{\partial p_1^i}{\partial x} + \right. \\ \left. v_1^i\ \frac{\partial p_1^i}{\partial \eta}\right) + \varepsilon_\mathrm{w}\ \frac{1}{Pr}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\omega+1}\left[\frac{\partial}{\partial \eta}\left(\left[(1-\varepsilon_\mathrm{w})^\omega + (1-(1-\varepsilon_\mathrm{w})^\omega)\,\mu_1^i\right]\ \frac{\partial \vartheta_1^i}{\partial \eta}\ \right)\right] + \\ (\gamma-1)M_\infty^2\ \left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\omega}\ \left[(1-\varepsilon_\mathrm{w})^\omega + (1-(1-\varepsilon_\mathrm{w})^\omega)\,\mu_1^i\right]\left[\left(\frac{\partial u_1^i}{\partial \eta}\right)^2 + \left(\frac{\partial w_1^i}{\partial \eta}\right)^2\right], \end{gathered} \tag{60}$$

$$\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}\ \left\{\frac{1}{1-\varepsilon_\mathrm{w}} - \frac{\varepsilon_\mathrm{w}}{1-\varepsilon_\mathrm{w}}\ \rho_1^i\right\}\cdot\left[1-\varepsilon_\mathrm{w}+\varepsilon_\mathrm{w}\vartheta_1^i\right] = 1 + (\gamma-1)M_\infty^2 p_1^i, \tag{61}$$

$$\left[(1-\varepsilon_\mathrm{w})^\omega + (1-(1-\varepsilon_\mathrm{w})^\omega)\,\mu_1^i\right] = \left[\left[1-\varepsilon_\mathrm{w}+\varepsilon_\mathrm{w}\vartheta_1^i\right]\right]^\omega. \tag{62}$$

The boundary conditions at the body surface

$$\eta = 0: \qquad u_1^i = v_1^i = 0, \quad w_1^i = w_\mathrm{w}, \quad \vartheta_1^i = 0. \tag{63}$$

For the non-hyposonic and isothermal boundary layer (when $M_\infty^2 \gg \varepsilon$ and $\varepsilon_\mathrm{w} = O(\varepsilon)$)

$$\frac{\partial}{\partial x}\left(u_1^i r_\mathrm{w}^j\right) + \frac{\partial}{\partial \eta}\left(v_1^i r_\mathrm{w}^j\right) = 0, \tag{64}$$

$$\begin{gathered}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}}\ \left(u_1^i\ \frac{\partial u_1^i}{\partial x} + u_1^i\ \frac{\partial u_1^i}{\partial x} + v_1^i \frac{\partial u_1^i}{\partial \eta}\ - j\frac{r'_\mathrm{w}}{r_\mathrm{w}} w_1^{i2}\right) = -\frac{\partial p_1^i}{\partial x} + \\ \left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\omega}\ \frac{\partial^2 u_1^i}{\partial \eta^2} + \begin{cases} \frac{1}{Fr}\ \left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}}\cos\theta, & \text{if } 1/Fr \gg \varepsilon, \\ 0, & \text{if } 1/Fr = O(\varepsilon), \end{cases}\end{gathered} \tag{65}$$

$$0 = -\ \frac{\partial p_1^i}{\partial \eta}, \tag{66}$$

$$j\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}-\omega}\left(u_1^i\ \frac{\partial w_1^i}{\partial x} + v_1^i\ \frac{\partial w_1^i}{\partial \eta} + u_1^i w_1^i\ j\frac{r'_\mathrm{w}}{r_\mathrm{w}}\right) = j\frac{\partial^2 w_1^i}{\partial \eta^2}, \tag{67}$$

$$0 = \left(u_1^i \; \frac{\partial p_1^i}{\partial x} + v_1^i \; \frac{\partial p_1^i}{\partial \eta} \right) + \left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right)^\omega \left[\left(\frac{\partial u_1^i}{\partial \eta} \right)^2 + \left(\frac{\partial w_1^i}{\partial \eta} \right)^2 \right], \tag{68}$$

$$\left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right)^{\frac{\gamma}{\gamma - 1}} = 1 + (\gamma - 1) M_\infty^2 p_1^i, \tag{69}$$

$$1 = 1. \tag{70}$$

The boundary conditions at the body surface

$$\eta = 0 : \qquad u_1^i = v_1^i = 0, \quad w_1^i = w_\mathrm{w}. \tag{71}$$

The last system can be simplified. From the equation (69) it follows that $p_1^i = \mathrm{const}$. As the result the equation (66) can be excluded from the system and the equation (68) degenerates to $0 = \left(\frac{\partial u_1^i}{\partial \eta} \right)^2 + \left(\frac{\partial w_1^i}{\partial \eta} \right)^2$, which denotes that $0 = \frac{\partial u_1^i}{\partial \eta}$ and $0 = \frac{\partial w_1^i}{\partial \eta}$.

Finally we receive

$$\frac{\partial}{\partial x} \left(u_1^i r_\mathrm{w}^j \right) + \frac{\partial}{\partial \eta} \left(v_1^i r_\mathrm{w}^j \right) = 0, \tag{72}$$

$$\begin{gathered} \left(u_1^i \; \frac{\partial u_1^i}{\partial x} + u_1^i \; \frac{\partial u_1^i}{\partial x} \; - j \frac{r'_\mathrm{w}}{r_\mathrm{w}} w_1^{i2} \right) = \\ \left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right)^{\omega - \frac{1}{\gamma - 1}} \cdot \frac{\partial^2 u_1^i}{\partial \eta^2} + \begin{cases} \frac{1}{Fr} \;\; \cos\theta, & \text{if } 1/Fr \gg \varepsilon, \\ 0, & \text{if } 1/Fr = O(\varepsilon), \end{cases} \end{gathered} \tag{73}$$

$$j \left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right)^{\frac{1}{\gamma - 1} - \omega} \left(u_1^i \; \frac{\partial w_1^i}{\partial x} + u_1^i w_1^i \; j \frac{r'_\mathrm{w}}{r_\mathrm{w}} \right) = j \frac{\partial^2 w_1^i}{\partial \eta^2}, \tag{74}$$

$$\left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right)^{\frac{\gamma}{\gamma - 1}} = 1 + (\gamma - 1) M_\infty^2 p_1^i, \tag{75}$$

For the hyposonic and non-isothermal boundary layer (when $M_\infty^2 = O(\varepsilon)$ and $\varepsilon_\mathrm{w} \gg \varepsilon$)

$$\frac{\partial}{\partial x} \left[(1 - \varepsilon_\mathrm{w} \rho_1^i) u_1^i r_\mathrm{w}^j \right] + \frac{\partial}{\partial \eta} \left[(1 - \varepsilon_\mathrm{w} \rho_1^i) v_1^i r_\mathrm{w}^j \right] = 0, \tag{76}$$

$$\begin{gathered} \left\{ \frac{1}{1 - \varepsilon_\mathrm{w}} - \frac{\varepsilon_\mathrm{w}}{1 - \varepsilon_\mathrm{w}} \; \rho_1^i \right\} \left[u_1^i \; \frac{\partial u_1^i}{\partial x} + u_1^i \; \frac{\partial u_1^i}{\partial x} + v_1^i \frac{\partial u_1^i}{\partial \eta} \; - j \frac{r'_\mathrm{w}}{r_\mathrm{w}} w_1^{i2} \right] = - \frac{\partial p_1^i}{\partial x} \; + \frac{\partial}{\partial \eta} \left[\left[(1 - \varepsilon_\mathrm{w})^\omega + \right. \right. \\ \left. \left. (1 - (1 - \varepsilon_\mathrm{w})^\omega) \;\; \mu_1^i \right] \frac{\partial u_1^i}{\partial \eta} \right] + \begin{cases} \frac{1}{Fr} \; \left\{ \frac{1}{1 - \varepsilon_\mathrm{w}} - \frac{\varepsilon_\mathrm{w}}{1 - \varepsilon_\mathrm{w}} \; \rho_1^i \right\} \cos\theta, & \text{if } 1/Fr \gg \varepsilon, \\ 0, & \text{if } 1/Fr = O(\varepsilon), \end{cases} \end{gathered} \tag{77}$$

$$0 = - \; \frac{\partial p_1^i}{\partial \eta}, \tag{78}$$

$$\begin{gathered} j \left\{ \frac{1}{1 - \varepsilon_\mathrm{w}} - \frac{\varepsilon_\mathrm{w}}{1 - \varepsilon_\mathrm{w}} \; \rho_1^i \right\} \left[u_1^i \; \frac{\partial w_1^i}{\partial x} + v_1^i \; \frac{\partial w_1^i}{\partial \eta} + u_1^i w_1^i \; j \frac{r'_\mathrm{w}}{r_\mathrm{w}} \right] = \\ j \frac{\partial}{\partial \eta} \left[\left[(1 - \varepsilon_\mathrm{w})^\omega + (1 - (1 - \varepsilon_\mathrm{w})^\omega) \;\; \mu_1^i \right] \frac{\partial w_1^i}{\partial \eta} \right], \end{gathered} \tag{79}$$

$$\begin{gathered} \left\{ \frac{1}{1 - \varepsilon_\mathrm{w}} - \frac{\varepsilon_\mathrm{w}}{1 - \varepsilon_\mathrm{w}} \; \rho_1^i \right\} \; \varepsilon_\mathrm{w} \cdot \left[u_1^i \; \frac{\partial \vartheta_1^i}{\partial x} + v_1^i \frac{\partial \vartheta_1^i}{\partial \eta} \; \right] = (\gamma - 1) M_\infty^2 \left(u_1^i \; \frac{\partial p_1^i}{\partial x} + v_1^i \; \frac{\partial p_1^i}{\partial \eta} \right) + \\ \frac{\varepsilon_\mathrm{w}}{Pr} \; \left[\frac{\partial}{\partial \eta} \left(\left[(1 - \varepsilon_\mathrm{w})^\omega + (1 - (1 - \varepsilon_\mathrm{w})^\omega) \;\; \mu_1^i \right] \; \frac{\partial \vartheta_1^i}{\partial \eta} \; \right) \right] + \\ (\gamma - 1) M_\infty^2 \;\; \left[(1 - \varepsilon_\mathrm{w})^\omega + (1 - (1 - \varepsilon_\mathrm{w})^\omega) \;\; \mu_1^i \right] \left[\left(\frac{\partial u_1^i}{\partial \eta} \right)^2 + \left(\frac{\partial w_1^i}{\partial \eta} \right)^2 \right], \end{gathered} \tag{80}$$

$$\left\{\frac{1}{1-\varepsilon_{\rm w}} - \frac{\varepsilon_{\rm w}}{1-\varepsilon_{\rm w}}\ \rho_1^i\right\} \cdot \left(1-\varepsilon_{\rm w}+\varepsilon_{\rm w}\vartheta_1^i\right) = 1 + (\gamma-1)M_\infty^2 p_1^i, \tag{81}$$

$$(1-\varepsilon_{\rm w})^\omega + (1-(1-\varepsilon_{\rm w})^\omega)\ \mu_1^i = \left(1-\varepsilon_{\rm w}+\varepsilon_{\rm w}\vartheta_1^i\right)^\omega. \tag{82}$$

The boundary conditions on the body surface

$$\eta = 0: \qquad u_1^i = v_1^i = 0, \quad w_1^i = w_{\rm w}, \quad \vartheta_1^i = 0. \tag{83}$$

For the hyposonic and isothermal boundary layer (when $M_\infty^2 = O(\varepsilon)$ and $\varepsilon_{\rm w} = O(\varepsilon)$)

$$\frac{\partial}{\partial x}\left[u_1^i r_{\rm w}^j\right] + \frac{\partial}{\partial \eta}\left[v_1^i r_{\rm w}^j\right] = 0, \tag{84}$$

$$u_1^i\ \frac{\partial u_1^i}{\partial x} + u_1^i \frac{\partial u_1^i}{\partial x} + v_1^i \frac{\partial u_1^i}{\partial \eta} - j\frac{r_{\rm w}'}{r_{\rm w}} w_1^{i2} = -\frac{\partial p_1^i}{\partial x} + \frac{\partial^2 u_1^i}{\partial \eta^2} + \begin{cases} \frac{1}{Fr}\left\{\frac{1-\varepsilon_{\rm w}\rho_1^i}{1-\varepsilon_{\rm w}}\right\}\cos\theta, & \text{if } 1/Fr \gg \varepsilon, \\ 0, & \text{if } 1/Fr = O(\varepsilon), \end{cases} \tag{85}$$

$$0 = -\ \frac{\partial p_1^i}{\partial \eta}, \tag{86}$$

$$j\left[u_1^i\ \frac{\partial w_1^i}{\partial x} + v_1^i\ \frac{\partial w_1^i}{\partial \eta} + u_1^i w_1^i\ j\frac{r_{\rm w}'}{r_{\rm w}}\right] = j\frac{\partial^2 w_1^i}{\partial \eta^2}, \tag{87}$$

$$0 = 0, \tag{88}$$

$$1 = 1, \tag{89}$$

$$1 = 1. \tag{90}$$

The boundary conditions at the body surface

$$\eta = 0: \qquad u_1^i = v_1^i = 0, \quad w_1^i = w_{\rm w}. \tag{91}$$

7. The missing boundary conditions

To obtain the missing conditions at the outer boundary of the boundary layer in the problem of the first approximation the principle of the asymptotic matching of the outer and the inner expansions, formulated by Van Dyke [4], is used. The procedure of the matching of the one-term outer and one-term inner expansions applied for the traditional dimensionless variables gives us [3] the following relations

$$u_1^i(x,\infty) = u_1^o(x,0), \quad w_1^i(x,\infty) = w_1^o(x,0), \quad T_1^i(x,\infty) = T_1^o(x,0), \quad \rho_{1old}^i(x,\infty) = \rho_{1old}^o(x,0). \tag{92}$$

In addition for the incompressible gas it is also used $p_{1old}^i(x,\infty) = p_{1old}^o(x,0)$.

Rewrite these relations for T and ρ in new dimensionless variables (the relations for the velocity components and the pressure are the same).

Compare the expansions for the gas parameters in the inviscid area

$$h^o = T^o = T_1^o + \varepsilon T_2^o + \ldots, \qquad h^o = 1 + \frac{\gamma-1}{2}M_\infty^2(\vartheta_1^o + \varepsilon\vartheta_2^o + \ldots),$$

$$\rho^o = \rho_{1old}^o + \varepsilon\rho_{2old}^o + \ldots, \qquad \rho^o = 1 + A_\rho(\rho_1^o + \varepsilon\rho_2^o + \ldots),$$

and inside the boundary layer

$$h^i = T^i = T_1^i + \varepsilon T_2^i + \ldots, \qquad h^i = B_{h1} + B_{h2}(\vartheta_1^i + \varepsilon\vartheta_2^i + \ldots),$$

$$\rho^i = \rho_{1old}^i + \varepsilon\rho_{2old}^i + \ldots, \qquad \rho^i = B_{\rho1} + B_{\rho2}(\rho_1^i + \varepsilon\rho_2^i + \ldots),$$

As the result we obtain

$$1 + \frac{\gamma-1}{2}M_\infty^2\ \vartheta_1^o(x,0) = B_{h1} + B_{h2}\ \vartheta_1^i(x,\infty), \quad 1 + A_\rho\ \rho_1^o(x,0) = B_{\rho1} + B_{\rho2}\ \rho_1^i(x,\infty). \tag{93}$$

Rewrite these equalities for all cases considered in this paper:

$$
\begin{cases}
1+\frac{\gamma-1}{2}M_\infty^2\ \vartheta_1^o(x,0)=\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\cdot \\
\qquad\qquad \left[1-\varepsilon_{\rm w}+\varepsilon_{\rm w}\vartheta_1^i(x,\infty)\right], & \text{if } M_\infty^2\gg\varepsilon,\ \varepsilon_{\rm w}\gg\varepsilon, \\
1+\frac{\gamma-1}{2}M_\infty^2\ \vartheta_1^o(x,0)=\left(1+\frac{\gamma-1}{2}M_\infty^2\right), & \text{if } M_\infty^2\gg\varepsilon,\ \varepsilon_{\rm w}=O(\varepsilon), \\
1=\left[1-\varepsilon_{\rm w}+\varepsilon_{\rm w}\vartheta_1^i(x,\infty)\right], & \text{if } M_\infty^2=O(\varepsilon),\ \varepsilon_{\rm w}\gg\varepsilon, \\
1=1, & \text{if } M_\infty^2=O(\varepsilon),\ \varepsilon_{\rm w}=O(\varepsilon),
\end{cases}
\tag{94}
$$

$$
\begin{cases}
1+\left(\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}}-1\right)\rho_1^o(x,0)= \\
\quad \left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}}\cdot\left[\frac{1}{1-\varepsilon_{\rm w}}-\frac{\varepsilon_{\rm w}}{1-\varepsilon_{\rm w}}\ \rho_1^i(x,\infty)\right], & \text{if } M_\infty^2\gg\varepsilon,\ \varepsilon_{\rm w}\gg\varepsilon, \\
1+\left(\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}}-1\right)\rho_1^o(x,0)= \\
\qquad\qquad \left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{1}{\gamma-1}}, & \text{if } M_\infty^2\gg\varepsilon,\ \varepsilon_{\rm w}=O(\varepsilon), \\
1=\left[\frac{1}{1-\varepsilon_{\rm w}}-\frac{\varepsilon_{\rm w}}{1-\varepsilon_{\rm w}}\ \rho_1^i(x,\infty)\right], & \text{if } M_\infty^2=O(\varepsilon),\ \varepsilon_{\rm w}\gg\varepsilon, \\
1=1, & \text{if } M_\infty^2=O(\varepsilon),\ \varepsilon_{\rm w}=O(\varepsilon).
\end{cases}
\tag{95}
$$

8. Concluding remarks

The full asymptotic analysis of the Navier-Stokes equations for flows over blunt bodies with large Reynolds numbers is carried out under all known assumptions (1. flows with $M_\infty \sim 1$ and the non-isothermal boundary layer; 2. flows with $M_\infty \sim 1$ and the isothermal boundary layer; 3. hyposonic flows ($M_\infty^2 \ll 1$) with the non-isothermal boundary layer; 4. hyposonic flows ($M_\infty^2 \ll 1$) with the isothermal boundary layer). As a result, the model for gas flow in both areas is formulated and an attempt to construct the procedure of the agreement of the solutions in both areas is made. This model constructed for the cases 2) and 3) possibly can occupy the intermediate place between two classical approaches when a gas is considered as incompressible (case 4) or compressible one (case 1) over the whole flow field.

9. Acknowledgements

The author wishes to express her sincere gratitude to Prof. Yu. M. Tsirkunov who attracted an attention to the problem. This research was supported in part by the Russian Foundation for Basic Research through grant No. 05-08-50075.

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