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Domain Decomposition Method for a Semilinear Singularly Perturbed Elliptic Convection-Diffusion Equation with Concentrated Sources *

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Introduction

In the case of regular boundary value problems, effective numerical methods based on domain decomposition are well known (see, for example, [1, 2] and the list of references there). Such methods allows us to reduce the solution of complicated problems with several characteristic scales to solving simplified problems on simpler subdomains and, in particular, to implement parallel computations. Errors of standard numerical methods applied to singularly perturbed problems can many times exceed the solution itself for small values of the singular perturbation parameter ε. For this type of problems, it is of interest to develop numerical methods whose errors are independent of the parameter ε and defined only by the number of mesh points (that is, such methods are said to converge ε-uniformly). When using an iterative method for solving nonlinear problems and/or domain decomposition schemes, we will require that its accuracy is defined only by the number of mesh points, moreover, the number of iterations required for convergence is also independent of ε.

In the case of nonlinear singularly perturbed problems, the attainment of ε-uniform convergence of a numerical methods necessarily requires the use of special meshes whose step-size in a boundary (interior) layer is exceedingly small (see, e.g., [3]). When such problems are decomposed, the number of iterations in the corresponding iterative process can be large and essentially depend on the parameter ε. Therefore, it is important to develop numerical methods based on domain decomposition techniques that converge ε-uniformly. For singularly perturbed problems in a composed domain (in particular, with concentrated sources) whose solution has several singularities such as boundary and interior layers, it is of keen interest to construct such methods so that each subdomain in the domain decomposition contains no more than a single singularity, which essentially simplifies the solution of the problem under consideration.

In this paper, we develop monotone linearized schemes based on an overlapping Schwarz method for a semilinear singularly perturbed elliptic convection-diffusion equation in a composed domain (vertical strip) in the presence of a concentrated source acting on the interface. We first study a special (base) scheme comprising a standard finite difference operator on a piecewise-uniform mesh condensing in the boundary and interior layers, and then an overlapping domain decomposition scheme constructed on the basis of the former that converge ε-uniformly at the rates $O\left(N_1^{-1}\ln N_1 + N_2^{-1}\right)$ and $O\left(N_1^{-1}\ln N_1 + N_2^{-1} + q^t\right)$, respectively. Here N_1 and N_2 are the number of mesh intervals in x_1 and the minimal number of mesh intervals in x_2 on a unit interval, $q < 1$ is the common ratio of a geometric progression, independent of ε, t is the iteration count. For these nonlinear schemes, we construct monotone linearized schemes of the same ε-uniform accuracy, in which the unknown function in the nonlinear term is taken at the previous iteration. We give necessary and sufficient conditions under which the overlapping Schwarz

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method is robust in the sense that its solutions converge ε-uniformly as the number of mesh points and the iteration count grow.

The linearized schemes are monotone, which admits to construct their upper and lower solutions. The technique of upper and lower solutions used in the paper makes it possible to find *a posteriori* the optimal number of iterations T in the linearized scheme for which the accuracy of its solution is the same (up to a constant factor) as that for the base scheme, where $T = \mathcal{O}\big(\ln\big(\min[N_1,\, N_2]\big)\big)$ (see also [4] for a reaction-diffusion problem). Thus, the number of required iterations is independent of ε. With respect to total computational costs, the iterative method is close to a solution method for linear problems, since the number of iterations is only weakly depending on the number of mesh points used. The linearized iterative schemes inherit the ε-uniform rate of convergence of the nonlinear schemes.

The decomposition schemes developed in the paper can be computed sequentially and in parallel. It is convenient to decompose the base schemes so that the original domain is partitioned into subdomains each of which involves no more than a single singularity. Note that schemes of the overlapping domain decomposition Schwarz method were considered earlier by the authors in [5] for linear problems and in [6, 7, 8] for nonlinear problems.

1. Problem formulation. The aim of research

On the vertical strip composed of two subdomains

$$\overline{D} = D \cup \Gamma, \quad \overline{D} = \overline{D}_1 \cup \overline{D}_2,$$

where

$$D = \big\{x:\ x_1 \in (-d_0, d^0),\ x_2 \in I\!R\big\}, \quad D_1 = D \cap \{x_1 < 0\}, \quad D_2 = D \cap \{x_1 > 0\},$$

we consider the Dirichlet problem for the semilinear singularly perturbed elliptic convection-diffusion equation

$$L\,(u(x)) \equiv L^2 u(x) - f(x, u(x)) = 0, \qquad x \in D^{(*)}, \tag{1.1a}$$

$$u(x) \ = \ \varphi(x), \qquad x \in \Gamma. \tag{1.1b}$$

Inside the domain (in the plane $x_1 = 0$), the concentrated source of power $q(x)$ acts on the set $\Gamma^* = \{x_1 = 0\} \times I\!R$, i.e.,

$$[\,u(x)\,] = 0, \quad l\,u(x) \equiv \varepsilon \left[\frac{\partial}{\partial x_1}\, u(x)\right] = -q(x), \qquad x \in \Gamma^*. \tag{1.1c}$$

Here $D^{(*)} = D \setminus \Gamma^*$,

$$L^2 \equiv \varepsilon \sum_{s=1,2} a_s(x)\,\frac{\partial^2}{\partial x_s^2} + \sum_{s=1,2} b_s(x)\,\frac{\partial}{\partial x_s} - c(x), \quad x \in D^{(*)};$$

$$[\,v(x)\,] = \lim_{x_1 \to +0} v(x) - \lim_{x_1 \to -0} v(x), \quad x \in \Gamma^*,$$

where $[\,v(x)\,]$ is the jump of a function $v(x)$ when passing through the interface Γ^* from D_1 to D_2. The singular perturbation parameter ε takes arbitrary values in the half-open interval (0,1]. The functions $a_s(x)$, $c(x)$ and $f(x,u)$ are assumed to be sufficiently smooth on $\overline{D}_j$ and $\overline{D}_j \times I\!R$, $j = 1, 2$, and so are the functions $\varphi(x)$ and $q(x)$ on Γ and Γ^*, respectively, $a_s(x) = a_s^j(x)$, $c(x) = c^j(x)$, $f(x,u) = f^j(x,u)$, $x \in \overline{D}_j$, $j = 1, 2$, morevover,

$$0 < a_0 \le a_s(x) \le a^0, \quad 0 < b_0 \le b_s(x) \le b^0, \quad s = 1, 2, \quad 0 \le c(x) \le c^0, \quad x \in \overline{D}_j,$$

$$|f(x,u)| \le M, \quad 0 \le c(x) + \frac{\partial}{\partial u}\, f(x,u) \le c^1, \quad (x,u) \in \overline{D}_j \times I\!R, \quad j = 1, 2; \tag{1.2}$$

$$|\varphi(x)| \le M, \quad x \in \Gamma, \quad |q(x)| \le M, \quad x \in \Gamma^*.$$

As $\varepsilon \to 0$, there appears a regular boundary layer in a neighbourhood of the outflow boundary $\Gamma_1 = \{x\colon\ x_1 = d_0,\ x_2 \in I\!R\}$ (through which the flow leaves the domain. In a neighbourhood of the set Γ^* from the side of the flow incoming to Γ^*, a transient (interior) layer generated by the concentrated source arises for small ε. The interior layer is strong. Here Γ_1 and Γ_2 are the left and right parts of the boundary Γ; $\Gamma = \Gamma_1 \cup \Gamma_2$.

For the boundary value problem (1.1), it is required to construct a nonlinear finite difference (base) scheme and a linearized scheme of successive approximations and, for such schemes, to develop overlapping domain decomposition schemes. It is necessary that these schemes converge ε-uniformly.

2. Finite difference scheme for problem (1.1)

In this section, we construct a monotone (base) finite difference scheme for the boundary value problem (1.1). On the set $\overline{D}$, we introduce an arbitrary rectangular grid as

$$\overline{D}_h = \overline{\omega}_1 \times \omega_2, \tag{2.1}$$

where $\overline{\omega}_1$ and ω_2 are generally nonuniform meshes on $[-d_0, d^0]$ and on the x_2-axis, respectively; the point $x_1 = 0$ belongs to the mesh $\overline{\omega}_1$. We set $h_s^i = x_s^{i+1} - x_s^i$, $s = 1, 2$, x_1^i, $x_1^{i+1} \in \overline{\omega}_1$, and x_2^i, $x_2^{i+1} \in \omega_2$; let $h_s = \max_i h_s^i$, $h = \max_s h_s$. Assume $h \le M N^{-1}$, where $N = \min_s N_s$, N_1 and N_2 are the number of mesh intervals in $\overline{\omega}_1$ and the minimal number of mesh intervals in ω_2 on a unit interval. The node $x_1 = 0$ is denoted by x_1^{i0}.

The equations (1.1a), (1.1b) are approximated by the difference scheme [9]

$$\Lambda(z(x)) \equiv \Lambda^2\, z(x) - f(x, z(x)) = 0, \qquad x \in D_h^{(*)}, \tag{2.2a}$$

$$z(x) = \varphi(x), \qquad x \in \Gamma_h. \tag{2.2b}$$

Here $D_h^{(*)} = D^{(*)} \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$,

$$\Lambda^2 \equiv \sum_{s=1,2} \big(\varepsilon a_s(x)\, \delta_{\overline{xs}\,\widehat{xs}} + b_s(x)\, \delta_{xs}\big) - c(x), \quad x \in D_h^{(*)},$$

$\delta_{\overline{xs}\,\widehat{xs}}\, z(x)$ is the second (centred) difference derivative on a nonuniform mesh [9], for example, $\delta_{\overline{x1}\,\widehat{x1}}\, z(x) = 2\big(h_1^i + h_1^{i-1}\big)^{-1}\big[\,\delta_{x1}\, z(x) - \delta_{\overline{x1}}\, z(x)\big]$, $x = (x_1^i, x_2) \in D_h$; $\delta_{xs}\, z(x)$ and $\delta_{\overline{xs}}\, z(x)$ are the first (forward and backward) difference derivatives.

We approximate condition (1.1c) on $\Gamma_h^* = \Gamma^* \cap \overline{D}_h$ by the discrete equation

$$\lambda\, z(x) \equiv \varepsilon\big(\,\delta_{x1} - \delta_{\overline{x1}}\,\big)\, z(x) = -q(x), \qquad x \in \Gamma_h^*, \quad x = (x_1^{i0}, x_2). \tag{2.2c}$$

The base nonlinear scheme (2.2), (2.1) is monotone [9] ε-uniformly.

The solution of the boundary value problem (1.1) is bounded ε-uniformly, however, the solution of the difference scheme (2.2), (2.1) is not ε-uniformly bounded.

The condition $\big(h = \mathcal{O}(\varepsilon)\big)$

$$N^{-1} = \mathcal{O}\big(\varepsilon\big) \tag{2.3}$$

is sufficient for the solutions of scheme (2.2), (2.1) to be ε-uniformly bounded.

The scheme (2.2), (2.1) converges for fixed values of the parameter ε; in particular, in the case of uniform meshes we have the error bound

$$|u(x) - z(x)| \le M\left[\varepsilon^{-1} N_1^{-1} + N_2^{-1}\right], \quad x \in \overline{D}_h^{\,u},$$

where $\overline{D}_h^u$ is a uniform mesh with respect to both variables x_1 and x_2.

On the set $\overline{D}$, we place a special grid condensing in the layer regions (see, e.g., [3]), on which scheme (2.2) converges ε-uniformly:

$$\overline{D}_h = \overline{D}_h^S = \overline{\omega}_1^S \times \omega_2, \tag{2.4}$$

where ω_2 is a uniform mesh (with step-size $1/N_2$), and $\overline{\omega}_1^S$ is a piecewise uniform mesh condensing in neighbourhoods of the boundary and interior layers. We divide $[-d_0, d^0]$ into four intervals $[-d_0, -d_0+\sigma_1]$, $[-d_0+\sigma_1, 0]$, $[0, \sigma_2]$, and $[\sigma_2, d^0]$. The mesh size on these subintervals is constant and equal to $h_1^{(1)} = 4\sigma_1 N_1^{-1}$, $h_1^{(2)} = 4(d_0 - \sigma_1)N_1^{-1}$, $h_1^{(3)} = 4\sigma_2 N_1^{-1}$, and $h_1^{(4)} = 4(d^0 - \sigma_2)N_1^{-1}$, respectively. The mesh parameters σ_1 and σ_2 depend on ε, N_1: $\sigma_1 = \min\left[2^{-1}d_0,\, M\varepsilon \ln N_1\right]$, $\sigma_2 = \min\left[2^{-1}d^0,\, M\varepsilon \ln N_1\right]$, where $M \ge m^{-1}$, $0 < m < m_0$, $m_0 = \min\limits_{j,\,\overline{D}_j}\left\{a_1^{-1}(x)\, b_1(x)\right\}$.

In fact, the solution of the difference scheme (2.2) on the mesh (2.4) is ε-uniformly bounded; note that condition (2.3) still does not hold for the mesh (2.4).

Using the majorant function technique and a priori estimates [3], we find the ε-uniform error bound for the numerical solution on the mesh (2.4):

$$|u(x) - z(x)| \le M\left[N_1^{-1}\ln N_1 + N_2^{-1}\right], \quad x \in \overline{D}_h^S. \tag{2.5}$$

Theorem 1 *Let the data of the boundary value problem* (1.1) *satisfy condition* (1.2)*, and also* $a_s,\ b_s,\ c \in C^{l+\alpha}(\overline{D}_j)$*,* $f \in C^{l+\alpha}(\overline{D}_j \times I\!R)$*,* $\varphi \in C^{l+\alpha}(\Gamma)$*,* $q \in C^{l+\alpha}(\Gamma^*)$*,* $l = 6$*,* $\alpha > 0$*,* $s, j = 1, 2$*. Then, the difference scheme* (2.2), (2.4) *converges* ε*-uniformly at the rate* $\mathcal{O}\big(N_1^{-1}\ln N_1 + N_2^{-1}\big)$*; the numerical solutions satisfy the bound* (2.5).

3. Linearized iterative difference scheme

To linearize the nonlinear scheme (2.2), we construct an iterative (two-level) difference scheme in which the nonlinear term is computed from the unknown function taken at the previous iteration (see, e.g., [7, 8]).

To solve problem (1.1), we use the difference scheme

$$\begin{aligned} &\Lambda_{(3.1)}(z(x,t)) \equiv \Lambda^2_{(2.2)}\, z(x,t) - p\,\delta_{\overline{t}}\, z(x,t) - f(x, \check{z}(x,t)) = 0, \quad (x,t) \in G_h^{(*)}, \\ &\lambda\, z(x,t) = q(x), \quad (x,t) \in S_h^*, \quad z(x,t) = \varphi(x), \quad (x,t) \in S_h. \end{aligned} \tag{3.1a}$$

Here

$$\begin{aligned} &\overline{G}_h = G_h \cup S_h, \quad \overline{G}_h = \overline{D}_h \times \overline{\omega}_0, \quad G_h = D_h \times \omega_0, \\ &G_h^{(*)} = D_h^{(*)} \times \omega_0, \quad S_h^* = \Gamma_h^* \times \omega_0, \end{aligned} \tag{3.1b}$$

$\overline{\omega}_0$ is a uniform time-like mesh on the semiaxis $t \ge 0$ with the step-size $h_t = 1$, so that the variable $t = 0, 1, ... \in \overline{\omega}_0$ specifies iteration counts; $S_h = S_h^L \cup S_{h0}$, and $S_h^L = \Gamma_h \times \omega_0$ is the lateral boundary of $\overline{G}_h$; $\delta_{\overline{t}} z(x,t) = h_t^{-1}[z(x,t) - \check{z}(x,t)]$, $\check{z}(x,t) = z(x, t - h_t)$, $(x,t) \in G_h$; the coefficient p satisfies the condition

$$p - \frac{\partial}{\partial u} f(x,u) \ge p_0 > 0, \quad (x,u) \in \overline{D}_j \times I\!R;$$

the boundary data $\varphi(x)$, $x \in \overline{D}$, is a sufficiently smooth bounded function satisfying

$$\varphi_{(3.1)}(x) = \varphi_{(1.1)}(x), \quad x \in \Gamma, \quad l\,\varphi_{(3.1)}(x) = q(x), \quad x \in \Gamma^*.$$

The function $z(x,t)$, $(x,t) \in \overline{G}_h$, where $\overline{G}_h$ is generated by the mesh $\overline{D}_{h(2.1)}$, is called the solution of the difference scheme (3.1), (2.1).

Scheme (3.1), (2.1), linear at each iteration, is monotone. Its solution $z(x,t)$ as $t \to \infty$ converges to the solution $z(x)$ of the base scheme (2.2), (2.1) at a rate of a geometric progression for fixed values of the parameter ε. Under condition (2.3), the convergence of $z(x,t)$ to $z(x)$ is indeed ε-uniform:

$$|z(x) - z(x,t)| \le M\, q_1^t, \quad (x,t) \in \overline{G}_h,$$

where the common ratio $q_1 < 1$ is independent of ε.

In the case of the piecewise uniform mesh (2.4), the convergence of $z(x,t)$ to $z(x)$ is ε-uniform. On the mesh (2.4), taking account of the bound (2.5) we obtain the ε-uniform bound

$$|u(x) - z(x,t)| \le M\left[N_1^{-1}\ln N_1 + N_2^{-1} + q_1^t\right], \quad (x,t) \in \overline{G}_h^S. \tag{3.2}$$

Here $\overline{G}_h^S = \overline{G}_h\big(\overline{D}_{h(2.4)}^S\big)$, $q_1 \le p^0(c_1+p^0)^{-1}$, $p^0 = \max\big(p - \frac{\partial}{\partial u} f(x,u)\big)$, $c_1 = \min\big(c(x) + \frac{\partial}{\partial u} f(x,u)\big)$, $(x,u) \in \overline{D}_j \times R$.

4. Iterative difference scheme of the Schwarz method

4.1. For the base scheme (2.2), we describe an overlapping domain decomposition method [5, 10]. Let open subdomains D^k, $k = 1,\dots,K$ cover the domain D :

$$D = \bigcup_{k=1}^{K} D^k.$$

We denote the minimal overlap of the sets D^k and $D^{[k]} = \bigcup\limits_{i=1,\ i\ne k}^{K} D^i$ by Δ^k, and Δ denotes the smallest value of Δ^k, $k = 1,\dots,K$, i.e.,

$$\min_{k,\ x^1,\ x^2} \rho(x^1, x^2) = \Delta, \quad x^1 \in \overline{D}^k, \quad x^2 \in \overline{D}^{[k]}, \quad x^1,\ x^2 \notin \{D^k \cap D^{[k]}\},$$

where $\rho(x^1,x^2)$ is the distance between x^1 and x^2. Generally, $\Delta = \Delta(\varepsilon)$.

It is convenient to introduce the uniform "time" mesh $\omega_0 = \{t^n : t^n = nh_t,\ n = 1,2,\dots\}$, $\overline{\omega}_0 = \omega_0 \cup \{t=0\}$ with the step-size $h_t = 1$, by associating its nodes to the iteration number, and thus the semi-discrete set $\overline{G} = \overline{D}\times\overline{\omega}_0$, $G = D\times\omega_0$ with the boundary $S = S^L \cup S_0$, $S^L = \Gamma\times\omega_0$ being the lateral boundary.

Let each set D^k be partitioned into P disjoint (possibly empty) sets:

$$D^k = \bigcup_{p=1}^{P} D_p^k, \quad k = 1,\dots,K, \quad \overline{D}_i^k \bigcap \overline{D}_j^k = \emptyset, \quad i \ne j.$$

Assume $G_p^k = D_p^k \times \overline{\omega}_0$, $p = 1,...,P$, $k = 1,...,K$. On the sets $\overline{G}$ and $\overline{G}_p^k$, we construct the grids

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0, \quad \overline{G}_{ph}^k = \overline{D}_{ph}^k \times \overline{\omega}_0, \quad \overline{D}_{ph}^k = \overline{D}_p^k \bigcap \overline{D}_h \tag{4.1a}$$

where $\overline{D}_h$ is the grid (2.1) or (2.4). We suppose that the faces of the sets $\overline{D}_p^k$ pass through the nodes of the grid $\overline{D}_h$. By $z(x,t)$, $(x,t) \in \overline{G}_h$, $t = 0,1,2,\dots$, we denote approximations to the function $z(x)$, $x \in \overline{D}_h$. For $t = 0$ we define the function $z(x,t)$ by $z(x,0) = \varphi(x)$, $x \in \overline{D}_h$, where $\varphi(x) = \varphi_{(3.1)}(x)$, $x \in \overline{D}$.

On the set G_h, we introduce the operator

$$\Lambda_{(4.1b)}\, z(x,t) = \begin{cases} \Lambda_{(2.2)}\, z(x,t), & (x,t) \in G_h^{(*)}, \\ \lambda_{(2.2)}\, z(x,t) + q(x), & (x,t) \in S_h^*; \end{cases} \quad (x,t) \in G_h. \tag{4.1b}$$

We now determine the sequence of discrete functions $z(x,t)$, $(x,t)\in\overline{G}_h$, $t=1,2,\dots$. Before we find the sequence of auxiliary functions $z^{\frac{k}{K}}(x,t)$, $x\in\overline{D}_h$, $k=1,\dots,K$, $t=1,2,\dots$ by solving the boundary value problems

$$\left.\begin{aligned} \Lambda_{(4.1b)}\big(z_p^{\frac{k}{K}}(x,t)\big)=0, &\qquad x\in D^k_{ph},\\ z_p^{\frac{k}{K}}(x,t)=z^{\frac{k-1}{K}}(x,t), &\qquad x\in\Gamma^k_{ph}\setminus\Gamma_h,\\ z_p^{\frac{k}{K}}(x,t)=\varphi(x), &\qquad x\in\Gamma^k_{ph}\cap\Gamma_h, \end{aligned}\right\},\ p=1,\dots,P; \tag{4.1c}$$

$$z^{\frac{k}{K}}(x,t)=\left\{\begin{array}{ll} z_p^{\frac{k}{K}}(x,t), & x\in\overline{D}^k_{ph},\ p=1,\dots,P,\\ z^{\frac{k-1}{K}}(x,t), & x\in\overline{D}_h\setminus\overline{D}^k \end{array}\right\},\quad x\in\overline{D}_h,\quad k=1,\dots,K;$$

$$z^{\frac{k-1}{K}}(x,t)=z(x,t-1),\quad x\in\overline{D}_h\ \text{ for }\ k=1;$$

$$z(x,0)=\varphi(x),\quad x\in\overline{D}_h;$$

$$z(x,t)=z^{\frac{K}{K}}(x,t);\qquad t=1,2,\dots\ ; \tag{4.1d}$$

here $\Gamma^k_{ph}=\overline{D}^k_{ph}\setminus D^k_{ph}$. It is required to find the sequence of functions $z(x,t)$, $x\in\overline{D}_h$, $t=1,2,...$, which are solutions of the iterative scheme (4.1), (2.1) (or (4.1), (2.4)) using $P\ge 1$ solvers. The intermediate problems (4.1c) on the subdomains $\overline{D}^k_{ph}$, $p=1,...,P$, can be solved in parallel (for $P>1$), independently of each other, on P processors [10]. For $P=1$ the subproblems on $\overline{D}^k_h=\overline{D}^k\cap\overline{D}_h$ are solved sequentially. Scheme (4.1), (2.1) is nonlinear.

For

$$\Delta=\Delta(\varepsilon)>0,\quad \varepsilon\in(0,1],\qquad \inf_{\varepsilon\in(0,1]}\big[\varepsilon^{-1}\,\Delta(\varepsilon)\big]\ge m_1>0 \tag{4.2}$$

$z(x,t)$ on the mesh $\overline{G}_{h(3.1)}\big(\overline{D}_{h(2.1)}\big)$, under the additional condition (2.3), converges to the solution $z(x)$ of scheme (2.2), (2.1) ε-uniformly as $t\to\infty$:

$$|z(x)-z(x,t)|\le M\,q^t,\quad (x,t)\in\overline{G}_h,\quad \text{where}\ \ q\le 1-m, \tag{4.3}$$

$q=q(m_1)$, and $q(m_1)$ grows as $m_1\to 0$; in general, $q>q_{1(3.2)}$. Condition (4.2) is a necessary and, under condition (2.3), a sufficient condition for the ε-uniform convergence (as $t\to\infty$) of solutions of the iterative scheme (4.1) to the solution of the base scheme (2.2).

In the case of the mesh (2.4), condition (4.2) is necessary and sufficient for the ε-uniform convergence of $z(x,t)$ to $z(x)$ as $t\to\infty$. Taking into account estimates (4.3) and (2.5), for the solution of scheme (4.1), (2.4) we find the error bound similar to (3.2)

$$|u(x)-z(x,t)|\le M\left[N_1^{-1}\ln N_1+N_2^{-1}+q_2^t\right],\quad (x,t)\in\overline{G}^S_h,\quad q_2\le 1-m, \tag{4.4}$$

where, generally speaking, $q_2>q_{(4.3)}$. For a linear problem, $q_2=q_{(4.3)}$.

Theorem 2 *Let the assumptions of Theorem* 1 *and condition* (4.2) *be fulfilled. Then the Schwarz scheme* (4.1), (2.4) *as* $N_1,N_2,t\to\infty$ *converges* ε*-uniformly at the rate* $\mathcal{O}\big(N_1^{-1}\ln N_1+N_2^{-1}+q_2^t\big)$, $q_2\le 1-m$; *the numerical solutions satisfy the bound* (4.4).

4.2. Based on the linearized scheme (3.1), (2.4) it is possible to construct the monotone linearized scheme of the Schwarz method whose solution $z(x,t)$ converges, as $t\to\infty$, to the solution $z(x)$ of scheme (2.2), (2.4) from above and/or from below and, besides, it converges ε-uniformly to the solution of problem (1.1) at the same rate $\mathcal{O}\big(N_1^{-2}\ \ln^2 N_1+N_2^{-2}+q^t\big)$, $q\le 1-m$, as scheme (4.1), (2.4). When solving the subproblem at the intermediate (inner) iteration in this scheme, we take the unknown function in the nonlinear term at the previous intermediate iteration.

To solve problem (1.1), we use the difference scheme (4.1) on the meshes (2.1) or (2.4), where the operator $\Lambda_{(4.1b)}$ is now defined by

$$\Lambda_{(4.1b)}\, z(x,t) = \begin{cases} \Lambda_{(3.1)}\, z(x,t), & (x,t) \in G_h^{(*)}, \\ \lambda_{(2.2)}\, z(x,t) + q(x), & (x,t) \in S_h^*; \end{cases} \qquad (x,t) \in G_h. \tag{4.5}$$

For solutions of the linearized scheme of the Schwarz method (4.1), (4.5), (2.4), the conclusion of Theorem 2 holds.

Let $z(x,t)$, $(x,t) \in \overline{G}_h$ be a solution of the discrete Schwarz method, and $z^{(i)}(x,t)$, $(x,t) \in \overline{G}_h$, $i = 1,2$, be solutions of some difference scheme, and let the following inequality be satisfied for $t = 0$:

$$z^{(1)}(x,0) \le z(x,0) \le z^{(2)}(x,0), \quad x \in \overline{D}_h. \tag{4.6}$$

If the inequality $z^{(1)}(x,t) \le z(x,t) \le z^{(2)}(x,t)$, $(x,t) \in \overline{G}_h$, is true for $t > 0$, and also

$$\max_x \left| z^{(i)}(x,t) - z(x,t) \right| \to 0, \quad x \in \overline{D}_h \quad \text{for} \quad t \to \infty, \quad i = 1,2,$$

we call the functions $z^{(1)}(x,t)$ and $z^{(2)}(x,t)$ the lower and upper solutions of the discrete Schwarz method, respectively.

Since the Schwarz scheme (nonlinear and linearized) is monotone, its solutions $z^{(1)}(x,t)$ and $z^{(2)}(x,t)$, $(x,t) \in \overline{G}_h$, satisfying condition (4.6) for $t = 0$ are the lower and upper solutions. We use the upper and lower solutions to evaluate *a posteriori* the number of iterations for which the accuracy of the linearized scheme is the same (up to a factor) as that for the base scheme (2.2), (2.4) (see also [4]).

The error in the solution of the linearized Schwarz scheme on the mesh (2.4) can be represented in the form

$$z^{(i)}(x,t) - u(x) = \big(z(x) - u(x)\big) + \big(z^{(i)}(x,t) - z(x)\big), \quad (x,t) \in \overline{G}_h, \quad i = 1,2,$$

where $z^{(i)}(x,t)$ is the solution of the linearized Schwarz scheme satisfying the condition

$$z^{(1)}(x,0) \le z(x) \le z^{(2)}(x,0), \quad x \in \overline{D}_h.$$

Let T be the number of iterations (in t) in the linearized Schwarz scheme under which the error in the solution of the base scheme (2.2), (2.4) and the deviation of the solution of the linearized scheme from the solution of the base scheme are commensurable. We call the function $z^{(i)}(x,T)$, $x \in \overline{D}_h$, the solution (upper for $j = 2$ and lower for $j = 1$) of the linearized Schwarz scheme, *consistent* with respect to the accuracy (of the base scheme) and with respect to the number of iterations (of the linearized Schwarz scheme).

For the upper and lower solutions of the linearized Schwarz scheme on the mesh (2.4), we find the optimal value

$$T \tag{4.7}$$

of the number of iterations, that is, the least value of T_0 for which such a condition holds:

$$\max_{\overline{D}_h} \left[z^{(2)}(x,T_0) - z^{(1)}(x,T_0) \right] \le M_1 \left[N_1^{-1} \ln N_1 + N_2^{-1} \right], \quad x \in \overline{D}_h.$$

The iterative difference scheme in which the moment of the termination of iterates is determined using upper and lower solutions will be called *consistent*.

For the consistent solution of the linearized Schwarz scheme (4.1), (4.5), (4.7), (2.4) we obtain the bound

$$|u(x) - z^{(j)}(x,T)| \le M_2 \left[N_1^{-2} \ln^2 N_1 + N_2^{-2} \right], \quad x \in \overline{D}_h, \; j = 1,2,$$

with T satisfying the upper bound

$$T \leq M_3 \big(\ln q^{-1} \big)^{-1} \ln \big(\min [N_1, N_2] \big),$$

where $q \leq 1 - m$, and the constants M_1, M_2, M_3 are independent of q.

Thus, the number of required iterations is independent of ε. With respect to total computational costs, the iterative method (4.1), (4.5), (4.7), (2.4) is close to a solution method for linear problems, since the number of iterations is only weakly depending on the number of mesh points used. The subproblems on the decomposition subdomains, each of which contains no more than a single singularity, can be solved both sequentially and in parallel (independently of each other).

Remark 1 *If $p = 0$ in the operator $\Lambda_{(3.1)}$, the iterates in the linearized scheme for $t \to \infty$, in general, diverge, for example, under the condition $(\partial/\partial u) f(x,u) > c(x)$, $(x,u) \in \overline{D} \times R$.*

Remark 2 *In a similar way, ε-uniformly convergent consistent iterative schemes based on the domain decomposition method can be constructed in that case when $\overline{D}$ is the composed domain with several concentrated sources.*

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