A parameter-uniform numerical method for a system of singularly perturbed ordinary differential equations

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1 Introduction

In many fields of applied mathematics we often come across initial/boundary value problems with small positive parameter(s). In particular, systems of singularly perturbed first order ordinary differential equations occur in chemical reactor theory. Related works are found in [10],[9]. Parabolic and regular layers are typical for such problems [8],[6]. Matthews et.al [4], [3] have suggested parameter robust numerical methods for system of singularly perturbed second order ordinary differential equations with one or two small parameters. Possible approaches to the construction of such methods and also some special schemes are given in [1],[10],[5], [7]. Here we present a parameter robust computational method for solving an initial value problem(IVP) for a system of first order singularly perturbed ordinary differential equations of the form

$$L_\varepsilon u_\varepsilon(x) = \left\{ \begin{array}{lcl} (L_\varepsilon u_\varepsilon)_1(x) & = & \varepsilon D u_{\varepsilon,1}(x) + a_{11}(x)u_{\varepsilon,1}(x) + a_{12}(x)u_{\varepsilon,2}(x) \\ & & + \cdots + a_{1n}(x)u_{\varepsilon,n}(x) = f_1(x) \\ \vdots & \vdots & \vdots \\ (L_\varepsilon u_\varepsilon)_n(x) & = & \varepsilon D u_{\varepsilon,n}(x) + a_{n1}(x)u_{\varepsilon,1}(x) + a_{n2}(x)u_{\varepsilon,2}(x) \\ & & + \cdots + a_{nn}(x)u_{\varepsilon,n}(x) = f_n(x), x \in (0,X], X > 0 \end{array} \right\} \tag{1.1}$$

$$u_{\varepsilon,i}(0) = u_i^0, \qquad \text{for } i = 1(1)n \tag{1.2}$$

where $u_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}, \ldots, u_{\varepsilon,n})^T$ and $u_\varepsilon \in \mathcal{C}^{(1)}(\Omega)$, $\overline{\Omega} = [0, X]$ and D denotes $\frac{d}{dx}$. The functions $a_{ij}, \;\; f_i \in \mathcal{C}^{(2)}(\overline{\Omega}), \; i, j = 1, 2, \ldots, n$, satisfy the following inequalities

$$\left. \begin{array}{ll} (i) & a_{ii}(x) > \sum_{j=1, j\neq i}^{n} |a_{ij}(x)|, \quad i = 1(1)n \\ (ii) & a_{ij}(x) \leq 0, \quad i, j = 1(1)n \;,\; i \neq j \end{array} \right\} \quad \forall\, x \in [0, X] \tag{1.3}$$

We introduce the positive number

$$\alpha = \min \left\{ \sum_{j=1}^{n} a_{1j}, \sum_{j=1}^{n} a_{2j}, \ldots, \sum_{j=1}^{n} a_{nj} \right\} \tag{1.4}$$

and we assume henceforth that the singular perturbation parameter ε satisfies $0 < \varepsilon \leq 1$. Without loss of generality we take $X = 1$. We use the notations $\Omega_0 = (0, 1]$, $\Omega = (0, 1)$ and $\overline{\Omega} =$

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$[0,1]$. For a vector u with n components we use the vector norm $||u|| = \max_i \ |u_i| \ , i = 1(1)n$ and for a continuous function v defined on $\overline{\Omega}$ we use the continuous maximum norm

$$||v||_{\overline{\Omega}} = \max_{x \in \overline{\Omega}} |v(x)|.$$

Further we assume that a vector $u \geq 0$ if and only if $u_i \geq 0$ for $i = 1(1)n$. Appropriate numerical methods for generalizations of problems of the form (1.1)-(1.4) were presented in [11]. Extensive numerical results and detailed proofs for these numerical methods were given in [3].

2 Analytical results

In this section we present some analytical results for the above IVP, which include a maximum principle, uniform stability result and estimates of the derivatives of the solution. We begin with a maximum principle.

Lemma 2.1 *Consider the IVP* (1.1),(1.2). *Then* $u_{\varepsilon,i}(0) \geq 0, i = 1, \ldots, n$ *and* $(L_\varepsilon u_\varepsilon)_i(x) \geq 0$ *for all* $x \in \Omega_0, i = 1, \ldots, n$ *imply that* $u_{\varepsilon,i}(x) \geq 0, i = 1, \ldots, n$ *for all* $x \in \overline{\Omega}$.

Proof : Define a test function $s(x) = (s_1(x), s_2(x), \ldots, s_n(x))^T$ by $s_i(x) \equiv 1, \quad i = 1, \ldots, n$. Then it is clear that $s_i(x) > 0, \forall \, x \in \overline{\Omega}$ and $i = 1, 2, \ldots, n$, using (1.3),

$$(L_\varepsilon s)_i(x) = \varepsilon s_i'(x) + \sum_{j=1}^{n} a_{ij}(x) s_j(x) = \sum_{j=1}^{n} a_{ij}(x) > \ 0, \qquad \text{for } \ i = 1(1)n$$

Now assume that the lemma is not true and introduce the quantity

$$\xi \ = \ \max_i \left\{ \max_{x \in \overline{\Omega}} \left(\frac{-u_{\varepsilon,i}}{s_i} \right) (x) \right\}$$

Since the lemma is not true, there exists $x^* \in \Omega_0$ such that $u_{\varepsilon,i}(x^*) < 0$ for at least one i Clearly $\xi > 0$ and $(u_{\varepsilon,i} + \xi s_i)(x) \geq 0$ for all $i = 1, 2, \ldots, n$ and for all $x \in \overline{\Omega}$. Furthermore, there exists a point $x_0 \in \overline{\Omega}$ such that $(u_{\varepsilon,k} + \xi s_k)(x_0) = 0$ for some k. This means that $u_{\varepsilon,k} + \xi s_k$ attains a minimum at $x = x_0$. Then

$$\begin{aligned}
0 \ &< \ (L_\varepsilon(u_\varepsilon + \xi s))_k(x_0) \\
&= \ \varepsilon(u_{\varepsilon,k} + \xi s_k)'(x_0) + \sum_{j=1, j \neq k}^{n} a_{kj}(x_0)(u_{\varepsilon,j} + \xi s_j)(x_0) \\
&\leq \ 0, \ \text{ since } (u_{\varepsilon,k} + \xi s_k)'(x_0) = 0 \text{ if } x_0 \in \Omega \ \text{ and } (u_{\varepsilon,k} + \xi s_k)'(x_0) \ < \ 0 \text{ if } x_0 = 1
\end{aligned}$$

a contradiction. Hence, it can be concluded that $u_{\varepsilon,i}(x) \geq 0$, $i = 1(1)n$, for all $x \in \overline{\Omega}$. ∎

Using this maximum principle lemma, a uniform stability bound is obtained in the following lemma.

Lemma 2.2 *If* $u_\varepsilon(x)$ *is the solution of the IVP* (1.1),(1.2) *then*

$$||u_\varepsilon(x)|| \ \leq \ C \ \max \{||u_\varepsilon(0)|| \, , \ ||L_\varepsilon u_\varepsilon(x)||\} \, , \qquad \forall \, x \in \overline{\Omega}$$

where C *is a constant independent of* x *and* ε.

Proof : Defining barrier functions $\Psi^{\pm}(x) = \left(\Psi_1^{\pm}(x), \, \Psi_2^{\pm}(x), \ldots, \Psi_n^{\pm}(x)\right)^T$ by

$$\Psi_i^{\pm}(x) = M \pm u_{\varepsilon,i}(x), \qquad \text{for } i = 1(1)n$$

where $M = C' \ \max \{||u_\varepsilon(0)|| \, , \ ||L_\varepsilon u_\varepsilon(x)||\}$, $C'$is a constant Then the desired stability bound follows from Lemma 2.1, applied on $\Psi^{\pm}(x)$. ∎

In the next two lemmas we obtain bounds on the derivatives of the solution.

Lemma 2.3 *Let* $u_\varepsilon(x)$ *be the solution of the IVP* (1.1),(1.2) *then* $u_{\varepsilon,i}(x)$ *for* $i = 1, 2, \ldots, n$ *satisfy*

$$\left|u_{\varepsilon,i}^{(k)}(x)\right| \leq C\left(1+\varepsilon^{-k}e^{-\alpha x/\varepsilon}\right), \qquad \text{for } i = 1(1)n$$

for $0 \leq k \leq 2$, $x \in \overline{\Omega}$ *and* C *is a constant independent of* x *and* ε.

Proof The result is true for $k = 0$ by Lemma 2.2. We now verify the result for $k = 1$. Consider the equations given by (1.1)

$$(L_\varepsilon u_\varepsilon)_i \equiv \varepsilon u'_{\varepsilon,i} + \sum_{j=1}^{n} a_{ij} u_{\varepsilon,j} = f_i\ , \qquad \text{for } i = 1(1)n$$

Differentiating the above equations once we obtain

$$\varepsilon\left(u'_{\varepsilon,i}\right)' + \sum_{j=1}^{n} a_{ij} u'_{\varepsilon,j} = f'_i - \sum_{j=1}^{n} a'_{ij} u_{\varepsilon,j}\ , \qquad \text{for } i = 1(1)n$$

This implies that

$$\left|(L_\varepsilon u'_\varepsilon)_i(x)\right| \leq C\ , \qquad \text{for } i = 1(1)n \tag{2.6}$$

From (1.1) it is easy to check that

$$\left|u'_{\varepsilon,i}(x)\right| \leq C\varepsilon^{-1}, \qquad \text{for } i = 1(1)n \tag{2.7}$$

Using the maximum principle for

$$\psi_i^{\pm}(x) = C'\ \left(1+\varepsilon^{-1}e^{-\alpha x/\varepsilon}\right) \pm u'_{\varepsilon,i}(x), \qquad \text{for } i = 1(1)n$$

where C' is a constant independent of x and ε ,one can show that

$$\left|u'_{\varepsilon,i}(x)\right| \leq C\left(1+\varepsilon^{-1}e^{-\alpha x/\varepsilon}\right)\ , \qquad \text{for } i = 1(1)n$$

where C is a constant independent of x and ε. Proceeding on similar lines we can also verify the result for $k = 2$. ∎

In order to establish the parameter uniform convergence of the numerical scheme, sharper estimates of the derivatives are required. We consider a decomposition of the solution u_ε into its smooth and singular components, as given below:

$$u_\varepsilon = v_\varepsilon + w_\varepsilon = ((v_0)_\varepsilon + \varepsilon(v_1)_\varepsilon) + w_\varepsilon$$

where $(v_0)_\varepsilon = A^{-1}f$ is the solution of the reduced problem and

$$\begin{array}{rclrcl} L_\varepsilon(v_1)_\varepsilon &=& -D(v_0)_\varepsilon, & (v_1)_\varepsilon(0) &=& 0 \\ L_\varepsilon w_\varepsilon &=& 0, & w_\varepsilon(0) &=& u_\varepsilon(0) - (v_0)_\varepsilon(0) \end{array}$$

The estimates of the derivatives of these components are as presented in the following lemma, and the proof is straight.

Lemma 2.4 *The smooth and singular components of the solution of the IVP* (1.1),(1.2) *satisfy*

$$\left|v_\varepsilon^{(k)}(x)\right| \leq C\left(1+\varepsilon^{1-k}e^{-\alpha x/\varepsilon}\right), \qquad \left|w_\varepsilon^{(k)}(x)\right| \leq C\varepsilon^{-k}e^{-\alpha x/\varepsilon}$$

for all $x \in \overline{\Omega}$ *and* $0 \leq k \leq 2$,*where* C *is a constant independent of* x *and* ε.

3 The discrete problem

The IVP (1.1),(1.2) is discretised using a fitted mesh method composed of a classical finite difference operator on a piecewise uniform fitted mesh. The corresponding discrete problem is

$$L_\varepsilon^N U_\varepsilon(x_j) = \left\{ \begin{array}{rcl} (L_\varepsilon^N U_\varepsilon)_1(x_j) & = & \varepsilon D^- U_{\varepsilon,1}(x_j) + a_{11}(x_j)U_{\varepsilon,1}(x_j) + a_{12}(x_j)U_{\varepsilon,2}(x_j) \\ & & + \cdots + a_{1n}(x_j)U_{\varepsilon,n}(x_j) = f_1(x_j) \\ \vdots & & \vdots \qquad\qquad \vdots \\ (L_\varepsilon^N U_\varepsilon)_n(x_j) & = & \varepsilon D^- U_{\varepsilon,n}(x_j) + a_{n1}(x_j)U_{\varepsilon,1}(x_j) + a_{n2}(x_j)U_{\varepsilon,2}(x_j) \\ & & + \cdots + a_{nn}(x_j)U_{\varepsilon,n}(x_j) = f_n(x_j), j = 1(1)N \end{array} \right\} \quad (3.1)$$

$$U_{\varepsilon,i}(0) = u_{\varepsilon,i}(0), \qquad \text{for } i = 1(1)n \quad (3.2)$$

where

$$D^- U_{\varepsilon,i}(x_j) = \frac{U_{\varepsilon,i}(x_j) - U_{\varepsilon,i}(x_{j-1})}{x_j - x_{j-1}}, \qquad j = 1(1)N\ ,\ i = 1(1)n$$

and the fitted mesh $\overline{\Omega}_\sigma^N$ is defined by

$$x_j = \frac{2j\sigma}{N}\ ,\ j = 0(1)\tfrac{N}{2}\ , \qquad x_{\frac{N}{2}+j} = \sigma + \frac{2j\,(1-\sigma)}{N}\ , j = 1(1)\tfrac{N}{2}$$

with the transition parameter σ given by $\sigma = min\left\{ \frac{1}{2}\ ,\ \frac{\varepsilon}{\alpha} lnN \right\}$

3.1 DISCRETE MAXIMUM PRINCIPLE

We now state the discrete maximum principle and the stability result without proof as the proofs are analogous to the continuous case.

Lemma 3.1 *Consider the discrete IVP* (3.1)-(3.2). *Then* $U_{\varepsilon,i}(0) \geq 0,\ i = 1(1)n$ *and* $(L_\varepsilon^N U_\varepsilon)_i(x_j) \geq 0$ *for* $i = 1(1)n,\ j = 1(1)N$ *imply that* $U_{\varepsilon,i}(x_j) \geq 0$ *for* $i = 1(1)n,\ j = 0(1)N$.

Lemma 3.2 *If* $U_\varepsilon(x_j)$ *is any mesh function then*

$$||U_\varepsilon(x_j)|| \leq C \max\left\{\| U_\varepsilon(0) \|, \| L_\varepsilon^N U_\varepsilon(x_j) \|\right\},\ \ j = 0(1)N$$

where C *is a constant independent of* j *and* ε.

The main theoretical result, a parameter–uniform error estimate, is given in the following theorem.

Theorem 3.3 *The fitted mesh finite difference method* (3.1) -(3.2) *with the classical finite difference operator on the piecewise uniform fitted mesh* $\overline{\Omega}_\sigma^N$ *is* ε*-uniform for the IVP* (1.1),(1.2). *Moreover the solution* u_ε *of the continuous problem and the solution* U_ε *of the discrete problem satisfy the following* ε *- uniform error estimate*

$$\sup_{0<\varepsilon\leq 1} ||U_{\varepsilon,i} - u_{\varepsilon,i}||_{\overline{\Omega}_\sigma^N} \leq C\ N^{-1} ln\ \ N, \qquad i = 1(1)n$$

where C *is a constant independent of* N *and* ε.

Proof : Consider a decomposition of U_ε given by, $\quad U_\varepsilon = V_\varepsilon + W_\varepsilon \quad$ where V_ε is the solution of the problem

$$L_\varepsilon^N V_\varepsilon = f, \qquad V_\varepsilon(0) = v_\varepsilon(0)$$

and W_ε is the solution of the problem

$$L_\varepsilon^N W_\varepsilon = 0, \qquad W_\varepsilon(0) = w_\varepsilon(0)$$

The error due to the discretisation can be written in the form

$$U_\varepsilon - u_\varepsilon = (V_\varepsilon - v_\varepsilon) + (W_\varepsilon - w_\varepsilon)$$

and the errors in the smooth and singular components of the solution can be estimated separately. The estimate of the error in the smooth component is obtained using the following classical argument. From the differential and difference equations,

$$(L_\varepsilon^N (V_\varepsilon - v_\varepsilon))_i \;=\; f_i - (L_\varepsilon^N v_\varepsilon)_i = \left(\left(L_\varepsilon - L_\varepsilon^N\right) v_\varepsilon\right)_i \;=\; \varepsilon\,(D - D^-)\, v_{\varepsilon,i},\ i = 1(1)n$$

Now

$$\begin{aligned}\left|\varepsilon\left(D - D^-\right) v_{\varepsilon,i}(x_j)\right| &\le C\,\varepsilon\,(x_j - x_{j-1}) \max_{t\in[x_{j-1},x_j]} \left|v''_{\varepsilon,i}(t)\right| \\ &\le C\,\varepsilon N^{-1} \max_{t\in[x_{j-1},x_j]} \left(1 + \varepsilon^{-1} e^{-\alpha t/\varepsilon}\right) \\ &\le C\,N^{-1}, \qquad\qquad i = 1(1)n, j = 1(1)N\end{aligned}$$

Therefore

$$\left|\left|L_\varepsilon^N\,(V_\varepsilon - v_\varepsilon)\,(x_j)\right|\right| \le C\,N^{-1}, \qquad\qquad j = 1(1)N$$

As V_ε and v_ε agree at the initial point, by the discrete stability result

$$||(V_\varepsilon - v_\varepsilon)\,(x_j)|| \le C\,N^{-1}, \qquad\qquad j = 0(1)N$$

In estimating the error in the singular component of the solution, the arguments depend on whether $\sigma = \dfrac{1}{2}$ or $\sigma = \dfrac{\varepsilon}{\alpha}\,ln\ N$.

Case 1 : $\sigma = \dfrac{1}{2}$ In this case the mesh is uniform and $\varepsilon^{-1} \le C\ ln\ N$. Again a classical argument suffices to show that

$$\left|\left|L_\varepsilon^N\,(W_\varepsilon - w_\varepsilon)\,(x_j)\right|\right| \le C\,N^{-1}\,ln\ N, \qquad\qquad j = 1(1)N \tag{3.8}$$

Case 2 : $\sigma = \dfrac{\varepsilon}{\alpha}\,ln\ N$ In this case the mesh is piecewise uniform with the mesh spacing $\frac{2\,\sigma}{N}$ in the subinterval $[0,\sigma]$ and $\frac{2\,(1-\sigma)}{N}$ in the interval $[\sigma, 1]$. Also $e^{-\frac{\alpha\sigma}{\varepsilon}} = e^{-ln\ N} = N^{-1}$. For x_j lying in the interval $(0,\sigma]$ the argument is classical. We have

$$\left|\varepsilon\left(D - D^-\right) w_{\varepsilon,i}(x_j)\right| \;\le\; C\,\varepsilon\,(x_j - x_{j-1}) \max_{t\in[x_{j-1},x_j]} \left|w''_{\varepsilon,i}(t)\right|\ , \qquad\qquad i = 1(1)n$$

Since the mesh width is $\frac{2\,\sigma}{N}$ and the second derivative of $w_{\varepsilon,i}(x_j)$ is bounded by $C\varepsilon^{-2}$, we obtain

$$\begin{aligned}|\varepsilon\,(D - D^-)\,w_{\varepsilon,i}(x_j)| &\le \varepsilon\,\frac{2\,\sigma}{N}\,C\,\varepsilon^{-2} \;\le\; C\,N^{-1}\,\frac{\sigma}{\varepsilon} \\ &\le C\,N^{-1}\,ln\ N\end{aligned}$$

Hence

$$\left|\left|L_\varepsilon^N\,(W_\varepsilon - w_\varepsilon)\,(x_j)\right|\right| \le C\,N^{-1}\,ln\ N, \qquad x_j \in (0,\sigma] \tag{3.9}$$

On the other hand, for $x_j \in [\sigma, 1]$ the argument is non–classical. We have

$$
\begin{array}{llcl}
 & (L_\varepsilon^N\,(W_\varepsilon - w_\varepsilon))_i(x_j) & = & \varepsilon\,(D - D^-)\,w_{\varepsilon,i}(x_j), \qquad i = 1(1)n \\
\text{But} & \left|\varepsilon\left(D - D^-\right) w_{\varepsilon,i}(x_j)\right| & \le & \varepsilon\,\left[\left|Dw_{\varepsilon,i}(x_j)\right| + \left|D^- w_{\varepsilon,i}(x_j)\right|\right], \quad i = 1(1)n \\
\text{and} & \left|D^- w_{\varepsilon,i}(x_j)\right| & \le & \max_{t\in[x_{j-1},x_j]} \left|w'_{\varepsilon,i}(t)\right|, \qquad i = 1(1)n \\
\text{Therefore} & \left|\varepsilon\left(D - D^-\right) w_{\varepsilon,i}(x_j)\right| & \le & 2\,\varepsilon \max_{t\in[x_{j-1},x_j]} \left|w'_{\varepsilon,i}(t)\right| \\
 & & \le & 2C\,\varepsilon\left(\varepsilon^{-1} e^{-\alpha x_{j-1}/\varepsilon}\right) \\
 & & \le & Ce^{-\alpha x_{j-1}/\varepsilon}
\end{array}
$$

Since $x_j \in [\sigma, 1]$ [5]

$$\left|\varepsilon\left(D - D^-\right) w_{\varepsilon,i}(x_j)\right| \;\le\; C\,N^{-1}, \qquad i = 1(1)n$$

and so

$$\|\, L_\varepsilon^N\,(W_\varepsilon - w_\varepsilon)\,(x_j)\,\| \;\le\; C\,N^{-1}, \qquad x_j \in [\sigma, 1] \tag{3.10}$$

Combining (3.9) and (3.10)

$$\|\, L_\varepsilon^N\,(W_\varepsilon - w_\varepsilon)\,(x_j)\,\| \;\le\; C\,N^{-1}\,lnN, \qquad x_j \in (0, 1] \tag{3.11}$$

As the continuous and discrete solutions coincide at $x = 0$, applying discrete stability result to the mesh function $W_\varepsilon - w_\varepsilon$ leads to the required estimate of the error in the singular component of the solution

$$\|\,(W_\varepsilon - w_\varepsilon)\,(x_j)\,\| \le\; C\,N^{-1}\,ln\,N, \qquad j = 0(1)N \qquad \blacksquare$$

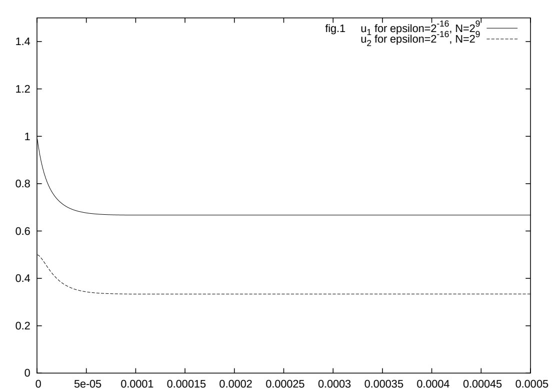
4 Numerical examples

In this section two examples are considered. All computations are performed in Fortran 77 with double precision in Linux using a Pentium PC.

Example 4.1 *Consider the IVP ,*

$$
\begin{aligned}
\varepsilon u'_1(x) + (2 + x)u_1(x) - u_2(x) &= 1 + x, \\
\varepsilon u'_2(x) - (1 + x)u_1(x) + (2 + x)u_2(x) &= x, \quad x \in (0, 1] \\
u_1(0) = 1 \;\;,\;\; u_2(0) &= 0.5
\end{aligned}
$$

This IVP is solved using the fitted mesh method presented in Section 3. In Table 1 we present the two mesh differences, the order of convergence and the error constant, which are calculated using the algorithm presented on page 166 of [2]. In Figures 1 and 2 the solution is displayed for particular values of ε and N.



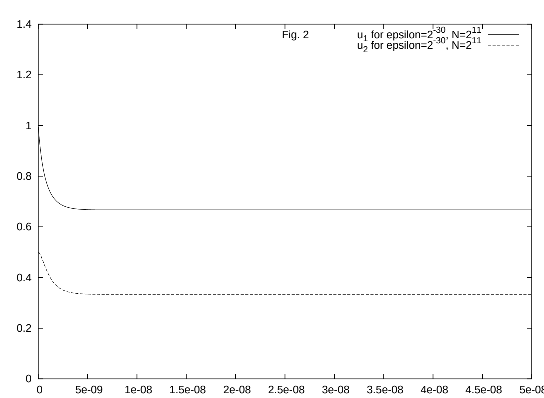


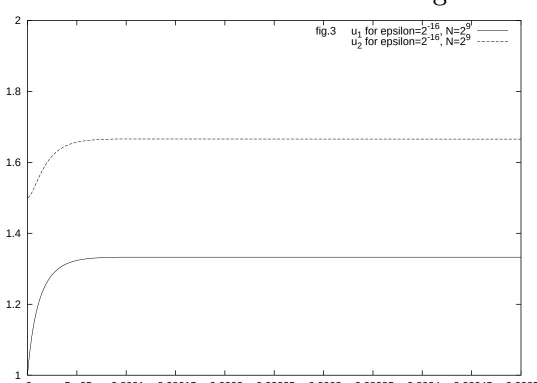
Table 1: *Two mesh differences D_ε^N for Example 4.1 for $\alpha = 1.0$ on Shishkin mesh.*

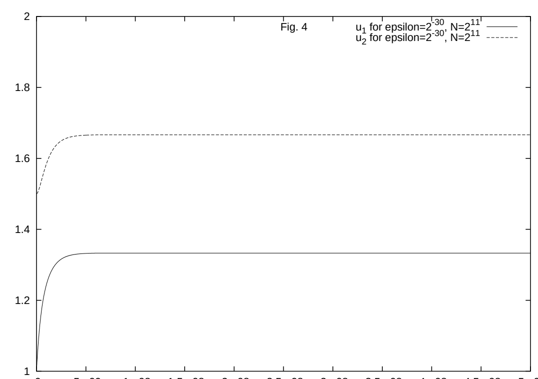
ε	Number of mesh points N								
	8	16	32	64	128	256	512	1024	2048
2^{-2}	0.175-1	0.104-1	0.569-2	0.299-2	0.153-2	0.776-3	0.391-3	0.196-3	0.982-4
2^{-6}	0.897-2	0.706-2	0.572-2	0.372-2	0.242-2	0.145-2	0.853-3	0.485-3	0.271-3
2^{-10}	0.875-2	0.692-2	0.561-2	0.366-2	0.237-2	0.143-2	0.841-3	0.478-3	0.267-3
2^{-14}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-18}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-22}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-26}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-30}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-34}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-38}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
D^N	0.175-1	0.104-1	0.607-2	0.391-2	0.255-2	0.153-2	0.893-3	0.509-3	0.297-3
p^N	0.746+0	0.783+0	0.632+0	0.613+0	0.740+0	0.777+0	0.810+0	0.773+0	
$C_{0.613}^N$	0.181+0	0.165+0	0.147+0	0.145+0	0.145+0	0.132+0	0.118+0	0.103+0	0.926-1
The order of convergence = 0.613									
The error constant = 0.181									

Example 4.2 *Consider the IVP ,*

$$
\begin{aligned}
\varepsilon u'_1(x) + 2u_1(x) - (1 + (x/2))u_2(x) &= 1, \\
\varepsilon u'_2(x) - u_1(x) + (2 + 2x)u_2(x) &= x + 2, \quad x \in (0, 1] \\
u_1(0) = 1 \ , \ \ u_2(0) &= 1.5
\end{aligned}
$$

This IVP is also solved using the fitted mesh method presented in Section 3. In Table 2 we present the numerical results and in Figures 3 and 4 we exhibit the solution of this IVP.





References

[1] E. P. Doolan, J. J. H. Miller, W. H. A. Schilders : Uniform numerical methods for problems with initial and boundary layers, Boole Press, Dublin, (1980)

[2] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, G. I. Shishkin : Robust computational techniques for boundary layers, Chapman and Hall/CRC, New York (2000)

[3] S. Matthews : Parameter robust methods for a system of coupled singularly perturbed ordinary differential reaction-diffution equations M.Sc thesis, Dublin City University (2000)

Table 2: *Two mesh differences D_ε^N for Example 4.2 for $\alpha = 1.0$ on Shishkin mesh.*

	Number of mesh points N								
ε	8	16	32	64	128	256	512	1024	2048
2^{-2}	0.161-1	0.930-2	0.505-2	0.264-2	0.134-2	0.682-3	0.343-3	0.172-3	0.861-4
2^{-6}	0.889-2	0.699-2	0.566-2	0.368-2	0.239-2	0.143-2	0.843-3	0.479-3	0.268-3
2^{-10}	0.874-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-14}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-18}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-22}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-26}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-30}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-34}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
2^{-38}	0.873-2	0.692-2	0.561-2	0.366-2	0.237-2	0.142-2	0.840-3	0.477-3	0.267-3
D^N	0.161-1	0.930-2	0.582-2	0.374-2	0.244-2	0.146-2	0.853-3	0.486-3	0.284-3
p^N	0.792+0	0.676+0	0.637+0	0.611+0	0.742+0	0.778+0	0.812+0	0.772+0	
$C^N_{0.611}$	0.166+0	0.146+0	0.140+0	0.137+0	0.137+0	0.126+0	0.112+0	0.977-1	0.874-1
The order of convergence = 0.611									
The error constant = 0.166									

[4] S. Matthews,J. J. H. Miller, E. O'Riordan, G. I. Shishkin : a parameter robust numerical method for a system of singularly perturbed ordinary differential equations, Nova Science Publishers,New York (2000)

[5] J. J. H. Miller, E. O'Riordan, G. I. Shishkin : Fitted numerical methods for singular perturbation problems. World scientific publishing Co.Pvt.Ltd., Singapore, (1996).

[6] O. A. Oleinik,V. N. Samokhin : Mathematical models in boundary layer theory, Series in Applied Mathematics and Scientific Computation, 15 Chapman&Hall,CRC Press Boca Raton(1999)

[7] H. G. Roos,M. Stynes and L. Tobiska : Numerical methods for singularly perturbed differential equations, Springer Verlag (1996)

[8] H. Schlichting : Boundary layer theory, McGraw Hill Series in Mechanical Engineering, McGraw Hill Book Company 7-th edition (1979)

[9] G. I. Shishkin : Approximation of solutions of singularly perturbed boundary value problems with a Parabolic boundary layer, USSR Comput.Maths.Math.Phys.,29,1-10(1989)

[10] G. I. Shishkin : Grid approximations of singularly perturbed elliptic and parabolic equations, Ural Branch of Russian Acad.Sci.,Ekaterinburg(1992)

[11] G. I. Shishkin : Mesh approximations of singularly perturbed boundary value problems for elliptic and parabolic equations,Comput.Maths.Math.Phys.,35(4):429-446(1995)

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