ABOUT UNSTEADY BOUNDARY LAYER ON A DIHEDRAL ANGLE

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Abstract

The considered unsteady flow of the viscous incompressible fluid is caused by the sudden motion of the dihedral angle with the constant velocity in the fluid being at rest. It is assumed, that the angle moves in the direction of the edge and the flow is layered. This flow simulates roughly a boundary layer in the neighborhood of the intersection the wing and the fuselage of an aircraft at enough distances from the leading and the trailing edges of the wing. In the case of right dihedral angle, the analytic solution of the considered problem is obtained, while in the case of arbitrary angle the problem for a function of three independent variables is reduced to a boundary value problem for an ordinary differential equation. The asymptotic behaviour of the solution of this equation by corresponding boundary conditions is investigated.

1. Introduction

The unsteady flow of the viscous incompressible fluid is considered. This flow is caused by the sudden motion of the dihedral angle Γ with the constant velocity U. We assume, that the angle moves in the direction of the edge and only one velocity component of fluid in this direction is different from zero. Such flows are called by layered ones [1].

The analytic solution of the considered problem was obtained in case of the right dihedral angle Γ while in case of arbitrary angle we reduced the problem to the self-similar boundary-value problem. Attention of author to this problem was attracted by Prof. Neyland in connection with the flow in the neighbourhood of the intersection wing and fuselage of an aircraft.

In the present work the power geometry methods [2] are used for obtaining self-similar solutions of boundary-value problems. These methods have simple algorithms. They were applied successfully both to linear and nonlinear problems in works [3] – [7] and others.

The unsteady layered flow, caused by the sudden motion of an infinite flat plate, was investigated for the first time by Stokes [8]. Steady boundary layer on the right dihedral angle was considered first by Loytsjansky (1936-1937) [9].

2. Initial equations

Let us denote cylindrical coordinates by r, θ, z; corresponding velocity components by v_r, v_ϑ, w; time, pressure, density and kinematic viscosity coefficient by t, p, ρ, ν. With these notations the equations of unsteady motion of viscous incompressible fluid and equation of continuity have the form:

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\vartheta}{r}\frac{\partial v_r}{\partial \vartheta} + w\frac{\partial v_r}{\partial z} - \frac{{v_\vartheta}^2}{r} =$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\bigtriangledown^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\vartheta}{\partial \vartheta}\right), \qquad (2.1)$$

$$\frac{\partial v_\vartheta}{\partial t} + v_r \frac{\partial v_\vartheta}{\partial r} + \frac{v_\vartheta}{r}\frac{\partial v_\vartheta}{\partial \vartheta} + w\frac{\partial v_\vartheta}{\partial z} + \frac{v_r v_\vartheta}{r} =$$

$$= -\frac{1}{\rho r}\frac{\partial p}{\partial \vartheta} + \nu\left(\bigtriangledown^2 v_\vartheta + \frac{2}{r^2}\frac{\partial v_r}{\partial \vartheta} - \frac{v_\vartheta}{r^2}\right), \tag{2.2}$$

$$\frac{\partial w}{\partial t} + v_r\frac{\partial w}{\partial r} + \frac{v_\vartheta}{r}\frac{\partial w}{\partial \vartheta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\bigtriangledown^2 w, \tag{2.3}$$

$$\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_\vartheta}{\partial \vartheta} + \frac{\partial (r w)}{\partial z} = 0, \tag{2.4}$$

where

$$\bigtriangledown^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} + \frac{\partial^2}{\partial z^2}.$$

Let the axis Oz be directed along the edge of the dihedral angle Γ having linear angle $\alpha \in (0, \pi/2]$. We shall consider layered flow setting $v_r = v_\vartheta = 0$. Then the equations (2.1) – (2.4) take the form:

$$\frac{\partial p}{\partial r} = 0, \tag{2.5}$$

$$\frac{1}{r}\frac{\partial p}{\partial \vartheta} = 0, \tag{2.6}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \vartheta^2}\right) - \frac{\partial w}{\partial t} \tag{2.7}$$

$$\frac{\partial w}{\partial z} = 0. \tag{2.8}$$

¿From the equations (2.5) and (2.6) it follows that left side of the equation (2.7) is independent on r and ϑ, but right side is independent on z according to (2.8). Consequently both sides of equation (2.7) are independent from space variables, but may depend only from time t. Let

$$\frac{\partial p}{\partial z} = F(t). \tag{2.9}$$

Usually the function $F(t)$ is given. Let us assume that $p \to p_0 = \text{const}$ both by $z \to -\infty$ and $z \to +\infty$. Then since the flow is caused only by the wall motion, we shall assume that $F(t) = 0$ and the equation (2.7) is written in the form:

$$\frac{\partial w}{\partial t} - \left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{s}\frac{\partial w}{\partial s} + \frac{1}{s^2}\frac{\partial^2 w}{\partial \vartheta^2}\right) = 0, \tag{2.10}$$

where $s = r/\sqrt{\nu}$.

We connect the frame of reference (r, ϑ, z) with dihedral angle Γ. In that case, the initial and boundary conditions for the function $w(s, \vartheta, t)$ are:

$$w = 0, \quad \text{as} \quad t \le 0, \tag{2.11}$$

$$w = 0, \quad \text{as} \quad t > 0 \quad \text{and} \quad \vartheta = 0 \quad \text{or} \quad \vartheta = \alpha, \tag{2.12}$$

$$w \to U, \quad \text{as} \quad \vartheta = \alpha/2, \quad s \to \infty. \tag{2.13}$$

3. The case of arbitrary dihedral angle

The boundary layer inside the dihedral angle Γ is interesting only in a neighbourhood the edge, where boundary layers on the faces influence each other. Outside this neighbourhood each of this boundary layers is described by well-known Blasius solution.

Let us reduce the equation (2.10) to the one having only two independent variables. This simple example demonstrates the application of the power geometry methods (v. Section 5) for the obtaining self-similar solutions.

The support of the differential polynomial in left hand side of the equation (2.10) has three different points: $Q_1 = (-1, 0, 0, 1)$, $Q_2 = (0, -2, 0, 1)$, $Q_3 = (0, -2, -2, 1)$ in the space (q_1, q_2, q_3, q_4), where q_1, q_2, q_3, q_4 correspond respectively to t, s, ϑ, w. The form of self-similar solution depends on the vector $P = (p_1, p_2, p_3, p_4)$. This vector must be perpendicular to the vectors $Q_2{}' = Q_2 - Q_1 = (1, -2, 0, 0)$, $Q_3{}' = Q_3 - Q_1 = (1, -2, -2, 0)$, $Q_4 = (0, 0, 0, 1)$. The last vector associates with nonzero boundary condition (2.13). As a result, we have $p_1 - 2p_2 = 0$, $p_3 = p_4 = 0$. Setting $p_1 = 1$, we obtain: $P = (1, 1/2, 0, 0)$; $\gamma_2 = p_2/p_1 = 1/2$, $\gamma_3 = p_3/p_1 = 0$, $\gamma_4 = p_4/p_1$=0; $\zeta_1 = s/t^{\gamma_2} = s/t^{1/2}$, $\zeta_2 = \vartheta/t^{\gamma_3} = \vartheta/t^0 = \vartheta$, $w = t^{\gamma_4}\varPhi_0(\zeta_1, \zeta_2) = U\varPhi(\zeta_1, \zeta_2)$.

Thus,

$$w = U\varPhi(\zeta_1, \zeta_2), \quad \zeta_1 = s/\sqrt{t}, \quad \zeta_2 = \vartheta. \tag{3.1}$$

Substituting these expressions into equation (2.10) we obtain:

$$\zeta_1{}^2 \frac{\partial^2 \varPhi}{\partial \zeta_1{}^2} + \zeta_1 \left(1 + \frac{1}{2}\zeta_1{}^2\right) \frac{\partial \varPhi}{\partial \zeta_1} + \frac{\partial^2 \varPhi}{\partial \zeta_2{}^2} = 0. \tag{3.2}$$

The separation of variables

$$\varPhi(\zeta_1, \zeta_2) = \varPhi_1(\zeta_1)\varPhi_2(\zeta_2), \tag{3.3}$$

lead to next equations:

$$\zeta_1{}^2 \varPhi_1{}'' + \zeta_1 \left(1 + \frac{1}{2}\zeta_1{}^2\right) \varPhi_1{}' - \lambda^2 \varPhi_1 = 0, \tag{3.4}$$

$$\varPhi_2{}'' + \lambda^2 \varPhi_2 = 0, \tag{3.5}$$

where $\lambda = \pi/\alpha$ in view of the boundary condition (2.12). A particular solution to the equation (3.5) has the form

$$\varPhi_2 = \sin \frac{\pi\vartheta}{\alpha}. \tag{3.6}$$

Let us obtain the asymptotics of the solution to the equation (3.4) with the boundary conditions:

$$\varPhi_1(0) = 0, \quad \varPhi_1(\infty) = 1. \tag{3.7}$$

Neglecting by $\zeta_1{}^2/2$ in comparison with 1 as $\zeta_1 \ll 1$, we reduce the equation (3.4) to the Euler equation:

$$\zeta_1{}^2 \varPhi_1{}'' + \zeta_1 \varPhi_1{}' - \lambda^2 \varPhi_1 = 0. \tag{3.8}$$

The general solution of this equation is

$$\Phi_1 = C_1{\zeta_1}^{\lambda} + C_2{\zeta_1}^{-\lambda} \ \ (C_1, C_2 - \text{arbitrary constants}).$$

Assuming $C_2 = 0$ we obtain $\Phi_1 = C_1{\zeta_1}^{\lambda}$ and

$$w = C_1{\zeta_1}^{\lambda}(1 + o(1))\sin\lambda\vartheta \ \text{ as } \ \zeta_1 \to 0. \tag{3.9}$$

We shall see in the next Section that $C_1 = 2/\pi$ if $\alpha = \pi/2$.

Let now $\zeta_1 \to +\infty$. Then using the Laurent series we find that

$$\Phi_1 = 1 - \frac{\lambda^2}{{\zeta_1}^2} + O\left(\frac{1}{{\zeta_1}^4}\right), \tag{3.10}$$

nd hence

$$w = U\left(1 - \frac{\lambda^2}{{\zeta_1}^2} + O\left(\frac{1}{{\zeta_1}^4}\right)\right)\sin\lambda\vartheta \ \text{ as } \ \zeta_1 \to \infty, \tag{3.11}$$

where $\lambda = \pi/\alpha$.

We shall see further that these asymptotics coinside with ones for the analytical solution in case of $\alpha = \pi/2$, i.e. $\lambda = 2$.

4. The case of right dihedral angle

In special case of $\alpha = \pi/2$ we consider our problem in Cartesian rectangular coordinate system (x, y, z). Let us denote corresponding velocity components by u, v, w. Then the equations of motion and equation of continuity have the form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\bigtriangledown^2 u, \tag{4.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\bigtriangledown^2 v, \tag{4.2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\bigtriangledown^2 w, \tag{4.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{4.4}$$

where

$$\bigtriangledown^2 = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right).$$

Let us consider layered flow setting $u = v = 0$. Then system of equations (4.1)–(4.4) may be written as

$$\frac{\partial p}{\partial x} = 0, \tag{4.5}$$

$$\frac{\partial p}{\partial y} = 0, \tag{4.6}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right). \tag{4.7}$$

$$\frac{\partial w}{\partial z} = 0. \tag{4.8}$$

In the Section 2 we saw that

$$\frac{\partial p}{\partial z} = F(t) = 0.$$

$$\frac{\partial w}{\partial t} = \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right). \tag{4.9}$$

This equation is classic heat conducting one. We find the self-similar variables with the help of power geometry (v. Section 5). These well-known variables are

$$\xi = \frac{x}{2\sqrt{\nu t}}, \quad \eta = \frac{y}{2\sqrt{\nu t}}, \quad \varphi(\xi, \eta) = \frac{w}{U}. \tag{4.10}$$

Change of variables in the equation (4.9) gives

$$\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \eta^2} + 2\xi \frac{\partial \varphi}{\partial \xi} + 2\eta \frac{\partial \varphi}{\partial \eta} = 0. \tag{4.11}$$

For this equation we have following boundary conditions:

$$\varphi = 0 \;\text{ as } \xi = 0 \;\text{ and as } \eta = 0, \tag{4.12}$$

$$\varphi = 1 \;\text{ as }\; \xi = \eta \to +\infty. \tag{4.13}$$

Separation of variables

$$\varphi(\xi, \eta) = \varphi_1(\xi)\varphi_2(\eta). \tag{4.14}$$

lead to equations

$$\varphi_1{}''(\xi) + 2\xi \varphi_1{}'(\xi) = 0, \tag{4.15}$$

$$\varphi_2{}''(\eta) + 2\eta \varphi_2{}'(\eta) = 0. \tag{4.16}$$

The solutions to these equations are expressed by means of Gauss error function:

$$\operatorname{erf} \gamma = \frac{2}{\sqrt{\pi}} \int_0^\gamma e^{-\sigma^2} \, d\sigma.$$

By substitution of these solutions into the expression (4.14) we obtain

$$\varphi(\xi, \eta) = \operatorname{erf} \xi \; \operatorname{erf} \eta. \tag{4.17}$$

and

$$w(x, y, t) = \frac{4U}{\pi} \int_0^\xi e^{-\sigma^2} \, d\sigma \int_0^\eta e^{-\sigma^2} \, d\sigma \tag{4.18}$$

where ξ and η are given in the form (4.10).

It is evidently that the boundary conditions (4.12)–(4.13) are satisfied since erf $(+\infty) = 1$.

It is not difficult to verify that in the case $\alpha = \pi/2$ the formulas (3.9) and (3.11) coincide with the corresponding asymptotics of the solution (4.18). This comparison shows that in (3.9) $C_1 = 2/\pi$ as $\alpha = \pi/2$. We do not make analogous comparison for $\alpha < \pi/2$ since separation of independent variables is impossible if we make use of oblique Cartesian coordinates.

5. On the power geometry

Let us give some information about the power geometry which we concern in the present paper. The details are in monograph [2].

We shall consider the boundary-value problems for the functions $x_3, ..., x_n$ of two independent variables x_1, x_2. Let $X = (x_1, ..., x_n)$. The differential monomial $a(X)$ is a product of the usual monomial $c\,{x_1}^{r_1}{x_2}^{r_2}...{x_n}^{r_n} = c\,X^R$ and a finite number of partial derivatives $\partial^l x_m/\partial x_1^{l_1}\partial x_2^{l_2}$, where $c = \text{const}$, $R = (r_1, ..., r_n) \in \mathbb{R}^n$, $m = 3, ..., n$, $l = l_1 + l_2$. To each differential monomial $a(X)$ there corresponds the vectorial power exponent $Q(a) \in \mathbb{R}^n$ according to the following rules: $Q(cX^R) = R$; $Q(\partial^l x_k/\partial x_1^{l_1}\partial x_2^{l_2}) = -l_1E_1 - l_2E_2 + E_k$, where E_j denotes the j-th unit vector; $Q(a_1a_2) = Q(a_1) + Q(a_2)$, a_1 and a_2 are differential monomials. The finite sum of differential monomials

$$f(X) = \sum a_k(X) \tag{5.1}$$

is called the differential polynomial. In $\mathbb{R}^n$ to the polynomial (5.1), there corresponds support $\mathbf{S}(f) = \{Q(a_k)\}$, which is the set of all vectorial power exponents of its monomials. The convex hull $\Gamma(f)$ of the support $\mathbf{S}(f)$ is called the Newton-Bruno polyhedron of the polynomial (2.1). Its boundary $\partial\Gamma(f)$ consists of faces $\Gamma_j^{(d)}$ where $d = \dim(\Gamma_j^{(d)})$. To each face $\Gamma_j^{(d)}$ there corresponds truncated polynomial $\hat{f}_j^{(d)}(X) = \sum a_k(X)$ over $k : Q(a_k) \in \Gamma_j^{(d)}$. Let $\mathbb{R}_*^n$ denote the space dual to the space $\mathbb{R}^n$. There exists the scalar product $\langle P, Q\rangle = p_1q_1 + \ldots + p_nq_n$ for $P = (p_1, \ldots, p_n) \in \mathbb{R}_*^n$ and $Q = (q_1, \ldots, q_n) \in \mathbb{R}^n$. For each face $\Gamma_j^{(d)}$, there exists such a vector $P \in \mathbb{R}_*^n$, that $\langle P, Q_1\rangle > \langle P, Q_2\rangle$ for any $Q_1 \in \Gamma_j^{(d)}$ and $Q_2 \in \Gamma \setminus \Gamma_j^{(d)}$. In $\mathbb{R}^n$ the hyperplane $\langle P, Q\rangle = \text{const} = \langle P, Q_1\rangle$ is supporting to the polyhedron $\Gamma(f)$ and the vector P is the exterior normal vector to the face $\Gamma_j^{(d)}$, i.e. directed outside of $\Gamma(f)$. For example let us consider the system of three differential equations

$$f_i(X) = 0, \quad i = 1, 2, 3, \tag{5.2}$$

where $f_i(X)$ are differential polynomials. To each of them there corresponds its support $\mathbf{S}(f_i)$, its polyhedron $\Gamma(f_i)$, its set of faces $\Gamma_{ij}^{(d)}$ and truncated equations $\hat{f}_{ij}^{(d)}(X) = 0$. The system of the equations

$$\hat{f}_{ij_i}^{(d_i)}(X) = 0, \quad i = 1, 2, 3, \tag{5.3}$$

is called the truncated system, if a vector $P = (p_1, ..., p_5) \in \mathbb{R}_*^5$ is the exterior normal vector to faces $\Gamma_{ij_i}^{(d_i)}$ of the polyhedrons $\Gamma(f_i)$ for $i = 1, 2, 3$. The corresponding truncated system (5.3) exists for each vector $P \neq 0$.

Let for $x_1 \to +\infty$ the system (5.2) have a solution of the form $x_m = x_1^{\gamma_m}\varphi_m(\zeta) + O({x_1}^{\gamma_m - \varepsilon})$, $m = 3, 4, 5$, where

$$\zeta = x_2x_1^{-\gamma_2}, \quad \varepsilon > 0, \quad \text{and} \quad \gamma_m = p_m/p_1, \quad m = 2, 3, 4, 5. \tag{5.4}$$

Then the truncation of the solution

$$x_m = x_1^{\gamma_m}\varphi_m(\zeta), \quad m = 3, 4, 5, \tag{5.5}$$

is a solution of the truncated system (5.3) [2,Ch.VI]. The truncated system (5.3) is quasihomogeneous, i.e. three faces $\Gamma^{(d_1)}_{1j_1}$, $\Gamma^{(d_2)}_{2j_2}$, $\Gamma^{(d_3)}_{3j_3}$ can be put into one and the same linear subspace $B \subset \mathbb{R}^5$ by means of parallel translations. Let the space B be two-dimensional subspace with the basis $B_1, B_2 \in \mathbb{R}^5$ and we have for example the boundary conditions

$$x_3 = c_3 x_2^{n_3}, \quad x_5 = c_5 x_2^{n_5} \ \text{ for } \ x_2 \to \infty, \quad c_3, c_5 = \text{const} \tag{5.6}$$

for the truncated system (5.3). Then this system has the self-similar solution (5.4), where the vector $P = (p_1, ..., p_5)$, is orthogonal to the vectors $B_1,\ B_2,\ -n_3E_2+E_3,\ -n_5E_2+E_5$. It is possible to to prove that $p_1 > 0$ if we take an interest in the asymptotics for $x_1 \to \infty$.

References

[1] *Schlichting, H.*, Grenzschicht-Theorie. Verlag G. Braun. Karlsruhe. 1951.

[2] *Bruno, A.D.*, Power geometry in algebraic and differential equations. Fizmatlit, Moscow, 1998 (Russian). = Elsevier Science, Amsterdam, 2000.

[3] *Bruno, A.D.*, Algorithmic analysis of singular perturbations and boundary layers by power geometry. Proc. of the International Conference on Boundary and Interior Layers (BAIL 2002), Perth, Western Australia, 2002, pp. 251-256.

[4] *Vasiliev, M.M.*, Asymptotics of some viscous, heat conducting gas flows //Proc. of the International Conference on Boundary and Interior Layers (BAIL 2002), Perth, Western Australia, 2002, pp. 251-256.

[5] *Vasiliev, M.M.*, About the obtaining self-similar solutions of the Navier-Stokes equations by methods of power geometry//Proc of ISAAC-2001, World Scientific, Singapore, 2003, pp. 93-101.

[6] *Bruno, A.D., Shadrina, T.V.*, The axially symmetric boundary layer on a needle //Proc. of the International Conference on Boundary and Interior Layers(BAIL 2004), Toulouse, France, 2004, 9 pp.

[7] *Vasiliev, M.M.*, On the self-similar solution of some magnetohydrodynamic problems //Proc. of the International Conference on Boundary and Interior Layers(BAIL 2004), Toulouse, France, 2004, 6 pp.

[8] *Stokes, G.G.*, On the effect of internal friction of fluid on the motion of pendumlums. Trans. Cambr. Phil.. IX, 8. 1851.

[9] *Loytsjansky, L.G.*, Laminar Boundary Layer. Fizmatgiz, Moscow, 1962 (Russian).