Computing Numerical Solutions of Lane-Emden Equation by Bifurcation Method *

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Abstract: Using the Liapunov-Schmidt method and symmetry-breaking bifurcation theory, we compute and visualize multiple solutions of Lane-Emden equation on a bounded domain of $\mathbb{R}^2$ with a homogeneous Dirichlet boundary condition, which plays an important role in stellar structure and evolution theory. The domains we consider here include the unit square and the square cut by small square. Along with the nontrivial solution branches of the corresponding nonlinear bifurcation problem bifurcated from its bifurcation points, numerical multiple solutions of Lane-Emden equation with different symmetry are obtained and visualized.
Key words: Lane-Emden equation, multiple solutions, symmetry-breaking, bifurcation, Liapunov-Schmidt reduction.

1 Introduction

In this paper, the Lane-Emden equations of index p

$$\begin{cases} \Delta u + u^p = 0, (x, y) \in \Omega \\ u|_{\partial\Omega} = 0, (x, y) \in \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded open domain in $\mathbb{R}^2, p > 0$, are concerned. Equation (1.1) describes the behavior of the density of a gas sphere in hydrostatic equilibrium in appropriate units. The index p, which is called the polytropic index in astrophysics, is larger than $\frac{1}{2}$. It means that no polytropic stellar system can be homogeneous in galactic dynamics([3],[7]).

The critical point theory was applied to prove the existence and multiplicity of solutions under various assumptions([2], [10]). But what distribution and structure they have and how to compute them have attracted the attention of many mathematicians, physicists and engineers. Due to the multiplicity, degeneracy and instability of the critical points with higher Morse

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index, the computation of multiple solutions encounters essential difficulties and is truly challenging. Since 90's of last century, numerical works to compute numerical solutions of (1.1) appeared in the literature. The mountain-pass algorithm, the scaling iterative algorithm ,the monotone iteration, the direct iteration algorithm and the research extension method([4],[5],[6]) are used to compute the solutions of (1.1). But in these algorithms " good guess of solution" of (1.1), which is a difficult task, is needed. Therefore only few solutions of (1.1) are computed yet.

In this paper, we try to use the bifurcation method to overcome this difficulty. Our main idea is to embed (1.1) into nonlinear elliptic BVP with parameter λ of the form

$$\begin{cases} F(u,\lambda) = \Delta u + \lambda u + u^p = 0, (x,y) \in \Omega, \\ u|_{\partial\Omega} = 0, (x,y) \in \partial\Omega. \end{cases} \tag{1.2}$$

According to the bifurcation theory (1.2) has nontrivial solution branches which bifurcate from its bifurcation points, so we can compute the solutions of (1.1) by using continuation method([1]) along these nontrivial solution branches of (1.2) until $\lambda = 0$. Many new solutions of (1.1) with different symmetry are computed by the bifurcation method.

An outline of the paper is as follows. In Section 2, the symmetry of solutions of (1.1) and (1.2) is discussed in detail and the symmetry-breaking bifurcation theory is used to classify the nontrivial solutions according to their symmetry. In Section 3, the numerical algorithm is given. Finally, in Section 4,numerical solutions of (1.1) with different symmetry for different p and different domain are visualized. Most of numerical results here are new.

2 Equivariance property of (1.2) and symmetry-breaking bifurcation

In this section the nonlinear bifurcation problems (1.2) in domains which have certain symmetry properties are discussed. For simplicity, we illustrate the procedure with $\Omega = [0,1] \times [0,1]$.

The symmetry group of a square is $D_4 = \{I, R_1, R_2, R_3, S_1, S_1', S_2, S_2'\}$, where

$$\begin{aligned} &Iu(x,y) = u(x,y), && S_1u(x,y) = u(x,1-y), \\ &S_1'u(x,y) = u(1-x,y), && S_2u(x,y) = u(y,x), \\ &S_2'u(x,y) = u(1-y,1-x), && R_1u(x,y) = u(1-y,x), \\ &R_2u(x,y) = u(1-x,1-y), && R_3u(x,y) = u(y,1-x), \\ &Z_2 = \{I,-I\}, && \Gamma = D_4 \times Z_2. \end{aligned}$$

Obviously, if p is *odd*, (1.2) is Γ-equavariant, i.e.

$$F(\gamma u, \lambda) = \gamma F(u, \lambda), \forall \gamma \in \Gamma$$

and if p is *even*, (1.2) is D_4-equavariant. Notice $u \equiv 0$ is trivial solution of (1.2) with Γ-symmetry, $\forall \lambda \in \mathbb{R}$.

The isotropy subgroups of D_4-equivariant problem (1.2) are

$$\begin{array}{ll} D_4 = \{I, R_1, R_2, R_3, S_1, S_1', S_2, S_2'\}, & \\ \Sigma_R = \{I, R_1, R_2, R_3\}, & \Sigma_r = \{I, R_2\}, \\ \Sigma_1 = \{I, S_1\}, & \Sigma_1' = \{I, S_1'\}, \\ \Sigma_2 = \{I, S_2\}, & \Sigma_2' = \{I, S_2'\}, \\ \Sigma_d = \{I, R_2, S_2, S_2'\}, & \Sigma_M = \{I, R_2, S_1, S_1'\}. \end{array}$$

The fixed point spaces of these isotropy subgroups and their bases list as follows:

Fixed point space	Orthogonal Bases
X^{D_4}	$sin(2k-1)\pi x sin(2k-1)\pi y$ or $sin(2k_1-1)\pi x sin(2k_2-1)\pi y + sin(2k_1-1)\pi y sin(2k_2-1)\pi x$
X^{Σ_d}	$sin 2k\pi x sin 2k\pi y$ or $sin 2k_1\pi x sin 2k_2\pi y + sin 2k_1\pi y sin 2k_2\pi x$
X^{Σ_1}	$sin 2k_1\pi x sin(2k_2-1)\pi y$
$X^{\Sigma_1'}$	$sin 2k_1\pi y sin(2k_2-1)\pi x$
X^{Σ_r}	$sin 2k_1\pi x sin 2k_2\pi y$
X^{Σ_M}	$sin(2k_1-1)\pi x sin(2k_2-1)\pi y$
X^{Σ_2} or $X^{\Sigma_2'}$	$sin 2k_1\pi x sin(2k_2-1)\pi y + sin 2k_1\pi y sin(2k_2-1)\pi x$

where $k, k_1, k_2 \in Z^+$, $k_1 \neq k_2$.

For each subgroup Σ of Γ the fixed point space X^Σ of Σ is invariant under (1.2) leading to the reduced problems

$$F_\Sigma(u, \lambda) = 0, (u, \lambda) \in X^\Sigma \times \mathbb{R}. \tag{2.1}$$

Solutions of (2.1) are precisely the solutions of(1.2) with at least the symmetry of X^Σ. The symmetry-breaking bifurcation theory([8],[9]) tells us that if there is a bifurcation point $\lambda = \lambda_0$ and the corresponding eigenfunction $\varphi_0 \in X^\Sigma$, then the nontrivial solution branch with Σ symmetry will bifurcate from the trivial solution at $\lambda = \lambda_0$. The reduced problem (2.1) is very useful in reducing the computational cost of determining the solution.

The reduced problems (2.1) in the fixed point space X^Σ will simplify the solved domain of the problem (1.1). At the same time, the boundary conditions should change according to the underlying symmetry Σ. The following is the table, which describes the solved domain and corresponding boundary condition for different Σ in the reduced problem (2.1).

symmetry	domain	boundary condition
Σ_1'	$[0,\frac{1}{2}]\times[0,1]$	$u\|_{x=0}=u\|_{y=0}=u\|_{y=1}=0, \frac{\partial u}{\partial x}\|_{x=\frac{1}{2}}=0$
Σ_1	$[0,1]\times[0,\frac{1}{2}]$	$u\|_{x=0}=u\|_{x=1}=u\|_{y=0}=0, \frac{\partial u}{\partial y}\|_{y=\frac{1}{2}}=0$
Σ_r	$[0,\frac{1}{2}]\times[0,1]$	$u\|_{x=0}=u\|_{y=0}=u\|_{y=1}=0, R_2u(x,y)=u(x,y)$
Σ_M	$[0,\frac{1}{2}]\times[0,\frac{1}{2}]$	$u\|_{x=0}=u\|_{y=0}=u\|_{x=\frac{1}{2}}=u\|_{x=\frac{1}{2}}=0$
Σ_R	$[0,\frac{1}{2}]\times[0,\frac{1}{2}]$	$u\|_{x=0}=u\|_{y=0}=0, R_iu(x,y)=u(x,y), i=1,2,3$
Σ_2	$0\le x\le 1$, $0\le y\le x$	$u\|_{x=1}=u\|_{y=0}=0$ $\frac{\partial u}{\partial x}\|_{x=y}-\frac{\partial u}{\partial y}\|_{x=y}=0$
Σ_2'	$0\le x\le 1$, $1-x\le y\le 1$	$u\|_{x=1}=u\|_{y=1}=0$ $\frac{\partial u}{\partial x}\|_{x+y=1}+\frac{\partial u}{\partial y}\|_{x+y=1}=0$
Σ_d	$0\le x\le 1$, $0\le y\le \frac{1}{2}$, $y\le x$, $y\le 1-x$,	$u\|_{x=0}=0$ $\frac{\partial u}{\partial x}\|_{x=y}-\frac{\partial u}{\partial y}\|_{x=y}=0$ $\frac{\partial u}{\partial x}\|_{x+y=1}+\frac{\partial u}{\partial y}\|_{x+y=1}=0$
D_4	$0\le x\le \frac{1}{2}$ $0\le y\le x$	$u\|_{y=0}=\frac{\partial u}{\partial x}\|_{x=\frac{1}{2}}=0$ $\frac{\partial u}{\partial x}\|_{x=y}-\frac{\partial u}{\partial y}\|_{x=y}=0$

Table 1 The reduced problems for different Σ

Table 2 shows the numbers of the nontrivial solution branches of (1.2) bifurcated from the bifurcation points $\lambda_{n,m}=(n^2+m^2)\pi^2$, (n,m=1,2,3,4) and their symmetry.

bifurcation points	nontrivial solution branches	
	number	symmetry
$(2\pi^2,0)$	1	D_4
$(5\pi^2,0)$	2	Σ_1,Σ_2
$(8\pi^2,0)$	1	Σ_d
$(10\pi^2,0)$	2	D_4,Σ_M
$(13\pi^2,0)$	2	Σ_1,Σ_2
$(17\pi^2,0)$	2	Σ_1,Σ_2
$(18\pi^2,0)$	1	D_4
$(20\pi^2,0)$	2	Σ_d,Σ_r
$(25\pi^2,0)$	2	Σ_1,Σ_2
$(32\pi^2,0)$	1	Σ_d

Table 2 Number and symmetry of nontrivial solution branches of (1.2)

3 Numerical algorithms

After discretization of (1.2) by the finite difference method, the Liapunov-Schmidt reduction and the numerical continuation method can be used to solve nontrivial solution of (1.1). In general, these numerical algorithms can be divided into three steps:

Step 1: The most important thing is to get nontrivial solution branches near bifurcation points. The Liapunov-Schmidt reduction principle is used to overcome the difficulty caused by the singularity near the bifurcation point. Let

$$u = \tau * \phi_0 + w, \eta = \lambda - \lambda_0$$

where λ_0 is a eigenvalue of the operator Δ on Ω, ϕ_0 is the corresponding eigenfunction and w has the same symmetry with the eigenfunction ϕ_0 satisfies $\langle \phi_0, w \rangle = 0$, τ is small parameter. Next we solve extended system:

$$\begin{cases} \Delta w + (\eta + \lambda_0) w + \eta \tau \phi_0 + (\tau \phi_0 + w)^p = 0, \\ w|_{\partial\Omega} = 0, \\ \langle \phi_0, w \rangle = 0. \end{cases} \tag{3.1}$$

After discretizing (3.1) with the five-points difference scheme on an equidistant mesh($h = \frac{1}{20}$), we can get the numerical solution of (3.1) by using the Gauss-Newton method for different τ. Continuing τ until u is far away from the trivial solution.

Step 2: Choose $u_{end} = \tau_{end}\phi_0 + w_{end}$, $\lambda_{end} = \lambda_0 + \eta_{end}$ as a start point, we solve

$$\begin{cases} F(u, \lambda) = \Delta_h u + \lambda u + u^p = 0, \\ u|_{\partial\Omega} = 0. \end{cases}$$

directly by the Gauss-Newton iteration, where λ is parameter.

Step 3: Continue λ until $\lambda = 0$, we get the solution u of (1.1) with different symmetry and plot it.

$\lambda_0 = \lambda_{n,m} = (n^2 + m^2)\pi^2$ and $\phi_0 = \phi_{n,m} = \sin(n\pi x) sin(m\pi y)$ are chosen for $\Omega = [0,1] \times [0,1]$. We have also computed the solutions of (1.1) in a modified unit square, which is the domain of an unit square cut by a small square. There λ_0 and ϕ_0 must be computed numerically.

4 Numerical results

4.1 p=3

We have computed many solutions of (1.1) with symmetries $D_4, \sum_2, \sum_1, \sum_d, \sum_M, \sum_r$ for$\Omega = [0,1] \times [0,1]$. The solutions are obtained by using continuation method along with the different branches of (1.2) bifurcated from the different bifurcation points λ_0. The solutions of (1.1) with different symmetries for$\Omega_i (i = 1, 2, ..., 9)$ are also computed by our methods ,whereΩ_1

is the domain of Ω cutted by the square [0.25,0.75]$\times$[0.25,0.75], Ω_2 is the domain of Ω cutted by the square[0.15,0.65]$\times$[0.25,0.75], Ω_3 is the domain of Ω cutted by the square [0.15,0.65]$\times$[0.15,0.65],Ω_4 is the domain of Ω cutted by the small square [0.45,0.55]$\times$[0.45,0.55], Ω_5 is the domain of Ω cutted by the small square [0.15,0.25]$\times$[0.15,0.25], Ω_6 is the domain of Ω cutted by the square [0,0.25]$\times$[0,0.25], Ω_7 is the domain of Ω cutted by the square [0,0.5]$\times$[0,0.5], Ω_8 is the domain of Ω cutted by the square [0,0.3]$\times$[0.35,0.65], Ω_9 is the domain of Ω cutted by two rectangles [0,0.3]$\times$[0,0.35] and [0,0.3] $\times$[0.65,1].

4.2 p=1.5

Five solutions of (1.1) are computed for $p = 1.5$.
TheD_4-symmetric solution of (1.1) on Ω,they are the D_4-symmetric solution of (1.1) on Ω_1,the Σ_1-symmetric solution of (1.1) on Ω_8,the Σ_1-symmetric solution of (1.1) on Ω_9,the Σ_2-symmetric solution of (1.1) on Ω_7.

Also, five solutions of (1.1) on different domains Ω, Ω_1, Ω_8, Ω_9 and Ω_7 are computed respectively for each case of $p = 2, 4, 5$.

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