Numerical Method for Blasius Equation on an infinite Interval *

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1 Introduction

Blasius problem on a half-infinite interval is considered. This problem has a place under mathematical modelling of viscid flow before thin plate. Blasius problem is a boundary value problem for a nonlinear third order ordinary differential equation on a half-infinite interval.

This problem was investigated in many articles. For example, G.I. Shishkin [1] studied asymptotic behavior of differential and difference solutions to get difference scheme with a finite number on nodes for enough long interval. We apply the method of marking of set of solutions, that satisfy the limit boundary condition at infinity to transform a problem under consideration to a problem for a finite interval [2]. To use that method we have problems, connected with nonlinearity of differential equation and with unbounded coefficient before second derivative. We offer to do transformation of independent variable to avoid a problem, connected with boundless of that coefficient.

We consider a Blasius problem as model nonlinear problem for application of developing technic.

2 Case of a linear problem

Consider a problem:

$$\varepsilon u''(x) + [a(x) + x]u'(x) = f(x), \tag{1}$$

$$u(0) = A, \ \lim_{x\to\infty} u(x) = 0. \tag{2}$$

Suppose,that $a(x) + x \geq 0, \varepsilon > 0$, functions $a(x), f(x)$ are smooth enough,

$$\exists \lim_{x\to\infty} a(x) = a_0, \ \lim_{x\to\infty} f(x) = 0.$$

Note, that domain is unbounded and solution $u(x)$ can be unbounded for a bounded function $f(x)$. Integrating equation (1), we get, that if $a(x) + x \geq \alpha > 0$, then

$$|u(x)| \leq |A| + \frac{1}{\alpha}\int_0^x |f(s)|\, ds.$$

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To avoid unboundedness of coefficient before first derivative, introduce new variable $t = \frac{x^2}{2}$. Then problem (1)-(2) become a form:

$$\varepsilon u''(t) + b(t)u'(t) = F(t),$$

$$u(0) = A, \ \lim_{t\to\infty} u(t) = 0, \tag{3}$$

where

$$b(t) = \frac{a(\sqrt{2t})}{\sqrt{2t}} + 1 + \frac{1}{2t}, \quad F(t) = f(\sqrt{2t})/(2t).$$

So, coefficient $b(t)$ is bounded for t, separated from zero. We can do some other replacement of x to do function $b(t)$ uniformly bounded, but it is not important for method, considered below.

By the next equation we mark solutions of differential equation (3), that satisfy the limit boundary condition at infinity [2, 3, 4]:

$$\varepsilon u'(t) + g(t)u(t) = \beta(t), \tag{4}$$

where $g(t)$ is solution of singular Cauchy problem for Riccati equation:

$$\varepsilon g'(t) + b(t)g(t) - g^2(t) = 0, \quad \lim_{t\to\infty} g(t) = 1, \tag{5}$$

$\beta(t)$ is solution of singular problem:

$$\varepsilon\beta'(t) + [b(t) - g(t)]\beta(t) = \varepsilon F(t), \quad \lim_{t\to\infty} \beta(t) = 0. \tag{6}$$

Thank to condition $\lim\limits_{t\to\infty} g(t) = 1$ every solution of equation (4) tends to zero at infinity and equation (4) picks that solutions of differential equation (3), that satisfy the limit boundary condition at infinity.

Lemma 1 *Let* $0 < b_1 \le b(t) \le b_2$. *Then* $b_1 \le g(t) \le b_2$.

Proof. At first prove, that $g(t) > 0, \quad 0 \le t < \infty$. Define

$$v(t) = \exp\left\{-\int_0^t \varepsilon^{-1} g(s)ds\right\}.$$

Obviously, $v(t)$ is solution of a problem:

$$\varepsilon v'' + b(t)v'(t) = 0,$$

$$v(0) = 1, \ \lim_{t\to\infty} v(t) = 0, \tag{7}$$

It follows from (7), that

$$v(t) > 0, \ v'(t) < 0, \ 0 < t < \infty.$$

Taking into account that $g(t) = -\varepsilon v'(t)/v(t)$, we get $g(t) > 0, \ 0 \le t < \infty$.

Prove, that $g(t) \le b_2$. Consider a problem:

$$\varepsilon g_2' + b_2 g_2 - g_2^2 = 0, \quad \lim_{t\to\infty} g_2(t) = b_2$$

with solution $g_2(t) = b_2$. Let $z = g_2 - g$. Then $z(t)$ is solution of a problem:

$$\varepsilon z'(t) - [g_2 - b(t) + g(t)]z(t) = (b(t) - g_2)g_2 \leq 0, \ \lim_{t\to\infty} z(t) = b_2 - b_\infty \geq 0. \tag{8}$$

If we suppose, that for some s $z(s) < 0$, then we shall have contradiction. It follows, that $g(t) \leq b_2$. In the same manner we can prove, that $g(t) \geq b_1$. Lemma is proved.

Using a problem (6), we can prove, that

$$||\beta||_\infty \leq ||g||_\infty \int\limits_0^\infty |F(s)|\, ds.$$

Using equation (4), transform a problem (3) to a problem for a finite interval:

$$\varepsilon u''(t) + b(t)u'(t) = f(t), \quad 0 < t < L,$$

$$u(0) = A, \ \varepsilon u'(L) + g(L)u(L) = \beta(L). \tag{8}$$

Prove, that problems (3) and (8) have a same solution for $0 \leq t \leq L$. Consider initial value problem:

$$\varepsilon u'(t) + g(t)u(t) = \beta(L), \ u(0) = A. \tag{9}$$

Taking into account equations (5),(6), we prove, that solution of a problem (9) satisfies to problems (3) and (8). Problems under consideration have unique solution, therefor problems (3) and (8) have a same solution for $0 \leq t \leq L$.

So, problem (3) is exactly reduced to a problem (8), formulated for a finite interval. We have only to find coefficients $g(L)$ and $\beta(L)$.

Functions $g(t)$, $\beta(t)$ as solutions of problems (5),(6) we can sick as asymptotic series:

$$g(t) \approx \sum_{n=0}^{N} \varepsilon^n g_n(t), \quad \beta(t) \approx \sum_{n=0}^{N} \varepsilon^n \beta_n(t), \tag{10}$$

or in a form:

$$g(t) \approx \sum_{n=0}^{N} (2t)^{-n/2} g_n, \ \beta(t) \approx \sum_{n=0}^{N} (2t)^{-n/2} \beta_n. \tag{11}$$

In case of representation (11) we suppose, that

$$a(\sqrt{2t}) = \sum_{n=0}^{N} (2t)^{-n/2} a_n + O\left((2t)^{-(N+1)/2}\right),$$

$$f(\sqrt{2t}) = \sum_{n=0}^{N} (2t)^{-n/2} f_n + O\left((2t)^{-(N+1)/2}\right).$$

In both cases (10) and (11) we got recurrent formulas on β_n, g_n. In particular, in case of representation (11) we have:

$$g_0 = 1, \ g_1 = a_0, \ g_2 = 1 + a_1, \ g_3 = -\varepsilon a_0 + 4a_0 + a_2 + 4a_0 a_1,$$

$$\beta_0 = 0, \ \beta_1 = -f_1, \ \beta_2 = -f_2 - \beta_1 \varepsilon^{-1}(g_3 - a_2).$$

Make inverse transformation of independent variable and get a next problem for a finite interval:

$$\varepsilon u''(x) + [a(x) + x]u'(x) = f(x), \quad 0 < x < L_0,$$
$$u(0) = A, \ \frac{\varepsilon}{L_0}u'(L_0) + g(L)u(L_0) = \beta(L_0). \tag{12}$$

Using maximum principle, we can prove next lemma.

Lemma 2 *Let $\tilde{u}(t)$ is solution of problem (12) with perturbed coefficients $\tilde{g}(L_0), \tilde{\beta}(L_0)$,*

$$|g(L_0) - \tilde{g}(L_0)|, \ |\beta(L_0) - \tilde{\beta}(L_0)| \le \Delta, \ \tilde{g}(L_0) \ge \sigma > 0.$$

Then

$$|u(x) - \tilde{u}(x)| \le \sigma^{-1}\Delta(|u(L_0)| + 1).$$

Consider a problem for numerical experiments:

$$u''(x) + \left[\frac{2x}{x+1} + x\right]u'(x) = 0,$$
$$u(0) = 1, \ \lim_{x\to\infty} u(x) = 0.$$

We compare different approaches for formulation of boundary condition at a finite point instead the limit boundary condition at infinity. Let $\Delta = \max_n |v_n^h - \tilde{v}_n^h|$, where $\tilde{v}_n^h$ is solution of the scheme of upwind differences for enough long interval $[0, L_0]$, $L_0 = 100$; v_n^h is solution of the same difference scheme on a shot interval depending on boundary condition. In Table 1 error Δ is represented for different approaches and intervals.

Table 1: Absolute errors for a linear problem

L	$w_n(L) = 0$	$\tilde{g}_n(L)$		
		g_0	$g_0 + g_1/L$	$g_0 + g_1/L + g_2/L^2$
2	$0.12e-01$	$0.95e-02$	$0.11e-02$	0.44e-04
3	$0.35e-03$	$0.20e-03$	$0.19e-04$	0.41e-05
5	$0.36e-07$	$0.13e-07$	$0.95e-09$	0.84e-10
7	$0.32e-12$	$0.86e-13$	$0.49e-14$	0.23e-15

3 Blasius problem

Consider Blasius problem:

$$u'''(x) + u(x)u''(x) = 0,$$
$$u(0) = 0, \ u'(0) = 0, \ \lim_{x\to\infty} u'(x) = 1. \tag{13}$$

Let

$$u(x) = v(x) + x, \quad w(x) = v'(x).$$

Then problem (13) can be written in a form:

$$v'(x) = w(x), \quad v(0) = 0,$$

$$w''(x) + [v(x) + x]w'(x) = 0,$$
$$w(0) = -1, \ \lim_{x\to\infty} w(x) = 0. \tag{14}$$

Consider iterative method for a problem (14):

$$v_n'(x) = w_n(x), \quad v_n(0) = 0,$$
$$w_n''(x) + [v_{n-1}(x) + x]w_n'(x) = 0, \ 0 < x < \infty,$$
$$w_n(0) = -1, \ \lim_{x\to\infty} w_n(x) = 0. \tag{15}$$

Lemma 3 *Let*

$$v_0(x) \geq v(x), \ v_0(x) \geq v_1(x), \ x > 0.$$

Then iterative method (15) converges.

Proof. Prove, that for every value of x sequences $v_n(x), \ w_n(x)$ are monotone decreasing and have low bounds.
Low bounds. Let $z_n(x) = v_n(x) - v(x), \ p_n(x) = w_n(x) - w(x)$. Compose a problem:

$$z_n'(x) = p_n(x), \quad z_n(0) = 0,$$
$$p_n''(x) + [v_{n-1}(x) + x]p_n'(x) = -z_{n-1}(x)w'(x), \ 0 < x < \infty,$$
$$p_n(0) = 0, \ \lim_{x\to\infty} p_n(x) = 0.$$

By induction prove, that for all $x > 0 \ \ z_n(x) \geq 0, \ p_n(x) \geq 0$. According to conditions of lemma $z_0(x) \geq 0$. Let $z_{n-1}(x) \geq 0$. Prove, that $p_n(x) \geq 0$. Suppose, that for some $s \ \ p_n(s) < 0$. Then there is point of negative minimum of function $p_n(x)$, it leads us to contradiction. So, $p_n(x) \geq 0$. It follows, that $z_n(x) \geq 0$. According to method of mathematical induction for all n $z_n(x) \geq 0, \ p_n(x) \geq 0$. It implies, that $v_n(x) \geq v(x), \ w_n(x) \geq w(x)$.

Monotony. Prove, that sequences $v_n(x), \ w_n(x)$ are monotone decreasing.

Let $z_n(x) = v_n(x) - v_{n+1}(x), \ p_n(x) = w_n(x) - w_{n+1}(x)$. Then $z_n(x) \ \ p_n(x)$ are solutions of a problem:

$$z_n'(x) = p_n(x), \quad z_n(0) = 0,$$
$$p_n''(x) + [v_{n-1}(x) + x]p_n'(x) = -z_{n-1}(x)w_{n+1}'(x), \ 0 < x < \infty,$$
$$p_n(0) = 0, \ \lim_{x\to\infty} p_n(x) = 0.$$

Using method of induction, prove, that $z_n(x) \geq 0, \ p_n(x) \geq 0$. According to conditions of lemma $z_0(x) \geq 0$. . It is follows from equation on $p_n(x)$, that if $z_{n-1}(x) \geq 0$, then $p_n(x) \geq 0$ for $x > 0$. Condition $p_n(x) \geq 0$ implies, that $z_n(x) \geq 0$. So, using method of mathematical induction, we proved, that for every $n \ \ z_n(x) \geq 0, p_n(x) \geq 0$.

We proved, that for every $x > 0$ sequences $v_n(x), \ w_n(x)$ are monotone decreasing and have low bounds. It's known, that in this case sequences under consideration have the property of convergence. Lemma is proved.

Consider a case $v_0(x) = 0$. Prove, that conditions of a lemma 3 are fulfilled. First condition has a place, because $v(x) \leq 0$. Verify the condition $v_0(x) \geq v_1(x), \ x > 0$. It follows from (15), that $w_1(x) \leq 0$ for $x > 0$. It implies, that $v_1(x) \leq 0 = v_0(x)$. So, conditions of lemma 3 are valid.

Consider a question of reduction Blasius problem to a finite interval. For every fixed n problem (15) is linear and we can use results, obtained in linear case. Second equation in (15) corresponds to (1) with $a(x) = v_{n-1}(x)$.

Transformed to a finite interval problem (15) has a form:

$$v_n'(x) = w_n(x),\ v_n(0) = 0,$$

$$w_n''(x) + [v_{n-1}(x) + x]w_n'(x) = 0,\ 0 < x < L,$$

$$w_n(0) = -1,\ \frac{1}{L}w_n'(L) + g_n(L)w_n(L) = 0. \tag{16}$$

Coefficient $g_n(L)$ can be calculated on base of asymptotic series (11) as it was discussed for linear case.

Consider results of numerical experiments. Write difference scheme for a problem (16):

$$v_n^k = v_n^{k-1} + \frac{w_n^{k-1} + w_n^k}{2}h,\ v_n^0 = 0,$$

$$\frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{h^2} + [v_{n-1}^k + x_k]\frac{w_n^{k+1} - w_n^k}{h} = 0,$$

$$w_n^0 = -1,\ \frac{1}{L}\frac{w_n^K - w_n^{K-1}}{h} + \tilde{g}_n(L)w_n^K = 0,$$

$$k = 1, 2, \ldots, K-1. \tag{17}$$

Compare errors, corresponding to different approaches for formulation of boundary condition instead of limit boundary condition at infinity. Let

$$\Delta = \max_n |v_n^h - \tilde{v}_n^h|,$$

where $\tilde{v}_n^h$ is solution of problem (17) for enough long interval $[0, L_0],\ L_0 \gg L$, when error of limit boundary condition transfer to point L_0 is inessential. Let $L_0 = 100,\ h = 0.1$ We continue iterations, if $\max_k |v_n^k - v_{n-1}^k| > \delta,\ \delta = 10^{-13}$. Initial iteration is defined as $v_0(x) = 0$. In Table 2 error Δ is presented for different L and $\tilde{g}_n(L)$ in compare with condition $w_n(L) = 0$. In this Table $g_0 = 1,\ g_1 = v_n(L)$. We have similar results in case $h = 0.01$.

Table 2: Absolute errors for Blasius problem

L	$w_n(L) = 0$	$\tilde{g}_n(L) = g_0$	$\tilde{g}_n(L) = g_0 + g_1/L$
3	$0.83e-1$	$0.24e-1$	$0.19e-1$
5	$0.76e-3$	$0.15e-3$	$0.47e-4$
7	$0.34e-6$	$0.53e-7$	$0.97e-8$
9	$0.12e-10$	$0.15e-11$	$0.18e-12$

It follows from numerical experiments, that using special boundary condition, based on equation (4), we get more accurate results in compare with classical approach.

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