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Inexact IRGNM under general source conditions

Abstract. In this paper we improve existing convergence and convergence rate results for the iteratively regularized Gauss-Newton method in two respects: First we show optimal rates of convergence under general source conditions, and second we assume that the linearized equations are solved only approximately in each Newton step. The latter point is important for large scale problems where the linearized equation can often only be solved iteratively, e.g. by the conjugate gradient method.

1 Introduction

Let us consider a nonlinear, ill-posed operator equation

$$F(x) = y,$$

where the operator $F : D(F) \to \mathcal{Y}$ is injective and continuously Fréchet differentiable on its domain $D(F) \subset \mathcal{X}$, and $\mathcal{X}, \mathcal{Y}$ are Hilbert spaces. We assume that there exists an $x^\dagger \in D(F)$ with

$$F(x^\dagger) = y \tag{1}$$

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and that only noisy data y^δ satisfying

$$\|y^\delta - y\| \leq \delta \tag{2}$$

are available. The nonnegative noise level δ is assumed to be known.

To iteratively compute an approximation to $x^\dagger$ we replace the nonlinear operator equation in the n-th Newton step by the linearized equation

$$F'[x_n^\delta]h_n = y^\delta - F(x_n^\delta), \qquad n = 1, 2, \ldots,$$

where h_n denotes the update $h_n = x_{n+1}^\delta - x_n^\delta$. Since in general the linearized equation inherits the ill-posedness of the equation $F(x) = y$, we apply Tikhonov regularization with initial guess $x_0 - x_n^\delta$, which leads to the regularized equation

$$(\gamma_n I + F'[x_n^\delta]^* F'[x_n^\delta])h_n = F'[x_n^\delta]^*(y^\delta - F(x_n^\delta)) + \gamma_n(x_0 - x_n^\delta), \qquad n = 1, 2, \ldots, \tag{3}$$

where γ_n is the regularization parameter. This algorithm is called iteratively regularized Gauss-Newton method (IRGNM) and was first studied by Bakushinskii [1]. In our case (γ_n) is a fixed sequence satisfying

$$\lim_{n\to\infty} \gamma_n = 0 \qquad \text{and} \qquad 1 \leq \frac{\gamma_n}{\gamma_{n+1}} \leq \gamma \tag{4}$$

for some $\gamma > 1$. As stopping rule for Newton's method we choose the well-known Morozov discrepancy principle, i.e. we stop the iteration at the first index N, for which the residual $\|F(x_N^\delta) - y^\delta\|$ satisfies

$$\|F(x_N^\delta) - y^\delta\| \leq \tau\delta < \|F(x_n^\delta) - y^\delta\|, \qquad 0 \leq n < N, \tag{5}$$

for a fixed parameter $\tau > 1$.

It is well known that for ill-posed problems convergence as the noise level tends to 0 can be arbitrarily slow unless a source condition is satisfied. For nonlinear problems these conditions have the form

$$x_0 - x^\dagger = f(F'[x^\dagger]^* F'[x^\dagger])w. \tag{6}$$

Here $f : [0, \|F'[x^\dagger]\|^2] \to [0, \infty)$ is an index function, i.e. f is increasing, continuous and $f(0) = 0$, and $w \in X$ is 'small', i.e. $\|w\| \leq \rho$ for a $\rho > 0$. Such conditions are also almost necessary for rates of convergence (see Bakushinskii & Kokurin [3]). So far the convergence of the IRGNM has been studied under Hölder type source conditions

$$f(t) := t^\nu, \qquad 0 < \nu \leq 1, \tag{7}$$

(see Bakushinskii [1] and Kaltenbacher, Neubauer & Scherzer [4]) and logarithmic source conditions

$$f(\lambda) := \begin{cases} (-\ln\lambda)^{-p}, & 0 < \lambda \leq \exp(-1), \\ 0, & \lambda = 0. \end{cases}, \qquad p > 0, \tag{8}$$

(see [8]). The former conditions are usually appropriate for mildly ill-posed problems, i.e. finitely smoothing operators F whereas the latter conditions (where the scaling condition $\|F'[x^\dagger\|^2 \leq \exp(-1)$ must be imposed) lead to natural smoothness conditions in terms of Sobolev spaces for a number of exponentially ill-posed problems.

Starting with the work of Mathé & Pereverzev [13] a series of papers has recently been devoted to the convergence of linear regularization methods under source conditions with general index functions. Prior to [13] the IRGNM under general source conditions was studied by Deuflhard, Engl & Scherzer [5], but no rates of convergence as the noise level δ tends to 0 were established. In this paper we show that the discrepancy principle leads to order optimal rates of convergence under the usual conditions concerning the finite qualification of Tikhonov regularization. In [14] Mathé & Pereverzev already showed that Tikhonov regularization with the discrepancy principle for linear inverse problems yields optimal rates of convergence under these conditions.

Up to the present convergence proofs for the iteratively regularized Gauss-Newton method have assumed that the linear equation (3) is solved exactly in each Newton step (see [1, 4, 8]). For large scale problems this is unrealistic. One usually computes just an approximation

$$h_n^{\mathrm{app}} \approx (\gamma_n I + F'[x_n^\delta]^* F'[x_n^\delta])^{-1} \left(F'[x_n^\delta]^* (y^\delta - F(x_n^\delta)) + \gamma_n (x_0 - x_n^\delta) \right)$$

in each Newton step. We formulate conditions under which this additional error does not impair the rate of convergence.

Finally we show in section 4 that the CG-method applied to (3) satisfies the assumptions of our convergence analysis for an appropriate stopping criterion. The CG-method has been shown to be an efficient choice for large scale, exponentially ill-posed problems, especially in combination with a preconditioner (see [9]), but in principle our convergence analysis applies to any iterative method.

An alternative approach is to apply an iterative method such as Landweber iteration, ν-method or CGNE directly to the Newton equation and use the regularizing properties of such methods with early stopping (see Bakushinskii [2], Kaltenbacher [11], Rieder [15, 16], Hanke [7]). However, for linear regularization methods the number of inner iterations typically grows exponentially with the Newton step. For the Newton-CG method no convergence rate results are available so far for weak source conditions (e.g. logarithmic and Hölder with small ν), and experimentally one often observes a slow-down of convergence after some initial good progress.

2 Convergence of the IRGNM for exact data

We first have to formulate some additional assumptions on the index function f and the operator F. To shorten notation we make the definitions

$$g_n(\lambda) := \frac{1}{\gamma_n + \lambda}, \quad r_n(\lambda) := 1 - \lambda g_n(\lambda), \quad A := F'[x^\dagger], \quad A_n := F'[x_n^\delta].$$

So g_n denotes the filter corresponding to Tikhonov regularization. To formulate our convergence result for general source conditions as presented in [13], we assume that that the index function f satisfies the inequality

$$\sup_{0<\lambda\leq\|F'[x^\dagger\|^2} \sqrt{\lambda}|r_n(\lambda)|f(\lambda) \;\leq\; c_f\sqrt{\gamma_n}f(\gamma_n), \qquad n \in \mathbb{N}_0. \tag{9a}$$

Since the classical qualification order of Tikhonov regularization is 1, this condition is satisfied if $\lambda \mapsto \lambda$ covers $\lambda \mapsto \sqrt{\lambda}f(\lambda)$ in the sense of [13, Definition 2]. Moreover, as $\lambda \mapsto \sqrt{\lambda}f(\lambda)$ covers $\lambda \mapsto f(\lambda)$ with constant 1, it follows from [13, Proposition 3] that

$$\sup_{0<\lambda\leq\|F'[x^\dagger\|^2} |r_n(\lambda)|f(\lambda) \;\leq\; c_f f(\gamma_n), \qquad n \in \mathbb{N}_0. \tag{9b}$$

We further assume that

$$\frac{f(\gamma\lambda)}{f(\lambda)} \;\leq\; C_f \qquad \text{for all} \qquad \lambda \in (0, \|A\|^2/\gamma]. \tag{9c}$$

This corresponds to the index function class $\mathcal{F}_{c_f,C_f}$ defined in [13]. Finally, we can assume w.l.o.g. that

$$f(\lambda) \leq 1 \qquad \text{for all} \qquad \lambda \in (0, \|A\|^2]. \tag{9d}$$

We will discuss the special index functions defined in (7) and (8) later.

As in [4, 8, 10] our analysis relies heavily on a local factorization of the operator F. We assume that for all $\bar{x}, x \in B(x^\dagger, E) := \{x : \|x - x^\dagger\| \leq E\}, E > 0$ there exist linear operators $R(\bar{x}, x) \in L(\mathcal{Y}, \mathcal{Y})$ and $Q(\bar{x}, x) \in L(\mathcal{X}, \mathcal{Y})$ such that

$$F'[\bar{x}] \;=\; R(\bar{x}, x)F'[x] + Q(\bar{x}, x) \tag{10a}$$

$$\|I - R(\bar{x}, x)\| \;\leq\; C_R \tag{10b}$$

$$\|Q(\bar{x}, x)\| \;\leq\; C_Q\|F'[x^\dagger](\bar{x} - x)\| \tag{10c}$$

for all $\bar{x}, x \in B(x^\dagger, E)$.

By x_{n+1}^δ we denote the computed new iterate and by $x_{n+1}^{\delta,\text{exc}}$ the new iterate for the exact update, i.e.

$$x_{n+1}^\delta = x_n^\delta + h_n^{\text{app}}, \qquad x_{n+1}^{\delta,\text{exc}} = x_n^\delta + h_n. \tag{11}$$

Hence, the computed new iterate can be written as

$$x_{n+1}^{\delta} = x_n^{\delta} + h_n^{\text{app}} = x_{n+1}^{\delta,\text{exc}} + (h_n^{\text{app}} - h_n).$$

A straightforward computation shows that the total error $e_n := x_n^\delta - x^\dagger$ for the iteratively regularized Gauss-Newton method can be decomposed into

$$\begin{aligned}
e_{n+1}^{\text{app}} &:= r_n(A^*A)f(A^*A)w, && \text{(12a)}\\
e_{n+1}^{\text{noi}} &:= g_n(A_n^*A_n)A_n^*(y^\delta - y), && \text{(12b)}\\
e_{n+1}^{\text{nl}} &:= (r_n(A_n^*A_n) - r_n(A^*A))\, f(A^*A)w, && \text{(12c)}\\
e_{n+1}^{\text{tay}} &:= g_n(A_n^*A_n)A_n^*(F(x^\dagger) - F(x_n^\delta) + A_n e_n), && \text{(12d)}\\
e_{n+1}^{\text{ls}} &:= h_n^{\text{app}} - h_n. && \text{(12e)}
\end{aligned}$$

Here e_n^{app} is the linear approximation error, e_n^{noi} is the propagated data noise error, e_n^{tay} involves the Taylor remainder, e_n^{nl} describes the nonlinearity effect that $A_n \neq A$ in general and e_n^{ls} is the error caused by the approximate solution of the linear system.

In the following lemma we present estimates for the error components and for their images under A defined in (12).

Lemma 1 *Assume that (1) – (6), (9) – (11) and $\|e_n\| \leq E$ hold. Then the following estimates hold for the error components defined above*

$$\begin{aligned}
\|e_{n+1}^{\text{app}}\| &\leq c_f f(\gamma_n)\rho && \text{(13a)}\\
\|e_{n+1}^{\text{noi}}\| &\leq \frac{1}{2\sqrt{\gamma_n}}\delta && \text{(13b)}\\
\|e_{n+1}^{\text{nl}}\| &\leq C_R \frac{\|Ae_{n+1}^{\text{app}}\|}{\sqrt{\gamma_n}} + \frac{3c_f}{2} C_Q \frac{\|Ae_n\|}{\sqrt{\gamma_n}} f(\gamma_n)\rho && \text{(13c)}\\
\|e_{n+1}^{\text{tay}}\| &\leq \frac{1}{2\sqrt{\gamma_n}}\left(2C_R + \frac{3}{2}C_Q\|e_n\|\right)\|Ae_n\| && \text{(13d)}
\end{aligned}$$

and for their images under A

$$\begin{aligned}
\|Ae_{n+1}^{\text{app}}\| &\leq c_f\sqrt{\gamma_n} f(\gamma_n)\rho && \text{(14a)}\\
\|Ae_{n+1}^{\text{noi}}\| &\leq \left(C_R + 1 + C_Q\frac{\|Ae_n\|}{2\sqrt{\gamma_n}}\right)\delta && \text{(14b)}\\
\|Ae_{n+1}^{\text{nl}}\| &\leq (C_R+1)\left[2C_R\|Ae_{n+1}^{\text{app}}\| + C_Q\|e_{n+1}^{\text{app}}\|\|Ae_n\|\right. && \text{(14c)}\\
&\quad \left. + \frac{\|Ae_{n+1}^{\text{app}}\|}{2\sqrt{\gamma_n}} C_Q\|Ae_n\|\right] + C_Q\|Ae_n\|\|e_{n+1}^{\text{nl}}\| &&\\
\|Ae_{n+1}^{\text{tay}}\| &\leq \left(C_R + 1 + C_Q\frac{\|Ae_n\|}{2\sqrt{\gamma_n}}\right)\left(2C_R + \frac{3}{2}\|e_n\|C_Q\right)\|Ae_n\|. && \text{(14d)}
\end{aligned}$$

Proof: We just spell out the proofs for the error components, in which the source condition appears. For the other components we refer to [4], where also the estimates for the latter error components are contained for the special case $f(t) = t^\nu$.
(13a) follows from (9b) and the isometry of the functional calculus. The proof of (14a) is analogous and uses (9a) and the identity $\|Az\| = \|(A^*A)^{1/2}z\|$, which holds for all $z \in \mathcal{X}$.
To shorten notation we define $T := (\gamma_n I + A^*A)$ and $T_n := (\gamma_n I + A_n^* A_n)$ and recall the important estimates

$$\|T_n^{-1}A_n^*\| \leq \frac{1}{2\sqrt{\gamma_n}}, \qquad \|A_nT_n^{-1}\| \leq \frac{1}{2\sqrt{\gamma_n}}, \tag{15}$$

$$\|T_n^{-1}\| \leq \frac{1}{\gamma_n}, \qquad \|A_nT_n^{-1}A_n^*\| \leq 1. \tag{16}$$

To show (13c) we estimate

$$\begin{aligned}
\|e_{n+1}^{\mathrm{nl}}\| &= \|\gamma_n T_n^{-1}(A^*A - A_n^*A_n)T^{-1}f(A^*A)w\| \\
&= \|\gamma_n T_n^{-1}[A_n^*(R(x^\dagger, x_n^\delta)^* - R(x_n^\delta, x^\dagger))A \\
&\quad + Q(x^\dagger, x_n^\delta)^*A - A_n^*Q(x_n^\delta, x^\dagger)]T^{-1}f(A^*A)w\| \\
&\leq C_R\frac{\|Ae_{n+1}^{\mathrm{app}}\|}{\sqrt{\gamma_n}} + \frac{3c_f}{2}C_Q\frac{\|Ae_n\|}{\sqrt{\gamma_n}}f(\gamma_n)\rho.
\end{aligned}$$

(14c) follows from

$$\begin{aligned}
\|Ae_{n+1}^{\mathrm{nl}}\| &\leq \|\gamma_n R(x^\dagger, x_n^\delta)A_nT_n^{-1}[A_n^*(R(x^\dagger, x_n^\delta)^* - R(x_n^\delta, x^\dagger))A \\
&\quad + Q(x^\dagger, x_n^\delta)^*A - A_n^*Q(x_n^\delta, x^\dagger)]T^{-1}f(A^*A)w\| + C_Q\|Ae_n\|\|e_{n+1}^{\mathrm{nl}}\| \\
&\leq (C_R + 1)\left[2C_R\|Ae_{n+1}^{\mathrm{app}}\| + \frac{\|Ae_{n+1}^{\mathrm{app}}\|}{2\sqrt{\gamma_n}}C_Q\|Ae_n\| + C_Q\|Ae_n\|\|e_{n+1}^{\mathrm{app}}\|\right] \\
&\quad + C_Q\|Ae_n\|\|e_{n+1}^{\mathrm{nl}}\|.
\end{aligned}$$

□

Lemma 2 *Assume that (1) – (6), (9) – (11) and* $\|e_n\| \leq E$ *are satisfied for* $0 \leq n < N$ *and for some sufficiently large* $\tau > 1$. *Then the inequalities*

$$\|e_{n+1}\| \leq \|e_{n+1}^{\mathrm{ls}}\| + (c_f + C_Rc_f)\rho f(\gamma_n) + \frac{\overline{c}}{C_Q}\frac{\|Ae_n\|}{\sqrt{\gamma_n}} \tag{17}$$

$$\|Ae_{n+1}\| \leq \|Ae_{n+1}^{\mathrm{ls}}\| + \overline{a}\|Ae_{n+1}^{\mathrm{app}}\| + \overline{b}\|Ae_n\| + \frac{\overline{c}}{\sqrt{\gamma_n}}\|Ae_n\|^2 \tag{18}$$

$$\underline{a}\|Ae_{n+1}^{\mathrm{app}}\| \leq \|Ae_{n+1}^{\mathrm{ls}}\| + \|Ae_{n+1}\| + \overline{b}\|Ae_n\| + \frac{\overline{c}}{\sqrt{\gamma_n}}\|Ae_n\|^2 \tag{19}$$

with constants

$$\begin{aligned}
\overline{a} &:= 1 + 2C_R(C_R+1), \qquad \underline{a} := 1 - 2C_R(C_R+1) \\
\overline{b} &:= (C_R+1)\left(\frac{1 + C_R + \frac{1}{2}EC_Q}{\tau - 1} + \left(2C_R + \frac{3}{2}EC_Q\right) + C_Q\left(\frac{3c_f}{2} + C_R c_f\right)\rho\right) \\
\overline{c} &:= C_Q\left(\frac{1 + C_R + \frac{1}{2}EC_Q}{2(\tau - 1)} + \frac{3c_f}{2}C_Q\rho + \frac{1}{2}\left(2C_R + \frac{3}{2}EC_Q\right)\right)
\end{aligned}$$

hold.

Proof: Notice that from (10) and (5) we obtain (see [4])

$$\tau\delta \;\le\; \left(C_R + 1 + \frac{1}{2}\|e_n\|C_Q\right)\|Ae_n\| + \delta$$

and thus

$$\delta \le \frac{1}{\tau - 1}\left(C_R + 1 + \frac{1}{2}EC_Q\right)\|Ae_n\|. \tag{20}$$

Then the sum of the estimates (13) together with (20) leads to inequality (17). Analogously the sum of the estimates (14) together with the estimates (13) and (20) leads to inequality (18). To show (19) we use the equality

$$Ae^{\mathrm{app}}_{n+1} + Ae^{\mathrm{nl}}_{n+1} = Ae_{n+1} - Ae^{\mathrm{noi}}_{n+1} - Ae^{\mathrm{tay}}_{n+1} - Ae^{\mathrm{ls}}_{n+1}.$$

Writing

$$\begin{aligned}
Ae^{\mathrm{nl}}_{n+1} &= \gamma_n R(x^\dagger, x_n^\delta)A_n T_n^{-1}\left[A_n^*(R(x^\dagger, x_n^\delta)^* - R(x_n^\delta, x^\dagger))A\right]T^{-1}f(A^*A)w \\
&\quad + \gamma_n R(x^\dagger, x_n^\delta)A_n T_n^{-1}\left[Q(x^\dagger, x_n^\delta)^*A - A^*Q(x_n^\delta, x^\dagger)\right]T^{-1}f(A^*A)w \\
&\quad + Q(x^\dagger, x_n^\delta)e^{\mathrm{nl}}_{n+1}
\end{aligned}$$

we get

$$\begin{aligned}
&Ae^{\mathrm{app}}_{n+1} + \gamma_n R(x^\dagger, x_n^\delta)A_n T_n^{-1}\left[A_n^*(R(x^\dagger, x_n^\delta)^* - R(x_n^\delta, x^\dagger))A\right]T^{-1}f(A^*A)w \\
&\quad = -\gamma_n R(x^\dagger, x_n^\delta)A_n T_n^{-1}\left[Q(x^\dagger, x_n^\delta)^*A - A^*Q(x_n^\delta, x^\dagger)\right]T^{-1}f(A^*A)w \\
&\qquad - Q(x^\dagger, x_n^\delta)e^{\mathrm{nl}}_{n+1} + Ae_{n+1} - Ae^{\mathrm{noi}}_{n+1} - Ae^{\mathrm{tay}}_{n+1} - Ae^{\mathrm{ls}}_{n+1}.
\end{aligned}$$

Now the assertion follows by estimating and using on the left hand side the second triangle inequality.

□

With these lemmas we can prove the following convergence result. A similar result has been shown in [5] for the special case $C_{\mathrm{ls}} = 0$ and under a different nonlinearity condition.

Proposition 3 *Let (1) – (6) and (9) – (11) hold. Assume that the error* e^{ls}_{n+1} *and its image* $F'[x^\delta_n]e^{\mathrm{ls}}_{n+1}$ *satisfy*

$$\|e^{\mathrm{ls}}_{n+1}\| \leq C_{\mathrm{ls}} f(\gamma_n), \qquad 0 \leq n < N, \tag{21a}$$

$$\|F'[x^\delta_n]e^{\mathrm{ls}}_{n+1}\| \leq C_{\mathrm{ls}}\sqrt{\gamma_n} f(\gamma_n), \qquad 0 \leq n < N, \tag{21b}$$

and that $C_R, C_Q, C_{\mathrm{ls}}, \gamma, 1/\gamma_0$ *and* ρ *are sufficiently small. Then there exists* $E > 0$ *such that the inexact Gauss-Newton iterates* x^δ_n*,* $0 \leq n \leq N$*, given by (11) are well defined for every* $x_0 \in D(F)$ *satisfying*

$$\|x_0 - x^\dagger\| \leq E \tag{22}$$

if the stopping index $N = N(\delta, y^\delta)$ *is determined by (5). Moreover,*

$$\|x^\delta_n - x^\dagger\| = O(f(\gamma_n)) \qquad \text{for} \qquad 1 \leq n \leq N \tag{23}$$

with $N = \infty$ *for* $\delta = 0$*. If* $\delta > 0$ *and the* γ_n *are chosen by*

$$\gamma_n = \gamma_0 \gamma^{-n}, \qquad n = 0, 1, 2, \ldots,$$

the stopping index is finite and $N = O(-\ln(u^{-1}(\delta)))$ *where* $u(\lambda) := \sqrt{\lambda} f(\lambda)$*. Conditions specifying "sufficiently small" are given in the proof.*

Proof: We will use an induction argument to prove for $0 \leq n \leq N$ the estimates

$$\theta_n \leq C_\theta, \tag{24a}$$

$$\|e_n\| \leq E. \tag{24b}$$

for θ_n and C_θ defined by

$$\theta_n := \frac{\|Ae_n\|}{u(\gamma_n)}, \qquad C_\theta := \max\left\{\theta_0, \frac{2a}{1 - b + \sqrt{(1-b)^2 - 4ac}}\right\}$$

with constants $a := \sqrt{\gamma} C_f (c_f \rho \overline{a} + C_{\mathrm{ls}}(C_R + 1))$, $b := \sqrt{\gamma} C_f (\overline{b} + C_Q C_{\mathrm{ls}})$ and $c := \sqrt{\gamma} C_f \overline{c}$. Notice that (24b) implies $x^\delta_n \in B(x^\dagger, E)$. Hence, if (24) is true for some $n \in \{0, 1, \ldots, N-1\}$, the estimate (18) holds. From (21a) and (21b) using (10) we get

$$\begin{aligned} \|Ae^{\mathrm{ls}}_{n+1}\| &\leq \|R(x^\dagger, x^\delta_n)F'[x^\delta_n]e^{\mathrm{ls}}_{n+1}\| + \|Q(x^\dagger, x^\delta_n)\|\|e^{\mathrm{ls}}_{n+1}\| \\ &\leq (C_R + 1)C_{\mathrm{ls}}\sqrt{\gamma_n} f(\gamma_n) + C_Q C_{\mathrm{ls}} \|Ae_n\|. \end{aligned} \tag{25}$$

This together with the estimates

$$\frac{\sqrt{\gamma_n}}{\sqrt{\gamma_{n+1}}} \leq \sqrt{\gamma} \qquad \text{and} \qquad \frac{f(\gamma_n)}{f(\gamma_{n+1})} \leq \frac{f(\gamma\gamma_{n+1})}{f(\gamma_{n+1})} \leq C_f,$$

(9d) and (14a) gives us the recursive estimate

$$\theta_{n+1} \le a + b\theta_n + c\theta_n^2.$$

Let t_1 and t_2 be the solutions to $a + bt + ct^2 = t$, i.e.

$$t_1 = \frac{2a}{1 - b + \sqrt{(1-b)^2 - 4ac}}, \qquad t_2 = \frac{1 - b + \sqrt{(1-b)^2 - 4ac}}{2c},$$

and assume that the constants $C_R, C_Q, C_{\mathrm{ls}}, \gamma, 1/\gamma_0$ and ρ are sufficiently small such that the smallness conditions

$$b + 2\sqrt{ac} \;<\; 1 \tag{26a}$$

$$\theta_0 \;\le\; \frac{1 - b + \sqrt{(1-b)^2 - 4ac}}{2c} \tag{26b}$$

$$C_{\mathrm{ls}} + (c_f + C_R c_f)\rho + \frac{\overline{c}}{C_Q} C_\theta \;\le\; E \tag{26c}$$

hold. Now we can show (24). For $n = 0$ (24a) is true by the definition of C_θ and (24b) by virtue of (22). Assume that (24) is true for $n = k$, $k < N$. Then the assumptions of Lemma 2 are satisfied, and therefore the estimate

$$\theta_{k+1} \le a + b\theta_k + c\theta_k^2$$

is true. By virtue of assumption (26a) we have $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$. By the induction hypothesis (24a) either $0 \le \theta_k \le t_1$ or $t_1 < \theta_k \le \theta_0$. In the first case, the non-negativity of a, b, and c implies

$$\theta_{k+1} \le a + b\theta_k + c\theta_k^2 \le a + bt_1 + ct_1^2 = t_1,$$

and in the second case we use assumption (26b) and the fact that

$$a + (b-1)t + ct^2 \le 0, \quad t_1 \le t \le t_2,$$

to show that

$$\theta_{k+1} \le a + b\theta_k + c\theta_k^2 \le \theta_k \le \theta_0.$$

Thus, in both cases (24a) is true for $n = k + 1$.

Using (17), assumptions (21a) and (26c), and the induction hypothesis we get

$$\|e_{k+1}\| \;\le\; (C_{\mathrm{ls}} + (c_f + C_R c_f)\rho) f(\gamma_k) + \frac{\overline{c}}{C_Q} \frac{\|Ae_k\| f(\gamma_k)}{\sqrt{\gamma_k} f(\gamma_k)} \le E f(\gamma_k) \le E.$$

This proves (24b). Furthermore, from the last computation we have in particular

$$\|e_{n+1}\| \le \left(C_{\mathrm{ls}} + (c_f + C_R c_f)\rho + \frac{\overline{c}}{C_Q} C_\theta \right) f(\gamma_n), \qquad n = 0, 1, \ldots, N-1, \tag{27}$$

and the constant $C_{\rm ls} + (c_f + C_R c_f)\rho + \overline{c}C_\theta/C_Q$ does not depend on δ or y^δ. This shows convergence and (23) for the noise-free case $\delta = 0$. Since by (20) and (24)

$$\delta \le \left(\frac{1 + C_R + \frac{E}{2}C_Q}{\tau - 1}\right) C_\theta u(\gamma_{N-1}) \tag{28}$$

for $\delta > 0$, the stopping index is finite with $N = \mathrm{O}(-\ln(u^{-1}(\delta)))$ for the choice $\gamma_n = \gamma_0\gamma^{-n}$.

□

3 IRGNM with discrepancy principle for nonlinear problems

Since we assume in this section that the index function f is unknown, we replace the error bounds (21) on $e^{\rm ls}_{n+1}$ by the strongest possible bounds

$$\|e^{\rm ls}_{n+1}\| \le C_{\rm ls}\sqrt{\gamma_n}, \qquad 0 \le n < N, \tag{29a}$$
$$\|F'[x^\delta_n]e^{\rm ls}_{n+1}\| \le C_{\rm ls}\gamma_n, \qquad 0 \le n < N, \tag{29b}$$

considering that f satisfies (9a) and that the classical qualification order of Tikhonov regularization is 1.

To prove convergence rates of optimal order for IRGNM we have to impose a further condition on the index function f.

Assumption 4 *Let $f \in C[0, \|A\|^2]$ be any strictly, monotonically increasing index function for which the function $\Phi : [0, f(\|A\|^2)] \to [0, \|A\|^2 f(\|A\|^2)]$ defined by*

$$\Phi(t) := t(f \cdot f)^{-1}(t)$$

is convex and twice differentiable.

Under this assumption this following stability result holds true:

Lemma 5 *Assume that the index function f satisfies Assumption 4 and that $z \in \mathcal{X}$ satisfies the source condition*

$$z = f(A^*A)w \qquad \text{with} \qquad \|w\| \le \rho. \tag{30}$$

Then the estimate

$$\|z\|^2 \le \rho^2\Phi^{-1}\left(\frac{\|Az\|^2}{\rho^2}\right) = \rho^2 f^2\left(u^{-1}\left(\frac{\|Az\|}{\rho}\right)\right) \tag{31}$$

holds.

Proof: Due to the assumptions on f and Φ the function Φ is invertible and an application of Jensen's inequality gives us the estimate in (31) (see Mair [12, Theorem 2.10]). The equality in (31) is a consequence of the identity $\Phi^{-1}(t^2) = f^2(u^{-1}(t))$, which follows from

$$\Phi(f^2(u^{-1}(t))) = f^2(\xi)(f \cdot f)^{-1}(f^2(\xi)) = f^2(\xi)\xi = t^2$$

with $\xi = u^{-1}(t)$.

□

Recall that for linear inverse problems the optimal error bound under the source condition (30) is of order $f(u^{-1}(\delta))$ (see [13]). The following main result shows that this rate of convergence is achieved by the IRGNM with the discrepancy principle.

Theorem 6 *Let the assumptions of Proposition 3 hold and let f satisfy Assumption 4. Assume furthermore that the inequalities (29) and the smallness conditions*

$$q\gamma < 1 \tag{32a}$$

$$\frac{\overline{a}}{1-q\gamma} + q\left(1 + \frac{\|A\|^2}{\gamma_0}\right) + \frac{(\|A\|^2+\gamma_0)}{\|Ae_0\|}\frac{(C_R+1)C_{\text{ls}}}{1-\gamma q} < 2 \tag{32b}$$

are satisfied. Here the constant q is defined by $q := C_Q C_{\text{ls}} + \overline{b} + C_Q E$ with the notation of Lemma 2. Then the final iterates x_N^δ satisfy the order optimal estimate

$$\|x_N^\delta - x^\dagger\| = O(f(u^{-1}(\delta))), \quad \delta \to 0. \tag{33}$$

Proof: Due to (29b) we get as in (25) the estimate

$$\|Ae_{n+1}^{\text{ls}}\| \le (C_R+1)C_{\text{ls}}\gamma_n + C_Q C_{\text{ls}}\|Ae_n\|.$$

Then by (18) and (24)

$$\|Ae_{n+1}\| \le (C_R+1)C_{\text{ls}}\gamma_n + \overline{a}\|Ae_{n+1}^{\text{app}}\| + (C_Q C_{\text{ls}} + \overline{b} + \overline{c}C_\theta)\|Ae_n\|$$

holds. Since (26c) implies $\overline{c}C_\theta \le C_Q E$ and hence $C_Q C_{\text{ls}} + \overline{b} + \overline{c}C_\theta \le q$, it follows by induction that

$$\|Ae_{n+1}\| \le (C_R+1)C_{\text{ls}}\sum_{k=0}^{n} q^{n-k}\gamma_k + \overline{a}\sum_{k=0}^{n}\|Ae_{k+1}^{\text{app}}\|q^{n-k} + \|Ae_0\|q^{n+1}.$$

The inequality

$$r_k(\lambda) = \left(\frac{\gamma_k}{\gamma_k+\lambda}\right) \le \left(\frac{\gamma_k}{\gamma_{k+1}}\right)\left(\frac{\gamma_{k+1}}{\gamma_{k+1}+\lambda}\right) \le \gamma r_{k+1}(\lambda), \qquad \lambda \ge 0,$$

together with the isometry of the functional calculus imply that

$$\|Ae_{k+1}^{\text{app}}\| = \|Ar_{k+1}(A^*A)f(A^*A)w\| \le \gamma\|Ar_{k+2}(A^*A)f(A^*A)w\| = \gamma\|Ae_{k+2}^{\text{app}}\|. \tag{34}$$

Analogously the inequality

$$\sqrt{\lambda} \le \left(\frac{\gamma_n + \|A\|^2}{\gamma_n}\right)\left(\frac{\gamma_n}{\gamma_n+\lambda}\right)\sqrt{\lambda} \le \left(1+\frac{\|A\|^2}{\gamma_0}\right)\gamma^n\sqrt{\lambda}r_n(\lambda), \qquad 0\le\lambda\le\|A\|^2,$$

implies that

$$\|Ae_0\| \le \left(1+\frac{\|A\|^2}{\gamma_0}\right)\gamma^n\|Ae_{n+1}^{\mathrm{app}}\|.$$

Using assumption (32a) we obtain

$$\sum_{k=0}^{n} q^{n-k}\|Ae_{k+1}^{\mathrm{app}}\| \le \sum_{k=0}^{n} q^{n-k}\gamma^{n-k}\|Ae_{n+1}^{\mathrm{app}}\| \le \frac{1}{1-q\gamma}\|Ae_{n+1}^{\mathrm{app}}\|.$$

Combining the last inequalities, we have shown that

$$\|Ae_{n+1}\| \le (C_R+1)C_{\mathrm{ls}}\sum_{k=0}^{n} q^{n-k}\gamma_k + \left(\frac{\overline{a}}{1-q\gamma} + q\left(1+\frac{\|A\|^2}{\gamma_0}\right)\right)\|Ae_{n+1}^{\mathrm{app}}\|. \tag{35}$$

Now it follows from (19), (24a), assumption (26c) and (32a), $q<1$ (because $\gamma>1$), $\overline{a}+\underline{a}=2$ and the last inequality for $n=N-2$ that

$$\begin{aligned}
\|Ae_N\| \;&\ge\; \underline{a}\|Ae_N^{\mathrm{app}}\| - (\overline{b}+C_QC_{\mathrm{ls}})\|Ae_{N-1}\| - \frac{\overline{c}}{\sqrt{\gamma_{N-1}}}\|Ae_{N-1}\|^2 \\
&\quad -(C_R+1)C_{\mathrm{ls}}\gamma_{N-1} \\
&\ge\; \underline{a}\|Ae_N^{\mathrm{app}}\| - q\left(\frac{\overline{a}}{1-q\gamma}+q\left(1+\frac{\|A\|^2}{\gamma_0}\right)\right)\gamma\|Ae_N^{\mathrm{app}}\| \\
&\quad -(C_R+1)C_{\mathrm{ls}}\left(q\sum_{k=0}^{N-2} q^{N-2-k}\gamma_k + \gamma_{N-1}\right) \\
&\ge\; \left(\underline{a} - \frac{q\gamma\overline{a}}{1-q\gamma} - q^2\gamma\left(1+\frac{\|A\|^2}{\gamma_0}\right)\right)\|Ae_N^{\mathrm{app}}\| \\
&\quad -(C_R+1)C_{\mathrm{ls}}\sum_{k=0}^{N-1} q^{N-1-k}\gamma_k \\
&=\; \left(2 - \frac{\overline{a}}{1-q\gamma} - q^2\gamma\left(1+\frac{\|A\|^2}{\gamma_0}\right)\right)\|Ae_N^{\mathrm{app}}\| \qquad (36)\\
&\quad -(C_R+1)C_{\mathrm{ls}}\sum_{k=0}^{N-1} q^{N-1-k}\gamma_k.
\end{aligned}$$

Furthermore, as above the inequality $\sqrt{\lambda}r_{N-1}(\lambda) \ge \sqrt{\lambda}\left(\frac{\gamma_{N-1}}{\gamma_0+\|A\|^2}\right)$ for $0\le\lambda\le\|A\|^2$ implies

$$\|Ae_N^{\mathrm{app}}\| \ge \frac{\gamma_{N-1}}{\|A\|^2+\gamma_0}\|Ae_0\|. \tag{37}$$

From (5) we get as in (20)

$$\delta \geq \frac{1 - C_R - \frac{E}{2} C_Q}{\tau + 1} \|Ae_N\|. \tag{38}$$

It follows from the condition (32a), $\gamma > 1$, and the definition of $\overline{b}$ that

$$1 > q = C_Q C_{\rm ls} + \overline{b} + C_Q E > C_R + \frac{E}{2} C_Q. \tag{39}$$

Using the inequality $\gamma_k/\gamma_{N-1} \leq \gamma^{N-1-k}$ we get

$$\gamma_k q^{N-1-k} \leq \gamma_{N-1} (\gamma q)^{N-1-k}$$

and thus

$$\sum_{k=0}^{N-1} q^{N-1-k} \gamma_k \leq \gamma_{N-1} \sum_{k=0}^{N-1} (\gamma q)^{N-1-k} \leq \frac{\gamma_{N-1}}{1 - \gamma q}. \tag{40}$$

Then we can estimate using (36), (38), (40) and assumption (32b)

$$\begin{aligned}\delta \ \geq \ & \frac{1 - C_R - \frac{E}{2} C_Q}{\tau + 1} \left[\left(2 - \frac{\overline{a}}{1 - q\gamma} - q^2 \gamma \left(1 + \frac{\|A\|^2}{\gamma_0} \right) \right) \|Ae_N^{\rm app}\| \right. \\ & \left. - (C_R + 1) C_{\rm ls} \gamma_{N-1} \frac{1}{1 - \gamma q} \right].\end{aligned}$$

This, (37) and (39) imply

$$C_1 \gamma_{N-1} \ \leq \ \delta, \tag{41}$$

$$C_2 \|Ae_N^{\rm app}\| \ \leq \ C_3 \gamma_{N-1} + \delta, \tag{42}$$

with the constants

$$\begin{aligned} C_1 \ &:= \ \left(\frac{1-q}{\tau+1} \right) \left[\left(2 - \frac{\overline{a}}{1 - q\gamma} - q \left(1 + \frac{\|A\|^2}{\gamma_0} \right) \right) \frac{\|Ae_0\|}{\|A\|^2 + \gamma_0} - \frac{(C_R+1) C_{\rm ls}}{1 - \gamma q} \right], \\ C_2 \ &:= \ \left(\frac{1-q}{\tau+1} \right) \left(2 - \frac{\overline{a}}{1 - q\gamma} - q^2 \gamma \left(1 + \frac{\|A\|^2}{\gamma_0} \right) \right), \\ C_3 \ &:= \ \left(\frac{1-q}{\tau+1} \right) \frac{(C_R+1) C_{\rm ls}}{1 - \gamma q}, \end{aligned}$$

independent of δ and y^δ. (32b) implies $C_1 > 0$ and $C_2 > 0$, and so using (41) and (42) we conclude

$$\|Ae_N^{\rm app}\| \leq C_4 \delta \qquad \text{with} \qquad C_4 := \left(\frac{\frac{C_3}{C_1} + 1}{C_2} \right). \tag{43}$$

Now we can apply Lemma 5 with w replaced by $C_4^{-1} r_{N-1}(A^*A)w$ and (43) to obtain

$$\frac{C_4}{\rho}\left\|\left(\frac{\rho}{C_4}\right)e_N^{\mathrm{app}}\right\| \le C_4 f\left(u^{-1}\left(\left(\frac{\rho}{C_4}\right)\left(\frac{\|Ae_N^{\mathrm{app}}\|}{\rho}\right)\right)\right) \le C_4 f\left(u^{-1}\left(\delta\right)\right).$$

From (28) we have that

$$\delta \le C_5 u(\gamma_{N-1}), \quad \text{where} \quad C_5 := \left(\frac{1+C_R+\frac{E}{2}C_Q}{\tau-1}\right)C_\theta.$$

To obtain an estimate for $\sqrt{\gamma_{N-1}}$ in terms of δ we estimate

$$\begin{aligned}
\sqrt{\gamma_{N-1}} &= C_5\frac{\sqrt{\gamma_{N-1}}}{\delta}u\left(u^{-1}\left(\frac{\delta}{C_5}\right)\right)\\
&\le \frac{C_5}{C_1}\sqrt{\frac{u^{-1}\left(\frac{\delta}{C_5}\right)}{\gamma_{N-1}}}f\left(u^{-1}\left(\max\left\{1,\frac{1}{C_5}\right\}\delta\right)\right)\\
&\le \frac{C_5}{C_1}\max\left\{1,\frac{1}{C_5}\right\}f\left(u^{-1}\left(\delta\right)\right).
\end{aligned}$$

In the second line we have used the definition of u, (41) and the monotonicity of $f\circ u^{-1}$, and in the last line the inequality $f(u^{-1}(t\delta)) \le tf(u^{-1}(\delta)$, $t \ge 1$, which follows from concavity of $f\circ u^{-1}$ (see Assumption 4). Then from the last estimate and assumption (29a) we obtain that

$$\|e_N^{\mathrm{ls}}\| \le \frac{C_{\mathrm{ls}}}{C_1}\max\{1,C_5\}\,f\left(u^{-1}(\delta)\right).$$

Now it remains to be shown that the error components in (13b) – (13d) are of order $O(f(u^{-1}(\delta)))$. To estimate the right hand side of (13c) and (13d) we combine (34), (35) again for the case $n = N-2$, (40), (41) and (42) to conclude $\|Ae_{N-1}\| = O(\delta)$. Then an application of $\|e_N\| \le E$, (43) and (9d) together with the last result shows that $\|e_N^{\mathrm{nl}}\|$ and $\|e_N^{\mathrm{tay}}\|$ are of order $O\left(\delta/\sqrt{\gamma_{N-1}}\right)$. For $\|e_N^{\mathrm{noi}}\|$ this already follows from (13b). Now applying a similar idea as above, we have

$$\frac{\delta}{\sqrt{\gamma_{N-1}}} \le \max\{1,C_5\}f\left(u^{-1}(\delta)\right).$$

So, altogether we have proven

$$\|e_N\| = \|x_N^\delta - x^\dagger\| = O(f(u^{-1}(\delta))), \qquad \delta \to 0.$$

□

It is worthwhile to note that the convergence theorems of the IRGNM given here comprise the results formulated in [4] and [8], where the additional error term e_n^{ls} was not considered and the theorems were formulated for either Hölder source conditions or logarithmic source conditions. To get the results stated there one has to check that the conditions (9) are satisfied for the functions defined in (7) and (8). The proofs can be found in [4] and [8].

To recover the asymptotic for the stopping index N consider first the case where the index function is given by (7). Then the function u is given by $u(t) = t^{1/2+\nu}$ and so $u^{-1}(t) = t^{2/(1+2\nu)}$. Now using Theorem 3 we conclude

$$N = O(-\ln(u^{-1}(\delta))) = O(-\ln(\delta^{2/(1+2\nu)})).$$

In the case where f is given by (8) we have $u(t) = \sqrt{t}(-\ln t)^{-p} \geq \sqrt{t}$. Therefore $u^{-1}(t) \leq t^2$ and so

$$N = O(-\ln(\delta^2)) = O(-\ln \delta).$$

Analogously one can recover the convergence rates proven in [4] and [8].

4 Solving the linearized equation

This section deals with the solution of the linearized and regularized equation (3). In general our convergence result applies to every algorithm to solve (3) that guarantees the conditions (21) resp. (29) in each Newton step. Here we will study the conjugate gradient (CG) method (see [9]). Our aim is to formulate a truncation criterion for the CG-method such that (21) is satisfied.

For details of the CG-method we refer to [6]. To shorten notation we define for the following

$$G_n := \begin{pmatrix} F'[x_n^\delta] \\ \sqrt{\gamma_n} I \end{pmatrix} \in L(\mathcal{X}, \mathcal{Y} \times \mathcal{X}), \qquad x_n^\delta \in D(F),$$

and $g_n^\delta := \left(y^\delta - F(x_n^\delta), \sqrt{\gamma_n}(x_0 - x_n^\delta)\right)^T$. Then (3) can be written as a normal equation

$$G_n^* G_n h_n = G_n^* g_n^\delta, \tag{44}$$

and the CG-iteration applied to the system (44) can be coded as follows:

$h_n^0 = 0;\ d^0 = g_n^\delta;\ r^0 = G_n^* d^0;\ p^1 = r^0;\ k = 0;$

while $\|r^k\| > \varepsilon \gamma_n^{3/2}$

$k = k + 1;$

$q^k = G_n p^k;$

$\alpha_k = \|r^{k-1}\|_{\mathcal{X}}^2 / \|q^k\|_{\mathcal{Y}}^2;$

$h_n^k = h_n^{k-1} + \alpha_k p^k;$

$d^k = d^{k-1} - \alpha_k q^k;$
$r^k = G_n^* d^k;$
$\beta_k = \|r^k\|_{\mathcal{X}}^2 / \|r^{k-1}\|_{\mathcal{X}}^2;$
$p^{k+1} = r^k + \beta_k p^k.$

Note that our stopping criterion does not require knowledge of the source condition. Further recall the important equation

$$r^k \;=\; G_n^*(g_n^\delta - G_n h_n^k).$$

Theorem 7 *For the CG-method applied to (44) the stopping criterion*

$$\|r^k\| \le \varepsilon \gamma_n^{3/2} \tag{45}$$

is met after a finite number J of steps, and the update $h_n^{\mathrm{app}} := h_n^J$ satisfies the estimates

$$\begin{aligned} \|h_n^\dagger - h_n^J\| &\le \varepsilon\sqrt{\gamma_n}, \\ \|G_n(h_n^\dagger - h_n^J)\| &\le \varepsilon\gamma_n, \end{aligned}$$

where $h_n^\dagger$ denotes the true solution of (44). In particular (29) holds.

Proof: It follows from standard convergence theory of the CG-method that $\lim_{k\to\infty} h_n^k = h_n^\dagger$ and $\lim_{k\to\infty} r_k = 0$. So the stopping criterion (45) is met after a finite number J of steps.
Since $\langle G_n^* G_n x, x\rangle = \|F'[x_n^\delta]x\|^2 + \gamma_n\|x\|^2 \ge \gamma_n\|x\|^2$, we have $\|(G_n^* G_n)^{-1}\| \le \gamma_n^{-1}$. Hence, we conclude

$$\begin{aligned} \|h_n^\dagger - h_n^J\| &\le \|(G_n^* G_n)^{-1}\| \|G_n^* G_n(h_n^\dagger - h_n^J)\| \\ &\le \gamma_n^{-1}\|G_n^*(g_n^\delta - G_n h_n^J)\| \\ &\le \varepsilon\sqrt{\gamma_n}. \end{aligned}$$

This proves (29a) with the constant $C_{\mathrm{ls}} = \varepsilon$. Since $(G_n^*)^\dagger G_n^*$ is the orthogonal projection onto $\overline{R(G_n)}$ and

$$\|(G_n^*)^\dagger\| = \|(G_n^* G_n)^{-1}\| = \gamma_n^{-1}$$

we can estimate

$$\begin{aligned} \|G_n(h_n^\dagger - h_n^J)\| &= \|(G_n^*)^\dagger G_n^* G_n(h_n^\dagger - h_n^J)\| \\ &\le \|(G_n^*)^\dagger\| \|G_n^* G_n(h_n^\dagger - h_n^J)\| \\ &\le (\sqrt{\gamma_n})^{-1}\varepsilon\gamma_n^{3/2} \\ &\le \varepsilon\gamma_n. \end{aligned}$$

The estimate (29b) with the constant C_{ls} given by ε now follows from the inequality

$$\|F'[x_n^\delta]x\|^2 \leq \|G_n x\|^2,$$

which holds for all $x \in \mathcal{X}$.

$\square$

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