Georg-August-Universität

Göttingen



An iterative method for inverse medium scattering problems based on factorization of the far field operator

Thorsten Hohage

Preprint Nr. 2005-09

Preprint-Serie des Instituts für Numerische und Angewandte Mathematik Lotzestr. 16-18 D - 37083 Göttingen

An iterative method for inverse medium scattering problems based on factorization of the far field operator

Thorsten Hohage

Institute for Numerical and Applied Mathematics, Lotzestr. 16-18, D-37083 Göttingen, Germany

E-mail: hohage@math.uni-goettingen.de

Abstract. We propose an iterative regularization method for the solution of inverse medium scattering problems which takes advantage of factorizations of the far field operator. Numerical experiments with two and three dimensional acoustic and electromagnetic medium scattering problems show that our method is very competitve compared to other iterative regularization methods.

1. Introduction

Recently factorizations of the far field operator have become popular for the solution of timeharmonic inverse scattering problems. The far field operator of a scattering problem relates incoming to outgoing waves, i.e. it is a measurable quantity. It is the given measured data of the inverse scattering problem, which consists in finding the scatterer. The inverse scattering problem can therefore be interpreted as a nonlinear operator equation with an operator-valued nonlinear operator F which maps the unknown scatterer to the far field operator.

Starting with a paper by Kirsch [10] a lot of progress has been made in solving inverse scattering problems using factorizations of the far field operator into a product of three bounded linear operators. These methods have the attractive feature that they do not require the solution of the direct scattering problem. However, for forward medium scattering problems it is only possible to recover the support of the inhomeniety of the medium by these approaches (see Kirsch [11, 12, 13] and Haddar & Monk [5] for a similar method).

In this paper we propose an iterative method for the full reconstruction of the refractive index of a medium given the corresponding far field operator, which also takes advantage of a factorization of the far field operator into a product of three bounded linear operators. The right and the left operator are severely ill-posed and independent of the scatterer. We show that the reconstruction of the acoustic refractive index from the middle operator is a well-posed problem. Therefore, we only regularize the linear operator which maps the middle operator to the far field operator. This leads to a significant reduction of the number of forward problem solutions compared to iterative regularization methods which do not exploit the special structure of the far field operator (see [8, 6]). Kaczmarz-type algorithms have been used by Bao & Li [2] for multi-frequency data and by Natterer, Vögeler & Wübbeling [14, 18] for high-frequency data.

The plan of this paper is as follows: In the following section we describe the forward scattering problems, the corresponding far field operators and their factorizations. Moreover, we show that the refractive index in acoustic medium scattering problems can be recovered from the middle operator in a stable way. In section 3 we derive our iterative method for the solution of nonlinear operator equations for which the image space consists of operators allowing a factorization. Finally in section 4 we present numerical results for both acoustic and electromagnetic medium scattering problems.

2. Forward problems and factorization of far field operators

2.1. Acoustic scattering problems

The scattering of time-harmonic acoustic waves is governed by the Helmholtz equation

$$\Delta u + k^2 n(x)u = 0 \qquad \text{in } \mathbb{R}^m.$$
(1a)

Here $m \in \{2,3\}$ is the space dimension, u denotes a velocity potential, k is the wave number, and n the refractive index of the medium. We assume that n is a complex valued function of the form

$$n = 1 - a$$
, $\operatorname{supp} a \subset B_{\rho} := \{ x \in \mathbb{R}^m : |x| < \rho \},\$

which belongs to the Hölder space $C^{0,\alpha}(\mathbb{R}^m)$ and satisfies $\operatorname{Re}(n) > 0$ and $\operatorname{Im}(n) \ge 0$. The set of all functions $a \in C_0^{0,\alpha}(B_\rho)$ for which these conditions are satisfied will be denoted by \mathcal{U} . We look for solutions

$$u = u^{1} + u^{s} \tag{1b}$$

which are a superposition of a given incident wave u^{i} , which satisfies the homogeneous Helmholtz equation $\Delta u^{i} + k^{2}u^{i} = 0$ in \mathbb{R}^{m} , and a scattered wave u^{s} , which satisfies the Sommerfeld radiation condition

$$r^{\frac{m-1}{2}}\left(\frac{\partial u^{\mathrm{s}}}{\partial r} - iku^{\mathrm{s}}\right) \to 0 \qquad \text{as } r = |x| \to \infty$$
 (1c)

uniformly for all directions $\hat{x} = x/|x|$. It is well-known (see e.g. Colton & Kress [4]) that this problem has a unique solution $u \in C^2(\mathbb{R}^m)$ if the refractive index belongs to a Hölder space $n \in C^{0,\alpha}(\mathbb{R}^m), \alpha > 0$.

The forward problem (1) can be formulated equivalently as an integral equation of the second kind called the *Lippmann-Schwinger equation*:

$$u(x) + k^2 \int_{B_{\rho}} \Phi(x - y) a(y) u(y) \,\mathrm{d}y = u^{\mathrm{i}}(x), \qquad x \in \mathbb{R}^m.$$

$$(2)$$

Here Φ denotes the fundamental solution to the Helmholtz equation in free space, i.e. $\Phi(x) := \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|}$, $x \neq 0$ for m = 3 and $\Phi(x) := \frac{i}{4}H_0^{(1)}(k|x|)$, $x \neq 0$ for m = 2 where $H_0^{(1)}$ denotes the Hankel function of the first kind of order 0.

It is common practice in scattering theory to use far field patterns as data for inverse problems. The far field pattern $u^{\infty} : S^{m-1} \to \mathbb{C}$ of a scattered field u^{s} , which satisfies the homogeneous Helmholtz equation in $\mathbb{R}^m \setminus B_{\rho}$ and the Sommerfeld radiation condition, is defined by the asymptotic formula

$$u^{\mathrm{s}}(x) = \frac{e^{ik|x|}}{|x|} \left(u^{\infty}(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right) \right), \qquad |x| \to \infty.$$

As a consequence of the asymptotic behavior of the fundamental solution Φ and the Lippmann-Schwinger equation (2), the far field pattern of the solution u^{s} to the scattering problem (1) is given by

$$u^{\infty}(\hat{x}) = \gamma_m \int_{B_{\rho}} e^{-ik\hat{x}\cdot y} a(y)u(y) \,\mathrm{d}y, \tag{3}$$

with $\gamma_2 := -\frac{k^2 e^{i\pi/4}}{\sqrt{8\pi}}$ and $\gamma_3 := -\frac{k^2}{4\pi}$. We will consider the inverse problem to recover the refractive index *n* from far-field data for incident plane waves $u^{i}(x,d) := e^{-ikx \cdot d}$ from all directions $d \in S^{m-1}$. The corresponding solutions will be denoted by u(x,d), $u^{s}(x,d)$, and the far field patterns by $u^{\infty}(\hat{x},d)$. Given $u^{\infty}(\hat{x}, d)$ for all $\hat{x}, d \in S^{m-1}$ we also know the far-field pattern corresponding to a superposition

$$u_g^{i}(x) = \int_{S^{m-1}} u^{i}(x,d)g(d) \,\mathrm{d}s(d), \qquad x \in \mathbb{R}^m$$
(4)

of incident plane waves with a density $g \in L^2(S^{m-1})$. It is given by

$$(U^{\infty}g)(\hat{x}) := \int_{S^{m-1}} u^{\infty}(\hat{x}, d)g(d) \,\mathrm{d}s(d), \qquad \hat{x} \in S^{m-1}.$$
(5)

It can be shown that the far field pattern u^{∞} is infinitely smooth in both arguments. Therefore, $U^{\infty}: L^2(S^{m-1}) \to L^2(S^{m-1})$ is a compact linear operator called the *far field operator*.

To derive a factorization of the operator U^{∞} , we introduce the operator $Z_{\rm o}$: $L^2(B_{\rho}) \rightarrow$ $L^{2}(S^{m-1}),$

$$(Z_{o}u)(\hat{x}) := \int_{B_{\rho}} e^{-ik\hat{x}\cdot y} u(y) \,\mathrm{d}y, \qquad \hat{x} \in S^{m-1}$$

and its transposed

$$(Z_{\mathbf{i}}g)(x) := \int_{S^{m-1}} e^{-ikd \cdot y} g(d) \,\mathrm{d}s(d), \qquad x \in B_{\rho}.$$

Since $u_q^i = Z_i g$, the total field u_g corresponding to the incident field u_q^i satisfies the Lippmann-Schwinger equation

$$u_g + V M_a u_g = u_g^{\dagger}$$

with the volume potential operator and the multiplication operator $V, M_a : L^2(B_\rho) \to L^2(B_\rho)$ given by $(Vu)(x) := k^2 \int_{B_a} \Phi(x-y)u(y) \, \mathrm{d}y$ and $(M_a u)(x) := a(x)u(x)$. Together with eq. (3) we obtain the factorization

$$U^{\infty} = \gamma_m Z_0 M_a (I + V M_a)^{-1} Z_i.$$
⁽⁶⁾

Theorem 2.1 The mapping

$$G: \qquad \mathcal{U} \to L(L^2(B_\rho))$$
$$a \mapsto M_a(I + VM_a)^{-1}$$

is one-to-one, and its inverse is continuous with respect to the L^{∞} -norm.

Proof: Let $a \in \mathcal{U}$ and B := G(a). Then $M_a = B + BVM_a$ or

$$(I - BV)M_a = B. (7)$$

To show that the operator $I - BV : L^2(B_\rho) \to L^2(B_\rho)$ is one to one, assume that

$$v - BVv = 0 \tag{8}$$

for $v \in L^2(B_{\rho})$. Multiplying from the left by V we obtain that

$$0 = Vv - VM_a(I + VM_a)^{-1}Vv$$

= {(I + VM_a)(I + VM_a)^{-1} - VM_a(I + VM_a)^{-1}}Vv
= (I + VM_a)^{-1}Vv.

Hence Vv = 0. Plugging this into (8), it follows that v = 0, so I - BV is one-to-one.

Now it follows from Riesz theory and the compactness of $V \in L(L^2(B_{\rho}))$ that I - BV is onto and has a bounded inverse. Therefore, (7) is equivalent to

$$M_a = (I - BV)^{-1}B$$

In particular, M_a is uniquely determined by B. Moreover, M_a depends continuously on B since the composition of bounded linear operators is continuous and $(I-BV)^{-1}$ depends continuously on B by a Neumann series argument. Since the mapping $L^{\infty}(B_{\rho}) \to L(L^2(B_{\rho}))$, $a \mapsto M_a$ is linear and isometric with respect to the L^{∞} -norm, it follows that a is also uniquely determined by B and depends continuously on B with respect to the L^{∞} -norm.

2.2. electromagnetic scattering problem

We now discuss the scattering of electromagnetic waves in an inhomogeneous, nonmagnetic medium. Then Maxwell's equation reduce to the equation

$$\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 n(x)\mathbf{E} = 0 \qquad \text{in } \mathbb{R}^3$$
(9a)

for the space-dependent part $\mathbf{E} : \mathbb{R}^3 \to \mathbb{C}^3$ of the electric field. Here $k = \omega \sqrt{\epsilon_0 \mu_0}$ denotes the wave number and $n(x) = 1 - a(x) = \frac{1}{\epsilon_0} (\epsilon(x) + i \frac{\sigma(x)}{\omega})$ the refractive index with the standard notations for the electric permittivity $\epsilon > 0$, the magnetic permeability μ_0 , the conductivity $\sigma > 0$, and the angular frequency $\omega > 0$. We assume that $\operatorname{supp} a \subset B_\rho$ and $n \in C^{1,\alpha}(\mathbb{R}^3)$. The total electric field

$$\mathbf{E} = \mathbf{E}^{\mathbf{i}} + \mathbf{E}^{\mathbf{s}} \tag{9b}$$

is composed of an incident field \mathbf{E}^{i} and a scattered field \mathbf{E}^{s} . The former satisfies the equation curl curl $\mathbf{E}^{i} - k^{2}\mathbf{E}^{i} = 0$, and the latter the *Silver-Müller radiation condition*

$$\operatorname{curl} \mathbf{E}^{\mathrm{s}}(x) \times x - ik|x|\mathbf{E}^{\mathrm{s}}(x) \to 0, \qquad |x| \to \infty$$

$$(9c)$$

uniformly for all directions $\hat{x} = x/|x| \in S^2$. Let $\Phi(x) := e^{ik|x|}/(4\pi|x|), x \neq 0$ denote the fundamental solution to the Helmholz equation in \mathbb{R}^3 . The forward electromagnetic scattering problem (9) can be formulated equivalently as an integral equation

$$\mathbf{E}(x) + k^2 \int_{B_{\rho}} \Phi(x - y) a(y) \mathbf{E}(y) \, \mathrm{d}y$$

$$+ \operatorname{grad} \int_{B_{\rho}} \Phi(x - y) \frac{\operatorname{grad} a(y)}{1 - a(y)} \cdot \mathbf{E}(y) \, \mathrm{d}y = \mathbf{E}^{\mathrm{i}}(x), \quad x \in \mathbb{R}^3,$$
(10)

the electromagnetic Lippmann-Schwinger equation. This equation is known to have a unique solution $\mathbf{E} \in L^2(B_{\rho})$ for all right hand sides. It follows that the far field pattern $\mathbf{E}^{\infty} : S^2 \to \mathbb{C}^3$ defined by the asymptotic formula

$$\mathbf{E}^{\mathrm{s}}(x) = \frac{e^{ik|x|}}{|x|} \left(\mathbf{E}^{\infty}(\hat{x}) + \mathrm{O}\frac{1}{|x|} \right), \qquad |x| \to \infty$$

is given by

$$\mathbf{E}^{\infty}(\hat{x}) = -k^2 \int_{B_{\rho}} \frac{e^{-ik\hat{x}\cdot y}}{4\pi} a(y) \mathbf{E}(y) \,\mathrm{d}y - ik\hat{x} \int_{B_{\rho}} \frac{e^{-ik\hat{x}\cdot y}}{4\pi} \frac{\mathrm{grad}\,a(y)}{1 - a(y)} \cdot \mathbf{E}(y) \,\mathrm{d}y.$$

Since the far field pattern belongs to the set $L_T^2(S^2) := \{ \mathbf{g} \in (L^2(S^2))^3 : \hat{x} \times \mathbf{g}(\hat{x}) \times \hat{x} = \mathbf{g}(\hat{x}) \}$ of tangential fields on the unit sphere, it follows that $\mathbf{E}^{\infty} = -\frac{k^2}{4\pi} \mathbf{Z}_0(a\mathbf{E})$ with the operator $\mathbf{Z}_0 : L^2(B_\rho)^3 \to L_T^2(S^2)$ defined by

$$(\mathbf{Z}_{o}\mathbf{E})(\hat{x}) := \hat{x} \times \int_{B_{\rho}} e^{-ik\hat{x}\cdot y} \mathbf{E}(y) \,\mathrm{d}y \times \hat{x}, \qquad \hat{x} \in S^{2}.$$
(11)

The transposed $\mathbf{Z}_{i}: L^{2}_{T}(S^{2}) \to L^{2}(B_{\rho})^{3}$ of this operator is given by

$$(\mathbf{Z}_{i}\mathbf{g})(x) := \int_{S^{2}} e^{-ik\hat{x}\cdot y} \hat{x} \times \mathbf{g}(\hat{x}) \times \hat{x} \,\mathrm{d}s(\hat{x}), \qquad x \in B_{\rho}.$$
(12)

Consider incident fields of the form $\mathbf{E}^{i}(x,d;p) = pe^{-ikx \cdot d}$ with direction $d \in S^{2}$ and polarization $p \in \mathbb{C}^{3}$ such that $d \cdot p = 0$, and denote the corresponding far field patterns by $\mathbf{E}^{\infty}(\hat{x},d)$. Since the differential equation (9a) is linear and the far field pattern \mathbf{E}^{∞} depends linearly on the scattered field \mathbf{E}^{s} , we have

$$\mathbf{E}^{\infty}(\hat{x},d;p) = \mathbf{e}^{\infty}(\hat{x},d)p$$

for some 3×3 matrix $\mathbf{e}^{\infty}(\hat{x}, d)$, and $\mathbf{e}^{\infty}(\hat{x}, d)$ is uniquely determined if we impose the additional condition $\mathbf{e}^{\infty}(\hat{x}, d) \cdot d = 0$. With this notation we can define the *electromagnetic far field operator* $\mathcal{E}^{\infty}: L^2_T(S^2) \to L^2_T(S^2)$ by

$$(\mathcal{E}^{\infty}\mathbf{g})(\hat{x}) := \int_{S^2} \mathbf{e}^{\infty}(\hat{x}, d) \mathbf{g}(d) \, \mathrm{d}s(d), \qquad \hat{x} \in S^2.$$

Arguing as in subsection 2.1 we can derive the following factorization of the operator \mathcal{E}^{∞} :

$$\mathcal{E}^{\infty} = -\frac{k^2}{4\pi} \mathbf{Z}_{\mathrm{o}} \mathbf{M}_a (\mathbf{I} - \mathbf{W}_a)^{-1} \mathbf{Z}_{\mathrm{i}}.$$
(13)

Here the operators $\mathbf{M}_a, \mathbf{W}_a \in L(L^2(B_\rho)^3)$ are defined by $(\mathbf{M}_a \mathbf{E})(x) := a(x)\mathbf{E}(x)$ and $(\mathbf{W}_a \mathbf{E})(x) := k^2 \int_{B_\rho} \Phi(x-y)a(y)\mathbf{E}(y) \,\mathrm{d}y + \mathrm{grad} \int_{B_\rho} \Phi(x-y) \frac{\mathrm{grad}\,a(y)}{1-a(y)} \cdot \mathbf{E}(y) \,\mathrm{d}y.$

3. The iterative algorithm

3.1. general regularization methods

We first review some iterative methods for the solutions of general ill-posed operator equations

$$F(a) = y \tag{14}$$

with an operator $F : D(F) \subset \mathcal{X} \to \mathcal{Y}$ between Hilbert spaces \mathcal{X} and \mathcal{Y} which is Fréchetdifferentiable on its domain of definition D(F). The Fréchet derivative of F at a point $a \in D(F)$ will be denoted by $F'[a] : \mathcal{X} \to \mathcal{Y}$. The measured data are $y^{\delta} \in \mathcal{Y}$ where δ denotes a known error bound, $\|y^{\delta} - y\| \leq \delta$. Moreover, let $a_0 \in D(F)$ be an initial guess of the solution a.

Landweber iteration is defined by the formula

$$a_{n+1} = a_n - \mu F'[a_n]^* (F(a_n) - y^{\delta}), \qquad n \in \mathbb{N}_0$$
(15)

where μ is a scaling parameter satisfying $\mu < \|F'[a]^*F'[a]\|^{-1}$ for all a in a neighborhood of the solution. Since the speed of convergence of Landweber iteration depends critically on the choice of μ , it is worth to approximately compute $\|F'[a_0]^*F'[a_0]\|$ by some steps of the power method. As we will see, the convergence of Landweber iteration is very slow even with an optimal choice

of η . Faster convergence can be achieved by Newton type methods, which compute in the *n*th step an update $h_n = a_{n+1} - a_n$ from the linearized operator equation

$$F'[a_n]h_n = y^{\delta} - F(a_n).$$
(16)

Since eq. (16) inherits the ill-posedness of the nonlinear operator equation (14), some linear regularization method has to be used to solve (16) approximately. Such methods can be described by a family of bounded functions $q_{\alpha} : [0, \infty) \to \mathbb{R}$ satisfying $\lim_{\alpha \searrow 0} q_{\alpha}(\lambda) = 1/\lambda$ for all $\lambda > 0$. Bounded approximations to the unbounded operator $F'[a_n]^{-1}$ are given by $q_{\alpha}(F'[a_n]^*F'[a_n])F'[a_n]^*$, where this expression has to be interpreted in sense of the functional calculus of bounded self-adjoint operators. This leads to iteration formulas of the form

$$a_{n+1} = a_n + q_{\alpha_n}(F'[a_n]^*F'[a_n])F'[a_n]^*(y^{\delta} - F(a_n)).$$
(17)

In particular, for Tikhonov regularization we have $q_{\alpha}(\lambda) = 1/(\lambda + \alpha)$, and h_n is the unique global minimum of the functional

$$h \mapsto \|F'[a_n]h + F(a_n) - y^{\delta}\|^2 + \alpha_n \|h\|^2, \qquad h \in \mathcal{X}.$$
 (18)

The corresponding iteration is known as the Levenberg-Marquardt method. Another class of iteration formulas is obtained by taking x_0 as initial guess for each linearized equation (16):

$$a_{n+1} = a_0 + q_{\alpha_n} (F'[a_n]^* F'[a_n]) F'[a_n]^* \left(y^{\delta} - F(a_n) + F'[a_n](a_n - a_0) \right)$$

This was first suggested for Tikhonov regularization by Bakushinskii [1] and later generalized by Kaltenbacher [9]. The original method suggested by Bakushinskii is known as iteratively regularized Gauß-Newton method (short IRGNM). Here a_{n+1} can be characterized as the minimum of a functional of the form (18) with $||h||^2$ replaced by $||h + a_n - a_0||^2$. In our experiments we have always chosen the regularization parameters by the simple a-priori rule $\alpha_n = \alpha_0 \cdot (2/3)^n$. The actual regularization parameter is the stopping index, which is determined by the discrepancy principle.

For large scale problems the computation of the Jacobian F'[a] can be prohibitively expensive since typically the computation of one column of the matrix for F'[a] takes as much time as one evaluation of the operator F. Therefore, we have to solve the minimization problems (18) iteratively, e.g. by the conjugate gradient method. However, for small values of α_n this requires a large number of iterations since the systems become ill-conditioned. In [8] we have suggested a preconditioning technique which uses the close connection between the conjugate gradient method and the Lanczos iteration. This technique is only efficient for exponentially ill-posed problems, i.e. it exploits the severe ill-posedness of the problem.

3.2. an iterative method based on factorization of the far field operator

Let us consider the situation that the operator $F: D(F) \subset \mathcal{X} \to \mathcal{Y}$ is a composition of a nonlinear operator $G: D(F) \to \mathcal{Z}$ and a linear operator $T: \mathcal{Z} \to \mathcal{Y}$ such that the inversion of Gis well-posed or at least less severely ill-posed then the inversion of F. We assume that \mathcal{Y} and \mathcal{Z} are Hilbert spaces and \mathcal{X} is a normed space. A variant of nonlinear Tikhonov regularization for this situation has been investigated by Chavent & Kunisch [3] who coined the name *state space regularization*. Since the ill-posedness of the problem is only (or mainly) due to the operator T, it makes sense to regularize only T and consider iterative regularization methods of the form

$$a_{n+1} := \operatorname{argmin}_{a \in D(F)} \|P_{N(T)^{\perp}} \left\{ G'[a_n](a - a_n) + G(a_n) \right\} - q_{\alpha_n}(T^*T)T^*y^{\delta} \|^2$$
(19)

where $P_{N(T)^{\perp}}: \mathcal{Z} \to N(T)^{\perp}$ is the orthogonal projection.

To fit medium scattering problems in this framework, we assume that both \mathcal{Y} and \mathcal{Z} are Hilbert-Schmidt spaces of operators,

$$\mathcal{Y} = \mathcal{HS}(L^2(\Omega_1))$$
 and $\mathcal{Z} = \mathcal{HS}(L^2(\Omega_2))$

and that $T: \mathcal{Z} \to \mathcal{Y}$ has the form

$$T: \qquad \mathcal{HS}(L^2(\Omega_2)) \to \mathcal{HS}(L^2(\Omega_1))$$
$$K \mapsto Z_0 K Z_i$$

with compact linear operators $Z_i : L^2(\Omega_1) \to L^2(\Omega_2)$ and $Z_o : L^2(\Omega_2) \to L^2(\Omega_1)$. Recall that Hilbert-Schmidt spaces are Hilbert spaces of operators, and that every $K \in \mathcal{HS}(L^2(\Omega_j))$ can be written as an integral operator with kernel $k \in L^2(\Omega_j \times \Omega_j)$, such that $||K||_{\mathcal{HS}} = ||k||_{L^2}$ (see e.g. [16]). If we are dealing with L^2 spaces of vector-valued functions as for the electromagnetic medium scattering problem, the kernel k is matrix valued, and we have to take the Frobenius norm, e.g.

$$\|\mathcal{E}^{\infty}\|_{\mathcal{HS}}^2 = \int_{S^2} \int_{S^2} \|e^{\infty}(\hat{x}, d)\|_F^2 \,\mathrm{d}s(\hat{x}) \,\mathrm{d}s(d).$$

To keep notation as concise as possible we formulate our algorithm in a discrete setting. Assume that G and F are matrix-valued functions mapping a subset D(F) of the space \mathcal{X} to the matrix spaces $Mat(M_{o}, M_{i})$ and $Mat(N_{o}, N_{i})$ respectively, and assume that these mappings are related by

$$F(a) = Z_{o}G(a)Z_{i}, \qquad a \in D(F)$$

with $Z_i \in Mat(M_i, N_i)$ and $Z_o \in Mat(N_o, M_o)$. (By Mat(M, N) we denote the set of all $M \times N$ matrices.) We assume that $M_i \ge N_i$ and $M_o \ge N_o$ (typically $M_i \gg N_i$ and $M_o \gg N_o$ in our applications!) and that the matrices Z_i and Z_o have full rank. The spaces $\mathcal{Z} = Mat(M_o, M_i)$ and $\mathcal{Y} = Mat(N_o, N_i)$ are equipped with the Frobenius norm as discrete analog of the Hilbert-Schmidt norm.

An essential observation for the following algorithm is that a singular value decomposition of the linear mapping $T: K \mapsto Z_o K Z_i$ between the large spaces $Mat(M_o, M_i)$ and $Mat(N_o, N_i)$ can be expressed in terms of singular value decompositions of the much smaller matrices Z_i and Z_o . More precisely, if $\{(c_i^j, d_i^j, \sigma_i^j) : j = 1, ..., N_i\}$ and $\{(c_o^k, d_o^k, \sigma_o^k) : k = 1, ..., N_o\}$ are singular systems of Z_i and Z_o , then

$$\{(d_{o}^{k}d_{i}^{j^{*}}, c_{o}^{k}c_{i}^{j^{*}}, \sigma_{o}^{k}\sigma_{i}^{j}): j = 1, \dots, N_{i}, k = 1, \dots, N_{o}\}$$

is a singular system of T. In all our examples the time required to compute this singular value decomposition was smaller then the time for one evaluation of the operator F. Since a singular value decomposition of T is so easy to compute, we choose the truncated singular value decomposition as regularization method for T in (19). This corresponds to the choice $q_{\alpha}(\lambda) := 1/\lambda$ for $\lambda \geq \alpha$ and $q_{\alpha}(\lambda) := 0$ else. We will see that this choice significantly reduces the computational cost compared to other regularization methods. We write

$$Z_{\rm i} = D_{\rm i} \Sigma_{\rm i} C_{\rm i}^*$$
 and $Z_{\rm o} = C_{\rm o} \Sigma_{\rm o} D_{\rm o}^*$ (20)

where the columns of D_l are the vectors d_l^j , the columns of C_l are the vectors c_l^j , and the diagonal entries of Σ_l are the singular values σ_l^j , $l \in \{i, o\}$. We introduce the matrix $S \in \text{Mat}(N_o, N_i)$,

$$S_{k,j} := \sigma_{i}^{j} \sigma_{o}^{k} \tag{21}$$

and the symbol .* for the elementwise matrix multiplication, and note that

$$C_{o}^{*}F(a)C_{i} = (D_{o}^{*}G(a)D_{i}) \cdot S, \qquad a \in D(F).$$

Analogously, the symbol ./ denotes pointwise division of matrices. With these notations we can formulate the following algorithm:

iterative regularization method based on factorization

Input: $Y^{\delta} \in \operatorname{Mat}(N_{o}, N_{i})$ measured data, $\delta > 0$ noise level, $a_{0} \in D(F)$ initial guess compute singular value decompositions (20) of Z_{i} and Z_{o} ; n := 0; Define $P^{\alpha_{0}} \in \operatorname{Mat}(N_{o}, N_{i})$ by $P_{j,k}^{\alpha_{0}} := 1$ if $\sigma_{i}^{j} \sigma_{o}^{k} \ge \alpha_{0}$ and $P_{j,k}^{\alpha_{0}} := 0$ else; while $||F(a_{n}) \cdot P^{\alpha_{n}} - Y^{\delta}||_{F} > 2\delta$ do choose $\alpha_{n} > 0$ (see below!) and define $P^{\alpha_{n}} \in \operatorname{Mat}(N_{o}, N_{i})$ by $P_{j,k}^{\alpha_{n}} := 1$ if $\sigma_{i}^{j} \sigma_{o}^{k} \ge \alpha_{n}$ and $P_{j,k}^{\alpha_{n}} := 0$ else; $a_{n+1} := \operatorname{argmin}_{a \in D(F)}$ $\|\{D_{o}^{*}(G'[a_{n}](a - a_{n}) + G(a_{n}))D_{i} - (C_{o}^{*}Y^{\delta}C_{i})./S\} \cdot P^{\alpha_{n}}\|_{F};$ n := n + 1;end

For the medium scattering problems discussed in section 2 we have implemented this method as follows:

- Each column d_i^j of the matrix D_i represents a Herglotz wave function, i.e. an incident wave of the form (4) or (12), rsp. Multiplying G(a) with d_i^j essentially means solving the corresponding forward scattering problem. This was done using a fast iterative solver of the Lippmann-Schwinger equation based on FFT (see [8, 6, 17]).
- Since we use spectral cut-off as regularization method, we only have to solve a forward scattering problem for those columns of P^{α_n} which contain non-zero entries (see Fig. 3).
- To solve the linearized operator equations we use the conjugate gradient method applied to the normal equation.
- Choice of the regularization parameters α_n : We choose $\alpha_0 := \sigma_i^0 \sigma_o^0/2$. The residual can be decomposed as follows:

$$||F(a_n) \cdot * P^{\alpha_n} - Y^{\delta}||_F^2 = ||(F(a_n) - Y^{\delta}) \cdot * P^{\alpha_n}||_F^2 + ||Y^{\delta} \cdot * P^{\alpha_n} - Y^{\delta}||_F^2$$

If the first term (the "low frequency" components of the error) is larger than the second, we set $\alpha_n = \alpha_{n-1}$, otherwise we choose $\alpha_n = \alpha_{n-1}/2$. Thus we first fit the stable components of the data and then gradually the more unstable components.

4. Numerical examples

We first tested 4 different methods on a medium-size problem, the two-dimensional acoustic scattering problem discussed in subsection 2.1. We used 50 incident waves, the wave number k = 3, and the refractive index shown in Fig. 1. For all regularization methods the total computational cost was dominated by the number of pde solutions. This number is shown in Table 1 for different values of the noise level δ . Landweber iteration was clearly the slowest methods for all values of δ . For $\delta = 0.001$ we stopped the iteration after 20.000 Landweber steps corresponding to 1.000.000 pde solutions. The value of the relative residual was roughly 0.004 at the stopping index. The convergence of the IRGNM was much much faster. Even without preconditioning it outperformed Landweber iteration except for the noise level 10%. The new

Table 1. Comparison of the number of pde solutions required by different methods for different values of the relative noise level δ . The test problem was a two-dimensional inverse scattering problem with wave number k = 3, the refractive index shown in Fig. 1 and 50 incident waves at equidistant angles.

method	$\delta = 0.1$	$\delta = 0.01$	$\delta=0.001$
Landweber	1850	33450	> 2000000
IRGNM with CGNE	2350	6050	24550
preconditioned IRGNM	1850	3050	7350
new method	430	1162	2916



Figure 1. The refractive index in test example in Tab. 1.

method suggested in subsection 3.2 was clearly faster than all other methods for all values of δ . The quality of the final reconstructions was similar for all the methods.

Before proceeding to large scale problems we would like to point out that the degree of ill-posedness of inverse medium scattering problems depends strongly on the wave number k, the space dimension, and the pde. This is illustrated at the matrices S defined in (21) containing the singular values of T (see Fig. 3). For a given noise level one can obtain much more accurate reconstructions for large than for small wave numbers and better reconstructions for electromagnetic than for acoustic scattering problems.

We also tested our method for the three-dimensional electromagnetic scattering problem, the size of which demands a very efficient inversion scheme. The computations were carried out in parallel on a cluster of 12 Linux PCs. In the example shown in Fig. 2 the preconditioned IRGNM required the solution of 14200 pde's whereas the method proposed in section 3 needed only 4816 pde solutions. This is a remarkable improvement and illustrates the potential of the proposed method for large scale inverse scattering problems.

5. Conclusion

We have proposed an iterative method for the solution of inverse scattering problems in inhomogeneous acoustic and electromagnetic media. This method is based on the fact that these problems can be formulated as operator equations with a nonlinear operator-valued operator Fwhich permits a factorization of the form $F(a) = Z_0 G(a) Z_i$ with compact linear operators Z_0 and Z_i . We have shown that the inversion of G is well-posed and that the proposed method is significantly more efficient than iterative regularization method which do not use the special





Reconstruction of the refractive index from the electromagnetic far field operator with 1% noise. Left: true refractive index; middle: reconstruction by the preconditioned IRGNM; right: reconstruction by the method in subsection 3.2.



Figure 3. This figure illustrates the degree of ill-posedness of various scattering problems by showing (parts of) the matrices S containing the singular values of T (see (21)). The singular values are arranged in decreasing order, $\sigma_j^1 \ge \sigma_j^2 \ge \ldots, j \in \{i, o\}$. The dotted lines enclose the matrix entries which are $\ge 0.1\sigma_i^1\sigma_o^1$, the dashed lines those $\ge 0.01\sigma_i^1\sigma_o^1$, and the solid lines the ones $\ge 0.001\sigma_i^1\sigma_o^1$.

structure of the problem.

Since the operator Z_i does not have dense range and Z_o is not one-to-one, we cannot recover G(a) from F(a), but only $P_oG(a)P_i$ where P_i is the orthogonal projection onto $\overline{R(Z_i)}$, and P_o the orthogonal projection onto $N(Z_o)^{\perp}$. We do not know if the inversion of the mapping $a \mapsto P_oG(a)P_i$ is stable. If this question could be settled, one could possibly obtain a full convergence proof for the iterative regularization method proposed in this paper. This would be a remarkable achievement since iterative regularization method are widely used for the solution of inverse scattering problems, but except for some partial results (see [7, 15]) no convergence analysis is available so far.

References

- A. B. Bakushinskii. The problem of the convergence of the iteratively regularized Gauss-Newton method. Comput. Maths. Math. Phys., 32:1353–1359, 1992.
- [2] G. Bao and P. Li. Inverse medium scattering for three-dimensional time harmonic Maxwell equations. Inverse Problems, 20:L1–L7, 2004.

- [3] G. Chavent and K. Kunisch. Regularization in state space. RAIRO, Modélisation Math. Anal. Numér., 27:535–564, 1993.
- [4] D. Colton and R. Kreß. Inverse Acoustic and Electromagnetic Scattering Theory. Springer Verlag, Berlin Heidelberg New York, second edition, 1997.
- [5] H. Haddar and P. Monk. The linear sampling method for solving the electromagnetic inverse medium problem. *Inverse Problems*, 18:891–906, 2002.
- [6] T. Hohage. On the solution of the forward and inverse electromagnetic medium scattering problem in the resonance region. submitted.
- [7] T. Hohage. Logarithmic convergence rates of the iteratively regularized Gauß-Newton method for an inverse potential and an inverse scattering problem. *Inverse Problems*, 13:1279–1299, 1997.
- [8] T. Hohage. On the numerical solution of a three-dimensional inverse medium scattering problem. Inverse Problems, 17:1743–1763, 2001.
- B. Kaltenbacher. Some Newton-type methods for the regularization of nonlinear ill-posed problems. Inverse Problems, 13:729–753, 1997.
- [10] A. Kirsch. Characterization of the shape of the scattering obstacle by the spectral data of the far field operator. *Inverse Problems*, 14:1489–1512, 1998.
- [11] A. Kirsch. Factorizations of the far field operator for the inhomogeneous medium case and an application in inverse scattering theory. *Inverse Problems*, 15:413–429, 1999.
- [12] A. Kirsch. The MUSIC-algorithm and the factorization method in inverse scattering theory for inhomogeneous media. *Inverse Problems*, 18:1025–1040, 2002.
- [13] A. Kirsch. The factorization method for Maxwell's equations. Inverse Problems, 20:117–134, 2004.
- [14] F. Natterer and F. Wübbeling. A propagation backpropagation method for ultrasound tomography. Inverse Problems, 11:1225–1232, 1995.
- [15] R. Potthast. On the convergence of Newton's method in inverse scattering. Inverse Problems, 17:1419–1434, 2001.
- [16] M. Taylor. Partial Differential Equations: Basic Theory, volume 1. Springer Verlag, New York, 1996.
- [17] G. M. Vainikko. Fast solvers of the Lippmann-Schwinger equation. In R. P. Gilbert, J. Kajiwara, and Y. S. Xu, editors, *Direct and Inverse Problems of Mathematical Physics*. Kluwer Acad. Publ., Dordrecht, 1999.
- [18] M. Vögeler. Reconstruction of the three-dimensional refractive index in electromagnetic scattering using a propagation-backpropagation method. *Inverse Problems*, 19:739–753, 2003.

Institut fr Numerische und Angewandte	e Mathematik
Universitt Gttingen	
Lotzestr. 16-18	
D - 37083 Gttingen	
Telefon:	0551/394512
Telefax:	0551/393944
Email: trapp@math.uni-goettingen.de	URL: http://www.num.math.uni-goettingen.de

Verzeichnis der erschienenen Preprints:

2005-01	A. Schöbel, S. Scholl:	Line Planning with Minimal Traveling Time
2005-02	A. Schöbel	Integer programming approaches for solving the delay management problem
2005-03	A. Schöbel	Set covering problems with consecutive ones property
2005-04	S. Mecke, A. Schöbel, D. Wagner	Station location - Complex issues
2005-05	R. Schaback	Convergence Analysis of Methods for Solving General Equations
2005-06	M. Bozzini, L. Lenarduzzi, R. Schaback	Kernel B –Splines and Interpolation
2005-07	R. Schaback	Convergence of Unsymmetric Kernel–Based Meshless Collocation Methods
2005-08	T. Hohage, F.–J. Sayas	Numerical solution of a heat diffusion problem by boundary element methods using the Laplace transform
2005-09	T. Hohage	An iterative method for inverse medium scattering problems based on factorization of the far field operator