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A NEWTON METHOD FOR RECONSTRUCTING NON STAR-SHAPED DOMAINS IN ELECTRICAL IMPEDANCE TOMOGRAPHY

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Abstract. We study the reconstruction the shape of a perfectly conducting inclusion in three dimensional electrical impedance tomography (EIT) using a regularized Newton method. This method involves a least squares penalty in the form of an additional nonlinear operator to cope with the non-uniqueness of general parametrizations of the unknown boundary. We provide a convergence result for this method in the general framework of nonlinear ill-posed operator equations. Moreover, we discuss the evaluation of the forward operator in EIT, its derivative, and the adjoint of the derivative using a wavelet based boundary element method. Numerical examples illustrate the performance of our method.

1. Introduction

Electrical impedance tomography is used in medical imaging to reconstruct the electric conductivity of a part of the body from measurements of currents and voltages at the surface [20]. The same technique is also used in geophysical explorations. An important special case consists in reconstructing the shape of an unknown inclusion or void assuming (piecewise) constant conductivities. In this case only one pair of current/voltage measurements is necessary, in principle. All these problems are nonlinear and severely ill-posed.

The problem is described by the overdetermined boundary value problem

$$\Delta u = 0 \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \qquad \text{on } \Gamma, \tag{1.1b}$$

$$u = f \qquad \text{on } \Sigma, \tag{1.1c}$$

$$\frac{\partial u}{\partial \mathbf{n}} = g \qquad \text{on } \Sigma. \tag{1.1d}$$

in the geometry shown in Fig. 1 (see Section 4 for details). The voltage f and the current g on the exterior boundary Σ are given, and our aim is to reconstruct the interior boundary Γ. It can be shown that Γ is uniquely determined from $f \neq 0$ and g (see [24]).

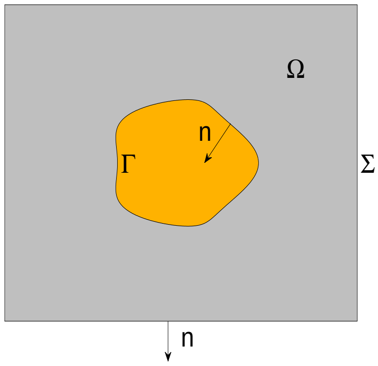

FIGURE 1. Geometry with unknown boundary Γ, exterior boundary Σ, and annular domain Ω.

It is often useful to formulate inverse problems as operator equations since a variety of efficient numerical methods, in particular Newton-type method are available for such problems. To describe an unknown boundary Γ by an element of a vector space, a parameterization is typically used. However, unless strong a priori information are available, e.g. that Γ is star-shaped, it is difficult to find a function space in which the representation of Γ is unique. In a previous paper [16] the authors considered generalized parametrization in the form of diffeomorphisms

$$\Psi : \Gamma_{\text{ref}} \to \Gamma \tag{1.2}$$

between a reference manifold Γ_{ref} and Γ and used a Sobolev space $H^s(\Gamma_{\text{ref}}; \mathbb{R}^3)$ as underlying vector space. We assume that one of the boundary values f and g is given exactly, and the other one (denoted by f^δ or g^δ, rsp.) is measured with errors. Introducing the operators $F_{\text{v}} : \Psi \mapsto u\big|_\Sigma$ and $F_{\text{c}} : \Psi \mapsto \frac{\partial u}{\partial \mathbf{n}}\big|_\Sigma$ where u is the unique solution to the boundary value problem (1.1a), (1.1b), (1.1d) or (1.1a), (1.1b), (1.1c), rsp. with $\Gamma = \Psi(\Gamma_{\text{ref}})$, the inverse problem can be formulated as operator equation

$$F_{\text{c}}(\Psi) = g^\delta \qquad \text{or} \qquad F_{\text{v}}(\Psi) = f^\delta, \text{ rsp.}$$

Whereas Γ is uniquely determined for exact data, Ψ is not. Standard regularized Newton-type methods tend to yield parameterizations which are either inadmissible or unfavorable for the evaluation of the forward operator. Therefore, in [16] a Newton method was suggested which is designed to find a parameterization Ψ, which minimizes a functional

$$\Psi \mapsto \|D\Psi(\mathbf{x})^\dagger\|^2_{L^2(\Gamma_{\text{ref}};\mathbb{R}^{3\times 2})} + \|\Psi - \Psi_0\|^2_{H^s(\Gamma_{\text{ref}};\mathbb{R}^d)}$$

among all parameterizations of Γ. Here Ψ_0 is an initial guess, $D\Psi(\mathbf{x})^\dagger = (D\Psi(\mathbf{x})^\star D\Psi(\mathbf{x}))^{-1} D\Psi(\mathbf{x})^\star$ denotes the Moore-Penrose inverse of $D\Psi(\mathbf{x})$, and the Frobenius norm is used on the space of 3×2 matrices. Hence, we not only try to avoid that $D\Psi(\mathbf{x})$ is singular for some $\mathbf{x} \in \Gamma_{\text{ref}}$, but also that $\|D\Psi(\mathbf{x})^{\dagger}\|$ is too large. A main contribution of the present paper is a convergence result for this method in the framework of abstract operator equations.

We do not aim to give a complete list of reference on obstacle problems in EIT and only refer to Hettlich and Rundell [17] and Chapko and Kress [7] for other iterative methods and to Brühl and Hanke [5, 6] for a non-iterative method using the complete Neumann-to-Dirichlet operator on Σ.

The plan of this paper is as follows: In Section 2 we show that the set of diffeomorphism of the form (1.2) for fixed Γ is a Banach submanifold of $H^s(\Gamma_{\text{ref}}; \mathbb{R}^d)$. This is used in Section 3 to derive a source condition under which a convergence result is shown. Section 4 contains characterizations of the Fréchet derivatives of the operators F_{c} and F_{v} and their adjoints. In Section 5 we summarize boundary integral formulations, which we use for the numerical solution of the forward problems, and provide some error estimates. Finally, in Section 6, we present numerical results.

2. Spaces of diffeomorphisms of submanifolds

Let $\Gamma_{\text{ref}}, \Gamma \subset \mathbb{R}^d$ be smooth closed submanifolds of dimension d' and consider diffeomorphisms $\Psi : \Gamma_{\text{ref}} \to \Gamma$. We call Γ_{ref} the reference manifold, which we assume to be infinitely smooth, and Ψ a (generalized) parameterization of Γ. Later we will always choose Γ_{ref} as the unit sphere $\mathbb{S}^2$, but the convergence result derived in Section 3 holds in much more general situations, e.g. closed curves in arbitrary dimensions.

Recall that for a smooth mapping $\Psi : \Gamma_{\text{ref}} \to \mathbb{R}^d$ the set $\Psi(\Gamma_{\text{ref}})$ is a smooth manifold if Ψ is injective and the derivative $D\Psi(\mathbf{x}) : T_{\mathbf{x}}\Gamma_{\text{ref}} \to \mathbb{R}^d$ is injective for all $\mathbf{x} \in \Gamma_{\text{ref}}$ (see e.g. [31, Theorem 73.E]). Here $T_{\mathbf{x}}\Gamma_{\text{ref}}$ denotes the tangent space to Γ_{ref} at the point $\mathbf{x} \in \Gamma_{\text{ref}}$. Since we want to work in a Hilbert space setting, we choose a Sobolev space $H^s(\Gamma_{\text{ref}}; \mathbb{R}^d)$ as underlying function space and define the following set of admissible parameterizations:

$$\begin{aligned} \mathcal{A}^s &:= \left\{ \Psi \in H^s(\Gamma_{\text{ref}}; \mathbb{R}^d) : \Psi \text{ and } D\Psi(\mathbf{x}) \text{ are one-to-one for all } \mathbf{x} \in \Gamma_{\text{ref}} \right\}, \\ s &> \frac{d'}{2} + 1. \end{aligned}$$

Lemma 1. *$\mathcal{A}^s$ is an open subset of $H^s(\Gamma_{\text{ref}}; \mathbb{R}^d)$.*

Proof. Let $\Psi_0 \in \mathcal{A}^s$. Since $s > d'/2 + 1$, $H^s(\Gamma_{\text{ref}}; \mathbb{R}^d)$ is continuously embedded in $C^1(\Gamma_{\text{ref}}; \mathbb{R}^d)$ ([30]). In particular, there exists a constant $c_e > 0$ such that $\sup_{\mathbf{x} \in \Gamma_{\text{ref}}} \|D\Psi(\mathbf{x})\| \leq c_e \|\Psi\|_{H^s}$ for all $\Psi \in H^s(\Gamma_{\text{ref}}; \mathbb{R}^d)$. Since Γ_{ref} is assumed to be compact, it follows that $C := \sup_{\mathbf{x} \in \Gamma_{\text{ref}}} \left\| [D\Psi_0(\mathbf{x})^* D\Psi_0(\mathbf{x})]^{-1} \right\|$ is finite. Therefore, there exists $\epsilon > 0$ such that

$$\sup_{\mathbf{x}\in\Gamma_{\mathrm{ref}}} \|D\Psi(\mathbf{x})^* D\Psi(\mathbf{x}) - D\Psi_0(\mathbf{x})^* D\Psi_0(\mathbf{x})\| < \frac{1}{C}$$

for all $\Psi \in H^s(\Gamma_{\mathrm{ref}};\mathbb{R}^d)$ with $\|\Psi - \Psi_0\|_{H^s} < \epsilon$. Hence, by a Neumann series argument, $D\Psi(\mathbf{x})^* D\Psi(\mathbf{x})$ is invertible for all $\mathbf{x} \in \Gamma_{\mathrm{ref}}$.
It remains to show that Ψ is injective if ϵ is chosen sufficiently small. First consider the special case $\Psi_0 = id$ and define $\mathbf{V} := \Psi - \Psi_0$. By Whitney's extension theorem ([29]) there exists a bounded linear extension operator $E : C^1(\Gamma_{\mathrm{ref}};\mathbb{R}^d) \to C^1(\mathbb{R}^d;\mathbb{R}^d)$ such that $(E\mathbf{V})|_{\Gamma_{\mathrm{ref}}} = \mathbf{V}$. (Here $\|\cdot\|_{C^1} := \|\cdot\|_{W^{1,\infty}}$.) If $\epsilon < 1/(c_e\|E\|)$, we have $\|E\mathbf{V}\|_{C^1} < 1$, and in particular the mapping $-E\mathbf{V} : \mathbb{R}^d \to \mathbb{R}^d$ is a contraction. Therefore, by Banach's fixed point theorem, the equation $\mathbf{x} = -(E\mathbf{V})(\mathbf{x}) + \mathbf{y}$ has a unique solution in $\mathbb{R}^d$ for all $\mathbf{y} \in \mathbb{R}^d$. Since $\mathbf{x} + (E\mathbf{V})(\mathbf{x}) = \Psi_0(\mathbf{x}) + \mathbf{V}(\mathbf{x}) = \Psi(\mathbf{x})$ for $\mathbf{x} \in \Gamma_{\mathrm{ref}}$, it follows that Ψ is injective.

Now assume that $\Psi_0 \in \mathcal{A}^s$ is arbitrary and set $\Gamma := \Psi_0(\Gamma_{\mathrm{ref}})$. Given $\Psi \in \mathcal{A}^s$ with $\|\Psi - \Psi_0\|_{H^s} < \epsilon$ define $\mathbf{V} \in C^1(\Gamma)$ by $\mathbf{V}(\Psi_0(\mathbf{x})) := \Psi(\mathbf{x}) - \Psi_0(\mathbf{x})$. Since $D\mathbf{V}(\Psi_0(\mathbf{x})) = (D\Psi(\mathbf{x}) - D\Psi_0(\mathbf{x}))\,[D\Psi_0(\mathbf{x})]^{-1}$ as a mapping from $T_{\Psi_0(\mathbf{x})}\Gamma$ to $\mathbb{R}^d$, it follows that $\|\mathbf{V}\|_{C^1} \le c\|\Psi - \Psi_0\|_{H^s}$ for some $c > 0$. Hence, the argument above with Γ_{ref} replaced by Γ shows that Ψ is injective if ϵ is sufficiently small. $\square$

A given manifold $\Gamma = \Psi_0(\Gamma_{\mathrm{ref}})$ parameterized by some $\Psi_0 \in \mathcal{A}^s$ has many parameterizations in $\mathcal{A}^s$. Let

$$\mathcal{A}^s_\Gamma := \{\Psi \in \mathcal{A}^s : \Psi(\Gamma_{\mathrm{ref}}) = \Gamma\}$$

denote the set of all such parameterizations. The following proposition shows that $\mathcal{A}^s_\Gamma$ locally looks like the space $H^s_{\tan}(\Gamma)$ of tangential vector fields in $H^s(\Gamma;\mathbb{R}^d)$. Here we impose the following assumption:

(2.1) There exist smooth mappings $\mathbf{n}_j : \Gamma \to \mathbb{S}^{d-1}$, $j = 1, \dots d - d'$ such that for all $\mathbf{x} \in \Gamma$ the vectors $\mathbf{n}_j(\mathbf{x})$ are pairwise orthogonal and orthogonal to $T_{\mathbf{x}}\Gamma$.

Note that (2.1) is satisfied with $\mathbf{n}_1$ as normal vector if Γ is a closed curve in $\mathbb{R}^2$ or a closed surface in $\mathbb{R}^3$. (2.1) is also satisfied for closed curves in $\mathbb{R}^3$ with $\mathbf{n}_1$ as the normal vector and $\mathbf{n}_2$ as the binormal vector of the Frenet-Serret frame.

Proposition 2. *Assume that $\Gamma := \Psi(\Gamma_{\mathrm{ref}})$ for some $\Psi \in \mathcal{A}^s$, and that Γ satisfies (2.1) with $\mathbf{n}_j \in C^k(\Gamma;\mathbb{R}^d)$, $k \in \mathbb{N}$, $k \ge s$. Then $\mathcal{A}^s_\Gamma$ is a C^1-Banach submanifold of $\mathcal{A}^s$, and the tangent space at any $\Psi \in \mathcal{A}^s_\Gamma$ is given by $T_\Psi\mathcal{A}^s_\Gamma = \{\mathbf{V} \in H^s(\Gamma_{\mathrm{ref}};\mathbb{R}^d) : \mathbf{V}\cdot(\mathbf{n}_j \circ \Psi) = 0, j = 1, \dots, d - d'\}$. In particular, for $\Psi_* \in C^k(\Gamma_{\mathrm{ref}};\mathbb{R}^d)$ we have*

$$T_{\Psi_*}\mathcal{A}^s_\Gamma = \left\{\tilde{\mathbf{V}} \circ \Psi_* : \tilde{\mathbf{V}} \in H^s_{\tan}(\Gamma)\right\}. \tag{2.2}$$

Moreover, for every $\Psi \in \mathcal{A}^s_\Gamma$ *there exists a diffeomorphism* $\varphi_\Psi : U \subset T_\Psi \mathcal{A}^s_\Gamma \to \mathcal{A}^s_\Gamma$ *from a neighborhood* U *of* 0 *in* $T_\Psi \mathcal{A}^s_\Gamma$ *to a neighborhood of* Ψ *in* $\mathcal{A}^s_\Gamma$*, which is of the form* $\varphi_\Psi(\mathbf{V}) = \Psi + \mathbf{V} + o(\|\mathbf{V}\|_{H^s})$.

Proof. We will characterize $\mathcal{A}^s_\Gamma$ as solution set of an equation

$$H(\Psi) = 0 \tag{2.3}$$

with a Fréchet differentiable mapping $H : U(\mathcal{A}^s_\Gamma) \subset \mathcal{A}^s \to H^s(\Gamma_{\text{ref}}; \mathbb{R})^{d-d'}$ defined on some neighborhood $U(\mathcal{A}^s_\Gamma)$ of $\mathcal{A}^s_\Gamma$ in $\mathcal{A}^s$. We will show that H is a submersion at every $\Psi \in \mathcal{A}^s_\Gamma$, i.e.

(i) $H'[\Psi] : H^s(\Gamma_{\text{ref}}; \mathbb{R}^d) \to H^s(\Gamma_{\text{ref}}; \mathbb{R})^{d-d'}$ is surjective.
(ii) There exists a continuous projection P of $H^s(\Gamma_{\text{ref}}; \mathbb{R}^d)$ onto the null space $N(H'[\Psi])$.

Then it follows from a general result on Banach manifolds ([31, Theorem 43.C]) that $\mathcal{A}^s_\Gamma$ is a C^1 Banach submanifold of $\mathcal{A}^s$ and

$$T_\Psi \mathcal{A}^s_\Gamma = N(H'[\Psi]). \tag{2.4}$$

Also the statement after (2.2) follows from this theorem.
To construct H, choose a functions $\eta_j \in C^k(U(\Gamma); \mathbb{R})$ defined in a neighborhood $U(\Gamma)$ of Γ in $\mathbb{R}^d$ such that

$$\eta_j(\mathbf{x}) = 0 \quad \text{and} \quad \operatorname{grad} \eta_j(\mathbf{x}) = \mathbf{n}_j(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Gamma, j = 1, \dots, d - d'.$$

The existence of such mappings η_j follows again from Whitney's extension theorem (see [29]). Moreover, by the implicit function theorem we have

$$\eta_1(\mathbf{x}) = \cdots = \eta_{d-d'}(\mathbf{x}) = 0 \quad \Leftrightarrow \quad \mathbf{x} \in \Gamma$$

for $\mathbf{x} \in U(\Gamma)$ if $U(\Gamma)$ is chosen sufficiently small. Now we define

$$\big(H(\Psi)\big)(\mathbf{x}) := [\eta_j(\Psi(\mathbf{x}))]_{j=1,\dots,d-d'}$$

and verify the properties above:
(Ad (2.3)): Since $H^s(\Gamma_{\text{ref}}; \mathbb{R}^d)$ is continuously embedded in $L^\infty(\Gamma_{\text{ref}}; \mathbb{R}^d)$, there exists a neighborhood $U(\mathcal{A}^s_\Gamma)$ of $\mathcal{A}^s_\Gamma$ in $H^s(\Gamma_{\text{ref}}; \mathbb{R}^d)$ such that $\Psi(\mathbf{x}) \in U(\Gamma)$ for all $\mathbf{x} \in \Gamma_{\text{ref}}$ and all $\Psi \in U(\mathcal{A}^s_\Gamma)$. Hence, H is well defined on $U(\mathcal{A}^s_\Gamma)$. If $U(\mathcal{A}^s_\Gamma)$ is chosen sufficiently small, then by Lemma 1 every solution Ψ to (2.3) must be parameterization of a manifold Γ', which is diffeomorphic to Γ_{ref} and contained in Γ. If Γ_{ref} is connected, this immediately shows that $\Gamma' = \Gamma$, otherwise apply the argument to each connected component of Γ_{ref}. Since conversely every $\Psi \in \mathcal{A}^s_\Gamma$ is a solution to (2.3), the solution

set of (2.3) is indeed $\mathcal{A}_\Gamma^s$.
(Ad (i)): The Fréchet derivative of H is given by

$$(H'[\Psi]\mathbf{V})_j(\mathbf{x}) = \operatorname{grad}\eta_j(\Psi(\mathbf{x}))\cdot\mathbf{V}(\mathbf{x}) = \mathbf{n}_j(\Psi(\mathbf{x}))\cdot\mathbf{V}(\mathbf{x}), \qquad j=1,\dots,d-d'$$

for $\mathbf{V}\in H^s(\Gamma_{\text{ref}};\mathbb{R}^d)$. Under the given smoothness assumptions $H'[\Psi]$ indeed maps to $H^s(\Gamma_{\text{ref}};\mathbb{R})^{d-d'}$ since $\mathbf{n}_j\circ\Psi\in H^s(\Gamma_{\text{ref}};\mathbb{R}^d)$ as $k\geq s$ and $(\mathbf{n}_j\circ\Psi)\cdot\mathbf{V}\in H^s(\Gamma_{\text{ref}};\mathbb{R})$ as $s>d'/2$. For $\varphi\in H^s(\Gamma_{\text{ref}};\mathbb{R})^{d-d'}$ the field $\mathbf{V}:=\sum_{j=1}^{d-d'}\varphi_j\,(\mathbf{n}_j\circ\Psi)$ belongs to $H^s(\Gamma_{\text{ref}};\mathbb{R}^d)$, and $H'[\Psi]\mathbf{V}=\varphi$. Hence, $H'[\Psi]$ is surjective.
(Ad (ii)): We have $N(H'[\Psi])=\{\mathbf{V}\in H^s(\Gamma_{\text{ref}};\mathbb{R}^d):\mathbf{V}\cdot(\mathbf{n}_j\circ\Psi)=0, j=1,\dots,d-d'\}$, and for $\Psi_*\in C^k(\Gamma_{\text{ref}};\mathbb{R}^d)$, the composition $\tilde{\mathbf{V}}\circ\Psi_*$ belongs to $H^s(\Gamma_{\text{ref}})$ for any $\tilde{\mathbf{V}}\in H^s_{\tan}(\Gamma)$, so $N(H'[\Psi])$ agrees with the right hand side of (2.2). P may be chosen as the orthogonal projection since $H^s(\Gamma_{\text{ref}};\mathbb{R}^d)$ is a Hilbert space and $N(H'[\Psi])$ is a closed subspace. □

3. Convergence of the regularized Newton method

We consider the following abstract setting: Let $\mathcal{X},\mathcal{Y}$, and $\mathcal{Z}$ be Hilbert spaces and $F:U\to\mathcal{Y}$, $G:U\to\mathcal{Z}$ Fréchet differentiable operators defined on a convex open subset $U\subset\mathcal{X}$ and $y\in R(F)$. Our aim is to find a solution $\Psi^\dagger$ the operator equation

$$F(\Psi)=y. \tag{3.1}$$

which minimizes the functional $\Psi\mapsto\|G(\Psi)\|^2+\|\Psi-\Psi_0\|^2$ among all solutions to (3.1):

$$\Psi^\dagger=\operatorname{argmin}_{\{\Psi:F(\Psi)=y\}}\left\{\|G(\Psi)\|^2+\|\Psi-\Psi_0\|^2\right\} \tag{3.2}$$

At our disposal are noisy data y^δ satisfying the error bound

$$\|y^\delta-y\|\leq\delta. \tag{3.3}$$

To this end we study the regularized Newton method as proposed in [16]. The k-th step of this method consists in solving the quadratic minimization problem

$$\left\|F'[\Psi_k]h+F(\Psi_k)-y^\delta\right\|_Y^2+\alpha_{k+1}\left\|G'[\Psi_k]h+G(\Psi_k)\right\|_Z^2+\alpha_{k+1}\left\|h+\Psi_k-\Psi_0\right\|_X^2=\min_{h\in\mathcal{X}}! \tag{3.4}$$

to compute the Newton update $h_k:=\Psi_{k+1}-\Psi_k$. Here

$$\alpha_k:=\alpha_0q^k \tag{3.5}$$

for some $q\in(0,1)$ and some $\alpha_0>0$. This is a variant of the iteratively regularized Gauss-Newton method (see [1, 3, 21]).

The essential assumption of our convergence analysis will be the source condition

$$\Psi_0-\Psi^\dagger-G'[\Psi^\dagger]^*G(\Psi^\dagger)=F'[\Psi^\dagger]^*w, \qquad w\in\mathcal{Y} \tag{3.6}$$

which is motivated and explained in the following lemma and remark:

Lemma 3. *Under the assumptions of Proposition 2 let $\mathcal{X} = H^s(\Gamma_{\mathrm{ref}};\mathbb{R}^d)$ and $\{\Psi \in \mathbb{D}(F) : F(\Psi) = y\} = \mathcal{A}^s_\Gamma$. Moreover, let $\Psi^\dagger$ be defined by (3.2). Then*

$$\Psi_0 - \Psi^\dagger - G'[\Psi^\dagger]^* G(\Psi^\dagger) \in \overline{R(F'[\Psi^\dagger]^*)}. \tag{3.7}$$

Proof. Note the derivative of the functional $J(\Psi) := \|G(\Psi)\|^2 + \|\Psi - \Psi_0\|^2$ is given by $J'[\Psi]h = 2\langle \mathbf{W}, h\rangle$ with $\mathbf{W} := \Psi^\dagger - \Psi_0 + G'[\Psi^\dagger]^* G(\Psi^\dagger)$. Assume that $\mathbf{W} \notin \overline{R(F'[\Psi^\dagger]^*)}$. Since $\overline{R(F'[\Psi^\dagger]^*)} = N(F'[\Psi^\dagger])^\perp$, there exists $h \in N(F'[\Psi^\dagger])$ such that $\langle \mathbf{W}, h\rangle < 0$. Let $\Psi_t := \varphi_{\Psi^\dagger}(th) \in \mathcal{A}^s_\Gamma$ for $t > 0$ sufficiently small with $\varphi_{\Psi^\dagger}(th) = \Psi^\dagger + th + O(t^2)$ as in Proposition 2. Then

$$J(\Psi_t) = J(\Psi^\dagger) + 2t\langle \mathbf{W}, h\rangle + o(t) \qquad \text{as } t \searrow 0.$$

As $\langle \mathbf{W}, h\rangle < 0$ this contradicts that $\Psi^\dagger$ is a minimum of J on $\mathcal{A}^s_\Gamma$. □

Remark 4. Since the range of $F'[\Psi^\dagger]^*$ is typically not closed for inverse problems, the source condition (3.6) does not follow from (3.7). As $F'[\Psi^\dagger]^*$ is typically a smoothing operator, (3.6) can be seen as a smoothness condition. This smoothness condition is quite restrictive in the case of exponentially ill-posed problems such as EIT, e.g. requiring the existence of analytic extensions (see [18]). Similar conditions have been used in [26]. Probably, the following analysis could be extended to weaker and more realistic logarithmic source conditions along the lines of [18]. However, as opposed to such an analysis the following results do not assume any restrictive condition on the degree of nonlinearity of the forward operator.

Introducing the operators

$$A_k := \begin{bmatrix} F'[\Psi_k] \\ \sqrt{\alpha_{k+1}} G'[\Psi_k] \end{bmatrix},$$

(3.4) can be rewritten as

$$\left\| \begin{bmatrix} A_k \\ \sqrt{\alpha_{k+1}} I \end{bmatrix} h - \begin{bmatrix} y^\delta - F(\Psi_k) \\ -\sqrt{\alpha_{k+1}} G(\Psi_k) \\ \sqrt{\alpha_{k+1}}(\Psi_0 - \Psi_k) \end{bmatrix} \right\|^2_{\mathcal{Y}\times\mathcal{Z}\times\mathcal{X}} = \min_{h\in\mathcal{X}}!$$

From the normal equation of this least squares problem we obtain the recursion formula

$$\Psi_{k+1} = \Psi_k + (\alpha_{k+1} I + A_k^* A_k)^{-1} \left(A_k^* \begin{bmatrix} y^\delta - F(\Psi_k) \\ -\sqrt{\alpha_{k+1}} G(\Psi_k) \end{bmatrix} + \alpha_{k+1}(\Psi_0 - \Psi_k) \right). \tag{3.8}$$

After straightforward manipulations it follows that

$$\begin{aligned}\Psi_{k+1}-\Psi^\dagger &= \alpha_{k+1}(\alpha_{k+1}I+A_k^*A_k)^{-1}(\Psi_0-\Psi^\dagger)\\ &\quad+(\alpha_{k+1}I+A_k^*A_k)^{-1}A_k^*\begin{bmatrix} y^\delta-F(\Psi_k)-F'[\Psi_k](\Psi^\dagger-\Psi_k)\\ \sqrt{\alpha_{k+1}}(-G(\Psi_k)-G'[\Psi_k](\Psi^\dagger-\Psi_k))\end{bmatrix}\\ &= \alpha_{k+1}(\alpha_{k+1}I+A_k^*A_k)^{-1}\left(\Psi_0-\Psi^\dagger-G'[\Psi_k]^*G(\Psi^\dagger)\right)\\ &\quad+(\alpha_{k+1}I+A_k^*A_k)^{-1}A_k^*\begin{bmatrix} y^\delta-F(\Psi_k)-F'[\Psi_k](\Psi^\dagger-\Psi_k)\\ \sqrt{\alpha_{k+1}}(G(\Psi^\dagger)-G(\Psi_k)-G'[\Psi_k](\Psi^\dagger-\Psi_k))\end{bmatrix}.\end{aligned}$$

Therefore, we can split up the error $E_{k+1}:=\Psi_{k+1}-\Psi^\dagger$ into the components

$$E_{k+1}=E_{k+1}^{\text{app}}+E_{k+1}^{\text{noi}}+E_{k+1}^{\text{nl},1}+E_{k+1}^{\text{nl},2}+E_{k+1}^{\text{nl},3}$$

defined by

$$\begin{aligned} E_{k+1}^{\text{app}} &:= \alpha_{k+1}(\alpha_{k+1}I+F'[\Psi^\dagger]^*F'[\Psi^\dagger])^{-1}\left(\Psi_0-\Psi^\dagger-G'[\Psi^\dagger]^*G(\Psi^\dagger)\right)\\ E_{k+1}^{\text{noi}} &:= (\alpha_{k+1}I+A_k^*A_k)^{-1}A_k^*[y^\delta-F(\Psi^\dagger),0]^\top\\ E_{k+1}^{\text{nl},1} &:= (\alpha_{k+1}I+A_k^*A_k)^{-1}A_k^*\begin{bmatrix} F(\Psi^\dagger)-F(\Psi_k)-F'[\Psi_k](\Psi^\dagger-\Psi_k)\\ \sqrt{\alpha_{k+1}}\left\{G(\Psi^\dagger)-G(\Psi_k)-G'[\Psi_k](\Psi^\dagger-\Psi_k)\right\}\end{bmatrix}\\ E_{k+1}^{\text{nl},2} &:= \alpha_{k+1}\left\{(\alpha_{k+1}I+A_k^*A_k)^{-1}-(\alpha_{k+1}I+F'[\Psi^\dagger]^*F'[\Psi^\dagger])^{-1}\right\}(\Psi_0-\Psi^\dagger-G'[\Psi^\dagger]^*G(\Psi^\dagger))\\ E_{k+1}^{\text{nl},3} &:= \alpha_{k+1}(\alpha_{k+1}I+A_k^*A_k)^{-1}\left(G'[\Psi^\dagger]^*-G'[\Psi_k]^*\right)G(\Psi^\dagger)\end{aligned}$$

The following essential lemma bounds the error $E_k^{\text{nl}}:=E_k^{\text{nl},1}+E_k^{\text{nl},2}+E_k^{\text{nl},3}$, which is due to the nonlinearity of the operators F and G in terms of the approximation error E_k^{app} and the propagated data noise error E_k^{noi}:

Lemma 5. *Let $y\in R(F)$ and $\Psi_0\in U$, and let* (3.3) *and* (3.5) *hold. Assume that there exists a solution $\Psi^\dagger$ to* (3.1) *satisfying the source condition* (3.6) *with $\|w\|\le\rho$. Moreover, assume that there are Lipschitz constants $L_F,L_G>0$ such that*

$$\begin{aligned}\|F'[\Psi_1]-F'[\Psi_2]\| &\le L_F\|\Psi_1-\Psi_2\|,\\ \|G'[\Psi_1]-G'[\Psi_2]\| &\le L_G\|\Psi_1-\Psi_2\|\end{aligned} \tag{3.9}$$

for all $\Psi_1,\Psi_2\in U$ and that G is scaled such that

$$L_G\|G(\Psi^\dagger)\|\le\frac{q}{3}\qquad\text{and}\qquad \|G'[\Psi^\dagger]^*G(\Psi^\dagger)\|\le\frac{1}{2}\|\Psi_0-\Psi^\dagger\|. \tag{3.10}$$

Then the Newton iterates Ψ_k defined by (3.8) *belong to $\mathbb{D}(F)$, and the error components satisfy the inequality*

$$\|E_k^{\text{nl}}\|\le\|E_k^{\text{app}}\|+\|E_k^{\text{noi}}\| \tag{3.11}$$

for $k = 1, \ldots, k_{\max}$ *where*

$$k_{\max} := \sup\left\{k \in \mathbb{N} : \frac{\delta}{\alpha_k} \le C_{\text{stop}}\right\} \tag{3.12}$$

if α_0 *is sufficiently large, and* $C_{\text{stop}}, \rho,$ *and* δ *are sufficiently small. (Precise bounds are given in the proof.)*

Proof. Let $\Psi^\dagger \in U$, and assume that the ball $B_R := \{\Psi \in \mathcal{X} : \|\Psi - \Psi^\dagger\| \le R\}$ is contained in U.

The proof is by induction in k: Assume that (3.11) holds true for some $k \in \{1, \ldots, k_{\max} - 1\}$. Since $E_k^{\text{app}} = r_k(F'[\Psi^\dagger]^* F'[\Psi^\dagger])z$ with $z := \Psi_0 - \Psi^\dagger - G'[\Psi^\dagger]^* G(\Psi^\dagger)$ and $r_k(\lambda) := \frac{\alpha_k}{\alpha_k + \lambda}$, it follows from the properties of the functional calculus and the inequality $0 \le r_{k-1}(\lambda) \le q^{-1} r_k(\lambda)$, which holds for all $\lambda \ge 0$, that

$$\|E_{k-1}^{\text{app}}\| \le q^{-1} \|E_k^{\text{app}}\|. \tag{3.13a}$$

Similarly the inequality $0 \le (\alpha_{k-1} + \lambda)^{-1} \le (\alpha_k + \lambda)^{-1}$ implies

$$\|E_{k-1}^{\text{noi}}\| \le \|E_k^{\text{noi}}\|. \tag{3.13b}$$

Together with the induction hypothesis we obtain

$$\|E_{k-1}\| \le 2\left(\|E_{k-1}^{\text{app}}\| + \|E_{k-1}^{\text{noi}}\|\right) \le 2\left(q^{-1}\|E_k^{\text{app}}\| + \|E_k^{\text{noi}}\|\right). \tag{3.14}$$

Due to the Lipschitz conditions (3.9) the Taylor remainder is bounded by $\|F(\Psi^\dagger) - F(\Psi_{k-1}) - F'[\Psi_{k-1}](\Psi^\dagger - \Psi_{k-1})\| \le \frac{L_F}{2}\|\Psi_{k-1} - \Psi^\dagger\|^2$, and similarly for G. Together with the standard bound

$$\|(\alpha_k I + A_k^* A_k)^{-1} A_k^*\| \le \frac{1}{2\sqrt{\alpha_k}} \tag{3.15}$$

and the notation $L := \sqrt{L_F^2 + \alpha_0 L_G^2}$ we obtain

$$\|E_k^{\text{nl},1}\| \le \frac{\sqrt{2}L}{4\sqrt{\alpha_k}}\|E_{k-1}\|^2 \le \frac{2\sqrt{2}L}{\sqrt{\alpha_k}}\left(q^{-2}\|E_k^{\text{app}}\|^2 + \|E_k^{\text{noi}}\|^2\right).$$

In the second step we have used (3.14) and the inequality $(\xi + \eta)^2 \le 2\xi^2 + 2\eta^2$. It follows from the source condition (3.6) that $\|E_k^{\text{app}}\| \le 2^{-1/2}\sqrt{\alpha_k}\rho$. Hence, $\frac{2\sqrt{2}L}{q^2\sqrt{\alpha_k}}\|E_k^{\text{app}}\| \le \frac{1}{3}$ if we choose $\rho \le q^2/(6L)$. Moreover, it follows from (3.15) and (3.12) that

$$\frac{2\sqrt{2}L}{\sqrt{\alpha_k}}\|E_k^{\text{noi}}\| \le \frac{\sqrt{2}L\delta}{\alpha_k} \le \sqrt{2}LC_{\text{stop}} \le \frac{1}{3}$$

if we choose $C_{\text{stop}} \le (3\sqrt{2}L)^{-1}$. Putting these estimates together yields

$$\|E_k^{\text{nl},1}\| \le \frac{1}{3}\left(\|E_k^{\text{app}}\| + \|E_k^{\text{noi}}\|\right).$$

To obtain a similar estimate for $\|E_k^{\mathrm{nl},2}\|$, we use the source condition (3.6) and either explicit elementary estimates or the Riesz-Dunford calculus as in [3, Section 4.1] to show that

$$\begin{aligned}\|E_k^{\mathrm{nl},2}\| &\le C_1\rho\,\|A_{k-1}^*A_{k-1} - F'[\Psi^\dagger]^*F'[\Psi^\dagger]\| \\ &\le C_1\rho\left(\alpha_{k-1}\|G'[\Psi_{k-1}]^*G'[\Psi_{k-1}]\| + \|F'[\Psi_{k-1}]^*F'[\Psi_{k-1}] - F'[\Psi^\dagger]^*F'[\Psi^\dagger]\|\right) \\ &\le C_2\rho\left(\alpha_k\|G'[\Psi^\dagger]^*G'[\Psi^\dagger]\| + \|E_{k-1}\|\right)\end{aligned}$$

for some constants $C_1, C_2 > 0$. For the last inequality we have used (3.9) again. Since $\|E_k^{\mathrm{app}}\| \ge \alpha_k\|(\alpha_0 I + F'[\Psi^\dagger]^*F'[\Psi^\dagger])^{-1}\left(\Psi_0 - \Psi^\dagger - G'[\Psi^\dagger]^*G(\Psi^\dagger)\right)\|$, we obtain using (3.14) that

$$\|E_k^{\mathrm{nl},2}\| \le \frac{1}{3}\left(\|E_k^{\mathrm{app}}\| + \|E_k^{\mathrm{noi}}\|\right)$$

for ρ sufficiently small. Finally, using (3.9), the induction hypothesis, (3.10), and (3.13) we obtain

$$\|E_k^{\mathrm{nl},3}\| \le L_G\|G(\Psi^\dagger)\|\,\|E_{k-1}\| \le \frac{1}{3}\left(\|E_k^{\mathrm{app}}\| + \|E_k^{\mathrm{noi}}\|\right).$$

Summing up these estimates for $\|E_k^{\mathrm{nl},j}\|$, $j = 1, 2, 3$ yields (3.11).

Next we show that $\Psi_k \in \mathbb{D}(F)$. Using (3.6) and the inequality $\alpha\sqrt{\lambda}/(\alpha+\lambda) \le \sqrt{\alpha}/2$, it follows that $\|E_k^{\mathrm{app}}\| \le \frac{1}{2}\sqrt{\alpha_k}\|w\| \le \frac{\rho\sqrt{\alpha_0}}{2}$, and hence $\|E_k^{\mathrm{app}}\| \le R/4$ for $\rho \le R/(2\sqrt{\alpha_0})$. Moreover,

$$\|E_k^{\mathrm{noi}}\| \le \|E_{k_{\max}}^{\mathrm{noi}}\| \le \frac{\delta}{2\sqrt{\alpha_{k_{\max}}}} \le \frac{R}{4}$$

if $\delta \le R\sqrt{\alpha_{k_{\max}}}/2$. Together with (3.11) this implies that $\Psi_k \in B_R(\Psi^\dagger) \subset U$.

The proof of the assertion for $k = 1$ is almost identical to the induction step. We have used the induction hypothesis only in the proof of inequality (3.14). For $k = 1$ this can be shown as follows: If $\alpha_1 = q\alpha_0 \ge q\|F'[\Psi^\dagger]\|^2/(1-q)$, then

$$1 \le \frac{\alpha_1}{q(\alpha_1 + \|F'[\Psi^\dagger]\|^2)} \le \frac{\alpha_1}{q(\alpha_1 + \lambda)} \qquad \text{for } \lambda \in [0, \|F'[\Psi^\dagger]\|^2],$$

and hence

$$\begin{aligned}\|E_0\| &\le q^{-1}\|\alpha_1(\alpha_1 I + F'[\Psi^\dagger]^*F'[\Psi^\dagger])^{-1}E_0\| \\ &\le q^{-1}\|E_1^{\mathrm{app}}\| - q^{-1}\|\alpha_1(\alpha_1 I + F'[\Psi^\dagger]^*F'[\Psi^\dagger])^{-1}G'[\Psi^\dagger]^*G(\Psi^\dagger)\| \\ &\le q^{-1}\|E_1^{\mathrm{app}}\| - \|G'[\Psi^\dagger]^*G(\Psi^\dagger)\|.\end{aligned}$$

Using (3.10) it follows that $\|E_0\| \le 2q^{-1}\|E_1^{\mathrm{app}}\|$. □

For both terms on the right hand side of (3.11) simple upper bounds are available from linear regularization theory: $\|E_n^{\mathrm{noi}}\| \le \frac{\delta}{2\sqrt{\alpha_n}}$, and $\|E_n^{\mathrm{app}}\| \le C\|\tilde{w}\|\alpha_n^\nu$ if the source condition

$$\Psi_0 - \Psi^\dagger - G'[\Psi^\dagger]^*G(\Psi^\dagger) = (F'[\Psi^\dagger]^*F'[\Psi^\dagger])^\nu\tilde{w} \tag{3.16}$$

with $\nu \in [\frac{1}{2}, 1]$ is satisfied for some $\tilde{w} \in \mathcal{X}$. ((3.16) is equivalent to (3.6) for $\nu = \frac{1}{2}$ and stronger than (3.6) for $\nu > \frac{1}{2}$.) This yields the following convergence theorem, the proof of which is completely analogous to the proof of Theorem 3.1 in [2]. (What remains to be shown is that the iterates are well defined, i.e. $\Psi_k \in U$ up to the optimal stopping index.)

Theorem 6. *In addition to the assumptions of Lemma 5 let* (3.16) *hold true. Then:*

(1) For exact data $y^\delta = y$ the iterates converge to $\Psi^\dagger$, and

$$\|\Psi_k - \Psi^\dagger\| = o\left(\alpha_k^\nu\right), \qquad k \to \infty.$$

(2) If $\nu > \frac{1}{2}$ the oracle stopping index

$$k_{\mathrm{or}} := \operatorname{argmin}_{k \in \mathbb{N}} \left(\|E_k^{\mathrm{app}}\| + \frac{\delta}{2\sqrt{\alpha_k}} \right) \tag{3.17}$$

satisfies (3.12) *for δ sufficiently small. In particular,*

$$\|\Psi_{k_{\mathrm{or}}} - \Psi^\dagger\| = O\left(\delta^{\frac{2\nu}{2\nu+1}}\right), \qquad \delta \to 0. \tag{3.18}$$

The last inequality remains valid if k_{or} is replaced by a stopping rule $k_* = k_*(\delta)$ satisfying $\alpha_{k_*} \sim \delta^{\frac{2}{2\nu+1}}$.

Of course, the oracle stopping rule (3.17) is not implementable. Instead, Lepskiĭ's *balancing principle* can be used:

$$k_{\mathrm{bal}} := \min\left\{ k \le k_{\max} : \|\Psi_k - \Psi_m\| \le \frac{4\delta}{\sqrt{\alpha_m}} \right\}. \tag{3.19}$$

From a general result in [2] (Theorem 4.1) we immediately obtain:

Theorem 7. *Under the assumptions of Lemma 5 there exists a constant $C > 0$ such that*

$$\|\Psi_{k_{\mathrm{bal}}} - \Psi^\dagger\| \le C \|\Psi_{k_{\mathrm{or}}} - \Psi^\dagger\| \tag{3.20}$$

if $k_{\mathrm{or}} \le k_{\max}$. In particular, (3.18) *holds with k_{or} replaced by k_{bal} under the assumptions of Theorem 6.*

4. The EIT problem: forward operator, derivative, and adjoint

Let $D \subset \mathbb{R}^3$ be a connected domain with Lipschitz boundary $\Sigma = \partial\Omega$, and assume that an unknown open void or inclusion $S \subset D$ with smooth boundary $\Gamma = \partial S$ is located inside the domain D satisfying $\operatorname{dist}(\Sigma, \Gamma) > 0$, see Figure 1. We also assume that $\Omega := D \setminus \overline{S}$ is connected and denote by $\mathbf{n}$ the outer normal vectors of Ω on Γ and Σ. The material in Ω has a constant conductivity wheras the inclusion S is perfectly conducting. To determine S or its boundary Γ, we assume to have available both a voltage distribution $f \in H^{1/2}(\Sigma)$ and a current distribution $g \in H_\diamond^{-1/2}(\Sigma) = \{g \in H^{-1/2}(\Sigma) : \int_\Sigma g\,dx = 0\}$ at the accessible boundary Σ. Hence, we are seeking the annular domain Ω and an associated harmonic function u, satisfying the system of equations (1.1).

We consider both the situation that a voltage distribution f is applied and a current distribution g^δ is measured, and the situation that a current distribution g is applied and a voltage distribution f^δ is measured. The suffix δ indicates that the corresponding boundary values contain measurement errors with noise level $\delta \geq 0$, i.e. $\|g - g^\delta\|_{L^2(\Sigma)} \leq \delta$ or $\|f - f^\delta\|_{L^2(\Sigma)} \leq \delta$, respectively. In practice typically currents are applied and voltages measured. Consider the forward operators $F_{\mathrm{c}}, F_{\mathrm{v}} : \mathbb{D}(F_{\mathrm{c/v}}) \to L^2(\Sigma)$ as defined in the introduction. Here $\mathbb{D}(F_{\mathrm{c/v}}) := \{\Psi \in \mathcal{A}^s : \Psi(\Gamma_{\mathrm{ref}}) \subset D\}$ using the notation of Section 2. F_{c} is defined for a given applied voltage $f \in H^1(\Sigma)$, and F_{v} is defined for a given applied current $g \in H_\diamond^{-1/2}(\Sigma)_\diamond$.

It can be shown that the operators F_{c} and F_{v} are Fréchet differentiable and that the derivative can be characterized by boundary-value problems (see e.g. [12, 23]). More precisely, for $\Psi \in \mathbb{D}(F_{\mathrm{c/v}})$ and $\mathbf{V} \in H^s(\Gamma_{\mathrm{ref}}; \mathbb{R}^3)$ the Fréchet derivative at Ψ in direction $\mathbf{V}$ is given by

$$F_{\mathrm{c}}'[\Psi](\mathbf{V}) = \frac{\partial u_{\mathbf{V}}'}{\partial \mathbf{n}}\Big|_\Sigma \qquad \text{or} \qquad F_{\mathrm{v}}'[\Psi](\mathbf{V}) = u_{\mathbf{V}}'\big|_\Sigma \tag{4.1a}$$

where $u_{\mathbf{V}}'$ is the solution of the following boundary value problem:

$$\begin{aligned} \Delta u_{\mathbf{V}}' &= 0 && \text{in } \Omega, \\ u_{\mathbf{V}}' &= -\frac{\partial u}{\partial \mathbf{n}}((\mathbf{V} \circ \Psi^{-1}) \cdot \mathbf{n}) && \text{on } \Gamma, \\ u_{\mathbf{V}}' &= 0 && \text{on } \Sigma \qquad (\text{for } F_{\mathrm{c}}), \\ \frac{\partial u_{\mathbf{V}}'}{\partial \mathbf{n}} &= 0 && \text{on } \Sigma \qquad (\text{for } F_{\mathrm{v}}) \end{aligned} \tag{4.1b}$$

Lemma 8. *The adjoint operators $F_{\mathrm{c/v}}'[\Psi]^* : L^2(\Sigma) \to H^s(\Gamma_{\mathrm{ref}}; \mathbb{R}^3)$ are given by*

$$F_{\mathrm{c/v}}'[\Psi]^* y = J_s^* \left[-\left((\det D\Psi) \frac{\partial p}{\partial \mathbf{n}} \frac{\partial u}{\partial \mathbf{n}} \mathbf{n} \right) \circ \Psi \right] \tag{4.2}$$

where $J_s : H^s(\Gamma_{\mathrm{ref}}; \mathbb{R}^3) \hookrightarrow L^2(\Gamma_{\mathrm{ref}}; \mathbb{R}^3)$ is the embedding operator and p is the solution to the boundary value problem

$$\begin{aligned} &\Delta p = 0 \ \textit{in } \Omega, \qquad p|_\Gamma = 0, \\ &p|_\Sigma = y \quad (\textit{for } F_{\mathrm{c}}) \qquad \textit{or} \qquad \frac{\partial p}{\partial \mathbf{n}}\Big|_\Sigma = -y \quad (\textit{for } F_{\mathrm{v}}), \ \textit{rsp.} \end{aligned} \tag{4.3}$$

Details in what sense the L^2 boundary values in (4.3) have to be understood can be found in [22], and a definition of $\det D\Phi$ is contained in [16].

Proof. According to the characterization (4.1) the Fréchet derivatives can be factorized as follows:

$$F'_{\mathrm{c/v}}[\Psi] = G_{\Psi}^{\mathrm{c/v}} A_{\Psi} J_s$$

Here $A_{\Psi} : L^2(\Gamma_{\mathrm{ref}}; \mathbb{R}^3) \to L^2(\Gamma; \mathbb{R})$ is defined by

$$(A_{\Psi}\mathbf{V})(\mathbf{y}) := -\frac{\partial u}{\partial \mathbf{v}}(\mathbf{y})\mathbf{V}(\Psi^{-1}(\mathbf{y}))\mathbf{n}(\mathbf{y}), \qquad \mathbf{y} \in \Gamma$$

and $G_{\Psi}^{\mathrm{c}} : L^2(\Gamma) \to L^2(\Sigma)$ is defined by $G_{\Psi}^{\mathrm{c}}\varphi := \frac{\partial v}{\partial \mathbf{n}}|_{\Sigma}$ where v is the unique solution to the boundary value problem

$$\Delta v = 0 \text{ in } \Omega, \qquad v|_{\Gamma} = \varphi, \qquad v|_{\Sigma} = 0. \tag{4.4}$$

G_{Ψ}^{v} is defined analogously, but $v|_{\Sigma} = 0$ is replaced by $\frac{\partial v}{\partial \mathbf{n}}|_{\Sigma} = 0$. Therefore,

$$F'_{\mathrm{c/v}}[\Psi]^* = J_s^* A_{\Psi}^* (G_{\Psi}^{\mathrm{c/v}})^*,$$

and we have to compute the adjoints of all factors. If p is a solution to (4.3), and v is a solution to (4.4), an application of Green's theorem yields

$$\langle y, G_{\Psi}^{\mathrm{c}}\varphi \rangle = \int_{\Sigma} p \frac{\partial v}{\partial \mathbf{n}}\, ds = \int_{\Gamma} \frac{\partial p}{\partial \mathbf{n}} v\, ds = \left\langle \frac{\partial p}{\partial \mathbf{n}}, \varphi \right\rangle,$$

and hence $(G_{\Psi}^{\mathrm{c}}) * y = \frac{\partial p}{\partial \mathbf{n}}$. Analogously, it can be shown that $(G_{\Psi}^{\mathrm{v}})^* y = p$. The adjoint of A_{Ψ} is derived in [16, Theorem 3]:

$$(A_{\Psi}^* f)(\mathbf{x}) = -\left(f(\mathbf{y}) \frac{\partial u}{\partial \mathbf{n}} \right) \mathbf{n}(\mathbf{y})(\det D\psi(\mathbf{x})), \qquad \mathbf{x} \in \Gamma_{\mathrm{ref}}$$

where $\mathbf{y} := \Psi(\mathbf{x})$. Putting these formulas together yields (4.2). ☐

5. Boundary Integral Equations

To evaluate the forward operators and their derivatives we use boundary integral equations. We introduce the single layer and the double layer operators with respect the boundaries $\Phi, \Psi \in \{\Gamma, \Sigma\}$ by

$$(\mathcal{V}_{\Phi\Psi} u)(\mathbf{x}) := \frac{1}{4\pi} \int_{\Phi} \frac{1}{\|\mathbf{x} - \mathbf{y}\|} u(\mathbf{y}) d\sigma_{\mathbf{y}}, \qquad \mathbf{x} \in \Psi,$$

$$(\mathcal{K}_{\Phi\Psi} u)(\mathbf{x}) := \frac{1}{4\pi} \int_{\Phi} \frac{\langle \mathbf{x} - \mathbf{y}, \mathbf{n}_{\mathbf{y}} \rangle}{\|\mathbf{x} - \mathbf{y}\|^3} u(\mathbf{y}) d\sigma_{\mathbf{y}}, \qquad \mathbf{x} \in \Psi.$$

Note that $\mathcal{V}_{\Phi\Psi}$ denotes an operator of order -1 if $\Phi = \Psi$, i.e. $\mathcal{V}_{\Phi\Phi} : H^{-1/2}(\Phi) \to H^{1/2}(\Phi)$, while it is an arbitrarily smoothing compact operator if $\Phi \neq \Psi$ since $\mathrm{dist}(\Gamma, \Sigma) > 0$. Likewise, if $\Sigma, \Gamma \in C^1$, the double layer operator $\mathcal{K}_{\Phi\Phi} : L^2(\Phi) \to L^2(\Phi)$ is compact while it smoothes arbitrarily if $\Phi \neq \Psi$. We refer the reader to [8, 25, 27] for a detailed description of boundary integral equations.

Introducing $\mathcal{V} := \left[\begin{smallmatrix} \mathcal{V}_{\Gamma\Gamma} & \mathcal{V}_{\Sigma\Gamma} \\ \mathcal{V}_{\Gamma\Sigma} & \mathcal{V}_{\Sigma\Sigma} \end{smallmatrix}\right]$ and $\mathcal{K} := \left[\begin{smallmatrix} \mathcal{K}_{\Gamma\Gamma} & \mathcal{K}_{\Sigma\Gamma} \\ \mathcal{K}_{\Gamma\Sigma} & \mathcal{K}_{\Sigma\Sigma} \end{smallmatrix}\right]$ the unknown Neumann data in the forward problem described by the operator F_{c} satisfy the system of equations

$$\mathcal{V} \begin{bmatrix} \frac{\partial u}{\partial \mathbf{n}}\big|_\Gamma \\ \frac{\partial u}{\partial \mathbf{n}}\big|_\Sigma \end{bmatrix} = \left(\frac{1}{2} I + \mathcal{K}\right) \begin{bmatrix} 0 \\ f \end{bmatrix}. \tag{5.1}$$

Note that the Dirichlet-to-Neumann map on $\Gamma \cup \Sigma$ is given by $\mathcal{V}^{-1}(\frac{1}{2}I + \mathcal{K})$. The unknown Neumann data of the local shape derivatives $u'_{\mathbf{V}}$ in (4.1) are analogously obtained from the boundary integral equations

$$\mathcal{V} \begin{bmatrix} \frac{\partial u'_{\mathbf{V}}}{\partial \mathbf{n}}\big|_\Gamma \\ \frac{\partial u'_{\mathbf{V}}}{\partial \mathbf{n}}\big|_\Sigma \end{bmatrix} = \left(\frac{1}{2} I + \mathcal{K}\right) \begin{bmatrix} -(\mathbf{V}\cdot\mathbf{n}) \frac{\partial u}{\partial \mathbf{n}}\big|_\Gamma \\ 0 \end{bmatrix}. \tag{5.2}$$

Rearranging these systems and introducing the operators $\mathcal{A} := \left[\begin{smallmatrix} \mathcal{V}_{\Gamma\Gamma} & -\mathcal{K}_{\Sigma\Gamma} \\ -\mathcal{V}_{\Gamma\Sigma} & 1/2I + \mathcal{K}_{\Sigma\Sigma} \end{smallmatrix}\right]$ and $\mathcal{B} := \left[\begin{smallmatrix} 1/2I + \mathcal{K}_{\Gamma\Gamma} & -\mathcal{V}_{\Sigma\Gamma} \\ -\mathcal{K}_{\Gamma\Sigma} & \mathcal{V}_{\Sigma\Sigma} \end{smallmatrix}\right]$ we obtain a boundary integral equation for the evaluation of the operator F_{v} and its Frechet derivative:

$$\mathcal{A} \begin{bmatrix} \frac{\partial u}{\partial \mathbf{n}}\big|_\Gamma \\ u|_\Sigma \end{bmatrix} = \mathcal{B} \begin{bmatrix} 0 \\ g \end{bmatrix}, \qquad \mathcal{A} \begin{bmatrix} \frac{\partial u'_{\mathbf{V}}}{\partial \mathbf{n}}\big|_\Gamma \\ u'_{\mathbf{V}}|_\Sigma \end{bmatrix} = \mathcal{B} \begin{bmatrix} -(\mathbf{V}\cdot\mathbf{n}) \frac{\partial v}{\partial \mathbf{n}}\big|_\Gamma \\ 0 \end{bmatrix}. \tag{5.3}$$

To approximately solve these boundary integral equations numerically we assume that the boundary manifold $\partial\Omega = \Gamma \cup \Sigma \subset \mathbb{R}^3$ is given as a parametric surface consisting of smooth patches. More precisely, let $\square := [0,1]^2$ denote the unit square. The manifold $\partial\Omega$ is partitioned into a finite number of patches

$$\partial\Omega = \bigcup_{i=1}^{M} \Gamma_i, \qquad \Gamma_i = \gamma_i(\square), \qquad i = 1, 2, \ldots, M, \tag{5.4}$$

where each $\gamma_i : \square \to \Gamma_i$ defines a diffeomorphism of $\square$ onto Γ_i. The intersection $\Gamma_i \cap \Gamma_{i'}$, $i \neq i'$, of two patches Γ_i and $\Gamma_{i'}$ is assumed to be either $\emptyset$ or a common edge or vertex.

The construction of the parametric representation of the fixed manifold Σ is straightforward. To construct a parametric representation of the varying boundary Γ, it suffices to construct such a representation of the reference manifold Γ_{ref}. For the unit sphere $\Gamma_{\mathrm{ref}} = \mathbb{S}^2$ this can be done as follows: The surface of the cube $[-0.5, 0.5]^3$ consists of six patches. Each point $\mathbf{x} \in \partial([-0.5, 0.5]^3)$ can be lifted onto the boundary Γ via the operation $\mathbf{y}(\mathbf{x}) = \Psi(\mathbf{x}/\|\mathbf{x}\|) \in \Gamma$. That way, the surface Γ is subdivided into $M = 6$ patches, the parametric representations $\gamma_i : \square \to \Gamma_i$ of which can easily be derived.

We introduce a mesh of level j on the unit square by dyadic subdivisions of depth j into 4^j squares. On this mesh we consider piecewise bilinear nodal basis functions

$\{\phi_{j,k}^{\square} : k \in \triangle_j^{\square}\}$, where $\triangle_j^{\square}$ denotes a suitable index set satisfying $|\triangle_j^{\square}| = (2^j+1)^2$. We define the set of basis functions on the surface Γ via parameterization

$$\phi_{i,j,k}(\mathbf{x}) := \begin{cases} \phi_{j,k}^{\square}(\mathbf{s}), & \mathbf{x} = \gamma_i(\mathbf{s}) \in \Gamma_i, \\ 0, & \text{elsewhere,} \end{cases}$$

where $i = 1, 2, \ldots, M$. Then, the trial spaces $V_j := \operatorname{span}\{\phi_{i,j,k} : (i,k) \in \triangle_j\}$, where $\triangle_j := \{(i,k) : i = 1, \ldots, M,\ k \in \triangle_j^{\square}\}$, are nested with respect to j.

For $\Phi, \Psi \in \{\Sigma, \Gamma\}$, we introduce the system matrices

$$\mathbf{V}_{\Phi\Psi} = \frac{1}{4\pi}\left[\int_{\Psi}\int_{\Phi} \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\theta_i^{\Phi}(\mathbf{y})\theta_j^{\Psi}(\mathbf{x})d\sigma_{\mathbf{y}}d\sigma_{\mathbf{x}}\right]_{i\in\triangle_j^{\Phi}, j\in\triangle_j^{\Psi}},$$

$$\mathbf{K}_{\Phi\Psi} = \frac{1}{4\pi}\left[\int_{\Psi}\int_{\Phi} \frac{\langle \mathbf{x}-\mathbf{y}, \mathbf{n}_{\mathbf{y}}\rangle}{\|\mathbf{x}-\mathbf{y}\|^3}\theta_i^{\Phi}(\mathbf{y})\theta_j^{\Psi}(\mathbf{x})d\sigma_{\mathbf{y}}d\sigma_{\mathbf{x}}\right]_{i\in\triangle_j^{\Phi}, j\in\triangle_j^{\Psi}},$$

and the mass matrices

$$\mathbf{M}_{\Phi} = \left[\int_{\Phi} \theta_i^{\Phi}(\mathbf{x})\theta_j^{\Phi}(\mathbf{x})d\sigma_{\mathbf{x}}\right]_{i,j\in\triangle_j^{\Phi}},$$

and the load vectors of Dirichlet and Neumann data f_Φ and g_Φ

$$\mathbf{f}_{\Phi} = \left[\int_{\Phi} \theta_i^{\Phi}(\mathbf{x})f(\mathbf{x})d\sigma_{\mathbf{x}}\right]_{i\in\triangle_j^{\Phi}}, \qquad \mathbf{g}_{\Phi} = \left[\int_{\Phi} \theta_i^{\Phi}(\mathbf{x})g(\mathbf{x})d\sigma_{\mathbf{x}}\right]_{i\in\triangle_j^{\Phi}}.$$

Then, in view of (5.1) and (5.2), the linear system of equations

$$\begin{bmatrix} \mathbf{V}_{\Gamma\Gamma} & \mathbf{V}_{\Sigma\Gamma} \\ \mathbf{V}_{\Gamma\Sigma} & \mathbf{V}_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} \mathbf{a}_{\Gamma} \\ \mathbf{a}_{\Sigma} \end{bmatrix} = \begin{bmatrix} 1/2\mathbf{M}_{\Gamma} + \mathbf{K}_{\Gamma\Gamma} & \mathbf{K}_{\Sigma\Gamma} \\ \mathbf{K}_{\Gamma\Sigma} & 1/2\mathbf{M}_{\Sigma} + \mathbf{K}_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{\Gamma}^{-1}\mathbf{f}_{\Gamma} \\ \mathbf{M}_{\Sigma}^{-1}\mathbf{f}_{\Sigma} \end{bmatrix}, \tag{5.5}$$

gives us the Neumann data $a_\Gamma = \sum_{i\in\triangle_j^{\Gamma}}[\mathbf{g}_\Gamma]_i\theta_i^{\Gamma}$ on Γ and $a_\Sigma = \sum_{i\in\triangle_j^{\Sigma}}[\mathbf{g}_\Sigma]_i\theta_i^{\Sigma}$ on Σ from the Dirichlet data on Γ and Σ. Likewise, the system

$$\begin{bmatrix} \mathbf{V}_{\Gamma\Gamma} & -\mathbf{K}_{\Sigma\Gamma} \\ -\mathbf{V}_{\Gamma\Sigma} & 1/2\mathbf{M}_{\Sigma} + \mathbf{K}_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} \mathbf{b}_{\Gamma} \\ \mathbf{a}_{\Gamma} \end{bmatrix} = \begin{bmatrix} 1/2\mathbf{M}_{\Gamma} + \mathbf{K}_{\Gamma\Gamma} & -\mathbf{V}_{\Sigma\Gamma} \\ -\mathbf{K}_{\Gamma\Sigma} & \mathbf{V}_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{\Gamma}^{-1}\mathbf{g}_{\Gamma} \\ \mathbf{M}_{\Sigma}^{-1}\mathbf{f}_{\Sigma} \end{bmatrix}, \tag{5.6}$$

yields the Dirichlet data $b_\Gamma = \sum_{i\in\triangle_j^{\Gamma}}[\mathbf{b}_\Gamma]_i\theta_i^{\Gamma}$ on Γ and the Neumann data $a_\Sigma = \sum_{i\in\triangle_j^{\Sigma}}[\mathbf{a}_\Sigma]_i\theta_i^{\Sigma}$ on Σ from the Neumann data $\mathbf{g}_\Gamma$ on Γ and the Dirichlet data $\mathbf{f}_\Sigma$ on Σ. Note that we plugged in the L^2-orthogonal projection involving $\mathbf{M}_\Phi^{-1}$ to decouple the data vectors from the boundary integral operators on the right hand side.

The following theorem, which is proved in the appendix, provides bounds on the errors in the evaluation of the forward operator F_{c} and its derivative if the boundary integral equations (5.1) and (5.2) are approximately solved by the boundary element method (5.5). An analogous result for the operator F_{v} is formulated and proved in the appendix.

Theorem 9. *Assume that $\Psi \in \mathbb{D}(F_c)$ is a smooth parameterization of the boundary Γ, $\partial\Omega \in C^\infty$, $f \in H^3(\Sigma)$, and $\mathbf{V} \in C^2(\Gamma_{\text{ref}};\mathbb{R}^3)$. If $h_j \sim 2^{-j}$ denotes the step width of the Galerkin discretization, then the discretization error in our approximations of the forward solution operator F_c and its Fréchet derivative satisfy the estimates*

$$\left\| F_c(\Psi) - F_{c,h_j}(\Psi) \right\|_{L^2(\Sigma)} = \mathcal{O}(h_j^2) \tag{5.7}$$

$$\left\| F_c'[\Psi]\mathbf{V} - F'[\Psi]_{c,h_j}\mathbf{V} \right\|_{L^2(\Sigma)} = \mathcal{O}(h_j) \tag{5.8}$$

uniformly in j.

Notice that we loose one order of convergence since the approximation of the Neumann data $\frac{\partial u}{\partial \mathbf{n}}$, which naturally belong to $H^{-1/2}(\Gamma)$, appear in the Dirichlet data, which naturally belong to $H^{1/2}(\Gamma)$.

Using the piecewise bilinear nodal basis functions leads to the traditional boundary element method. Then, the system matrices are densely populated and we end up with an at least quadratic complexity for computing the approximate solution of (5.5) and (5.6), i.e., the computational work scales like $\mathcal{O}\big((|\triangle_j^\Gamma|+|\triangle_j^\Sigma|)^2\big) = \mathcal{O}(16^j)$.

We employ instead appropriate biorthogonal spline wavelets as constructed in several papers, see e.g. [11, 14, 15]. Then, we obtain quasi–sparse system matrices having only $\mathcal{O}(|\triangle_j^\Gamma| + |\triangle_j^\Sigma|) = \mathcal{O}(4^j)$ relevant matrix coefficients. Applying the matrix compression strategy developed in [9, 28] combined with an exponentially convergent *hp*–quadrature method [13], the wavelet Galerkin scheme produces the approximate solutions of (5.5) and (5.6) within linear complexity. Especially, due to the norm equivalences of wavelet bases, the diagonally scaled system matrices are well posed [10, 28].

6. Numerical results

6.1. **The forward solver.** To validate the convergence rates of the forward solver, we consider the harmonic function $u(x,y,z) = 4x^2 - 3y^2 - z^2$ and the geometric situation shown in Fig. 2. For test reasons we prescribe the inhomogeneous Dirichlet data $u|_\Gamma$ on the obstacle's boundary Γ and either the Dirichlet data $v|_\Sigma$ or the Neumann data $\langle(\nabla u)|_\Sigma, \mathbf{n}\rangle$ on the fixed boundary Σ, which is chosen as the surface of the cube $[-1,1]^3$. Thus, the solution of the boundary value problems (1.1) is u.

We discretize the associated boundary integral equations as described in section 5 in order to compute the unknown Neumann or Dirichlet data on Σ. The corresponding L^2-errors are plotted in Fig. 2. In fact, we observe the predicted rate of convergence h_j^2 (indicated by the dashed line) even though the boundary Σ is only piecewise smooth.

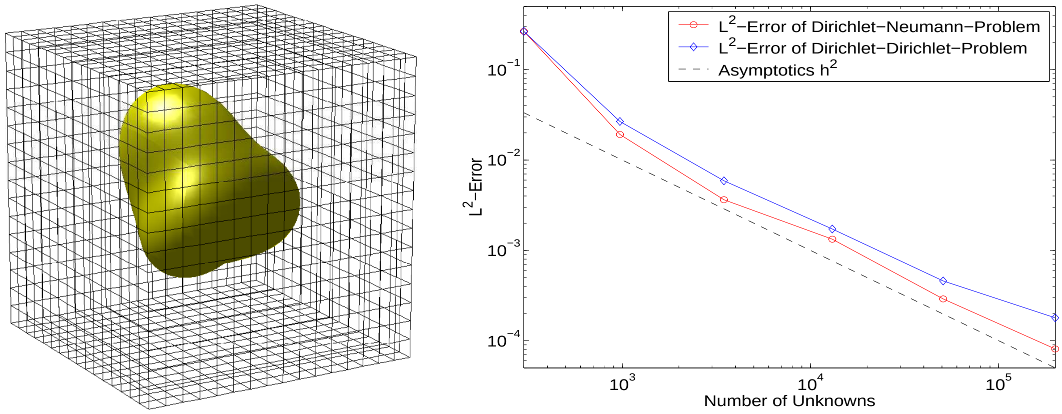

FIGURE 2. Left: Test configuration; right: $L^2(\Sigma)$-error of the forward solver.

6.2. **Reconstructing a pear-like inclusion.** In a second test we study the reconstruction of the pear-like inclusion shown in the Fig. 3, located inside the cube $[-1,1]^3$. To reconstruct this inclusion we apply the four linear independent voltage distributions $f(x,y,z) = 1, x, y, z$ and measure the associated current distributions g^δ.

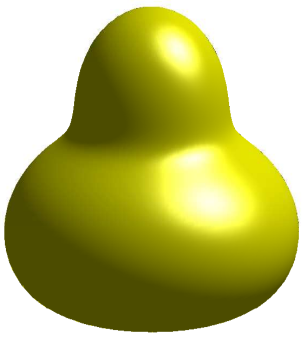

FIGURE 3. The true pear-like inclusion.

We computed the reconstruction for different scalings of the inclusion, always localized in the center of the cube. Each Cartesian component of the inclusion's parameterization has been discretized by spherical harmonics of order ≤ 20, resulting in $3 \cdot 20^2 = 1200$ unknown design parameters. Notice that this ansatz comprises in particular shifts of the inclusions.

To avoid an inverse crime the parameterization of the true boundary did not belong to the finite dimensional subspace. Moreover, we employed different discretization

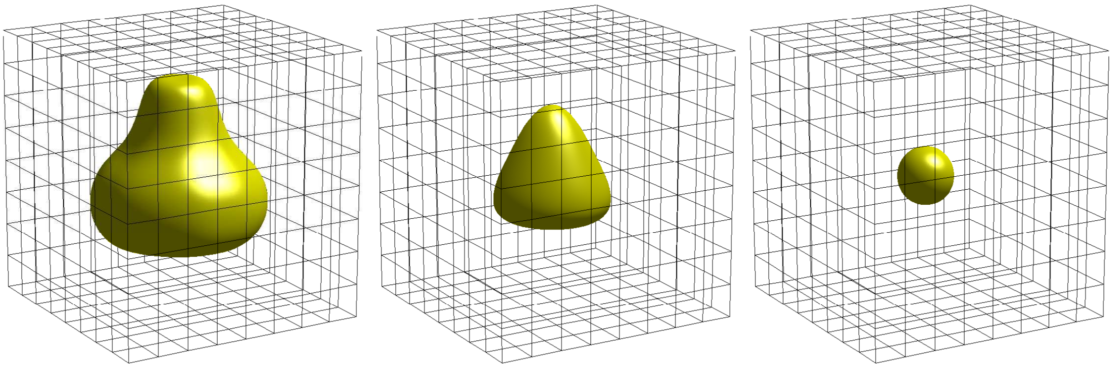

FIGURE 4. The reconstructions of the pear-like inclusion in case of different scalings.

levels for computing the synthetic data and for reconstructing the inclusion. The synthetic data are computed on level $j = 5$ (about 13000 boundary elements) while the reconstruction is performed on level $j = 4$ (about 3000 boundary elements). Here we added a small Gaußian noise with standard deviation 0.001 to the synthetic data at each grid point.

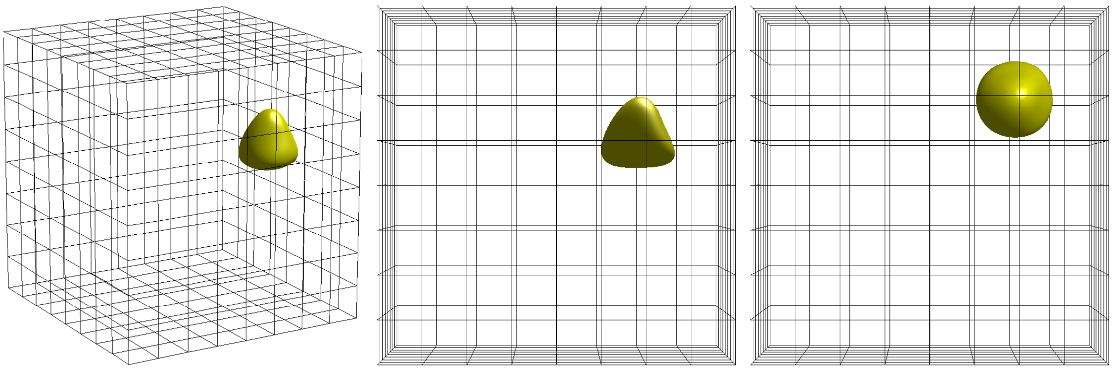

FIGURE 5. Reconstruction in case of a shifted inclusion.

As shown in Fig. 4 the reconstructions are the more accurate the larger the inclusion. In case of the a small inclusion we loose all geometric details and only find a small sphere. However, by moving a small inclusion closer to the boundary we are able to reconstruct also its geometric details, see Fig. 5. These observations illustrate the fact that the reconstruction of a surface is the more ill-posed the larger its distance from the outer boundary Σ. As initial guess we always used the sphere of radius 0.5 centered at the origin.

6.3. **Reconstructing an H-shaped inclusion.** The last test is dedicated to the reconstruction an H-shaped inclusion, located inside the cube $[-1, 1]^3$ as presented in Fig. 6. To reconstruct this inclusion we apply the ten linear independent current distributions $g(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\alpha}}$, $|\boldsymbol{\alpha}| \leq 2$, and measure the related voltage distributions f^{δ}. The synthetic data have been computed by using about 40 000 boundary elements.

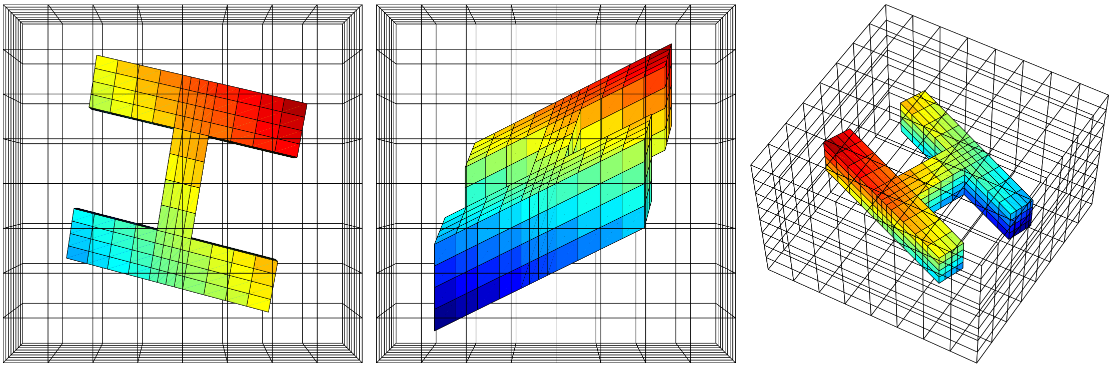

FIGURE 6. The true H-shaped inclusion.

Fig. 7 shows the reconstruction obtained by our algorithm. In the wavelet Galerkin method the discretization level has been chosen as $j = 3$ (about 750 boundary elements). The position and crude shape of the inclusion are found perfectly, whereas geometric details like the cross brace of the H are not reconstructed well.

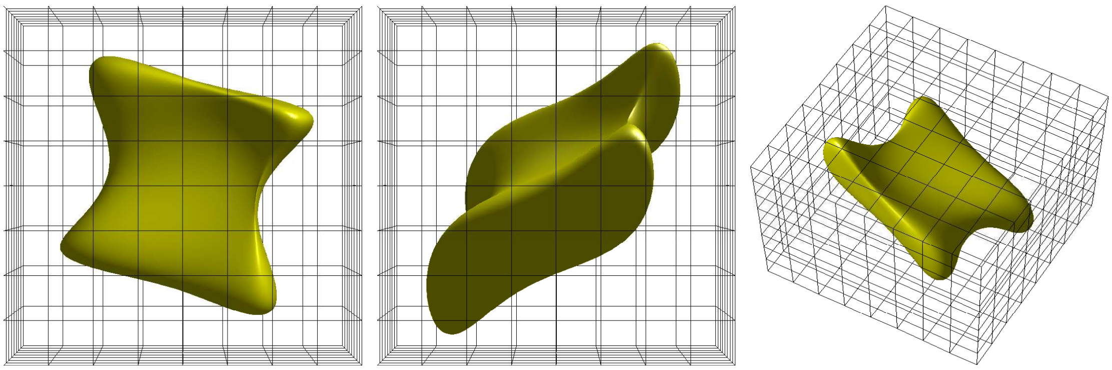

FIGURE 7. The reconstruction of the H-shaped inclusion.

APPENDIX A. CONVERGENCE ANALYSIS OF BEM

In this appendix we give the proof of Theorem 9.

Proof. We denote the exact and the approximate Neumann data of (5.1) by $g := \partial u/\partial \mathbf{n}$ and $g_j := [\partial u/\partial \mathbf{n}]_{h_j}^{\text{app}}$, respectively. In a first step we prove the estimate

$$\|g - g_j\|_{H^{-1/2}(\partial\Omega)} \lesssim h_j^{5/2} \|g\|_{H^2(\partial\Omega)}. \tag{A.1}$$

To this end, we will abbreviate the exact and the approximate right hand sides by $b := (1/2+\mathcal{K})f\chi_\Sigma$ and $b_j = (1/2+\mathcal{K}\Pi_j)f\chi_\Sigma$. Thus, we get the estimate

$$\|b-b_j\|_{H^{1/2}(\partial\Omega)} = \|\mathcal{K}(f-\Pi_j f)\chi_\Sigma\|_{H^{1/2}(\partial\Omega)} \lesssim \|(f-\Pi_j f)\chi_\Sigma\|_{H^{-1/2}(\partial\Omega)},$$

since $\mathcal{K}: H^{-1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega)$ is continuous if $\partial\Omega\in C^2$. Hence, using (A.4), we arrive at

$$\|b-b_j\|_{H^{1/2}(\partial\Omega)} \lesssim h_j^{5/2}\|f\chi_\Sigma\|_{H^2(\Sigma)} \lesssim h_j^{5/2}\|g\|_{H^1(\partial\Omega)}, \tag{A.2}$$

where the last estimate issues from the fact that the Dirichlet-to-Neumann map is a continuous operator of order 1.

By the First Strang Lemma

$$\|g-g_j\|_{H^{-1/2}(\partial\Omega)} \lesssim \inf_{v_j\in V_j}\|g-v_j\|_{H^{-1/2}(\partial\Omega)} + \sup_{v_j\in V_j}\frac{(b-b_j,v_j)_{L^2(\partial\Omega)}}{\|u_j\|_{H^{-1/2}(\partial\Omega)}} \tag{A.3}$$

we arrive at (A.1).

Finally, let Π_j denote the L^2-orthogonal projection onto V_j. Since for any $v\in L^2(\partial\Omega)$ the function $\Pi_j v$ is the Galerkin solution of the equation $Ix=v$ with respect to V_j, we arrive at the error estimate

$$\|v-\Pi_j v\|_{H^{-t}(\partial\Omega)} \lesssim h_j^{2+t}\|v\|_{H^d(\partial\Omega)}, \quad 0\le t\le 2, \tag{A.4}$$

provided that $v\in H^2(\partial\Omega)$. Thus, we derive the estimate

$$\begin{aligned}\|g-g_j\|_{L^2(\partial\Omega)} &\lesssim \|(I-\Pi_j)(g-g_j)\|_{L^2(\partial\Omega)} + \|\Pi_j(g-g_j)\|_{L^2(\partial\Omega)}\\ &\lesssim h_j^2\|g\|_{H^2(\partial\Omega)} + h_j^{-1/2}\|\Pi_j(g-g_j)\|_{H^{-1/2}(\partial\Omega)}\\ &\lesssim h_j^2\|g\|_{H^2(\partial\Omega)} + h_j^{-1/2}\big\{\|g-\Pi_j g\|_{H^{-1/2}(\partial\Omega)} + \|g-g_j\|_{H^{-1/2}(\partial\Omega)}\big\}\\ &\lesssim h_j^2\|g\|_{H^2(\partial\Omega)},\end{aligned}$$

which implies (5.7).

The estimate (5.8) follows as above by showing that

$$\left\|\frac{\partial u'_{\mathbf{V}}}{\partial\mathbf{n}} - \left[\frac{\partial u'_{\mathbf{V}}}{\partial\mathbf{n}}\right]^{\text{app}}_{h_j}\right\|_{H^{-1/2}(\partial\Omega)} \lesssim h^{3/2}\left\{\left\|\frac{\partial u'_{\mathbf{V}}}{\partial\mathbf{n}}\right\|_{H^1(\partial\Omega)} + \|g\|_{H^2(\partial\Omega)}\right\}. \tag{A.5}$$

Let $b := -(1/2+\mathcal{K})[(\mathbf{V}\cdot\mathbf{n})g\chi_\Gamma]$ denote the exact right hand side of (5.2) and $b_j := -(1/2+\mathcal{K})\Pi_j[(\mathbf{V}\cdot\mathbf{n})g_j\chi_\Gamma]$ its approximation. Using $\mathbf{V}\cdot\mathbf{n}\in C^1(\Gamma)$ and (A.1) we obtain

$$\begin{aligned}\|b-b_j\|_{H^{1/2}(\partial\Omega)} &\lesssim \|(\mathbf{V}\cdot\mathbf{n})(g-g_j)\chi_\Gamma\|_{H^{1/2}(\partial\Omega)} \lesssim \|\mathbf{V}\cdot\mathbf{n}\|_{C^1(\Gamma)}\|g-g_j\|_{H^{1/2}(\partial\Omega)}\\ &\lesssim h^{3/2}\|g\|_{H^2(\partial\Omega)}.\end{aligned}$$

Here the norm of $g-g_j$ in $H^{1/2}(\partial\Omega)$ is estimated using a similar argument as after (A.4). The First Strang Lemma (A.3) yields (A.5) which finishes the proof. □

We also give the analog of Theorem 9 for the operator $F_{\rm c}$:

Theorem 10. *Assume that $\Psi \in \mathbb{D}(F_{\rm v})$ is a smooth parameterization of the boundary Γ, $\partial\Omega \in C^\infty$, $g \in H^1(\Sigma)$, and $\mathbf{V} \in C^2(\Gamma_{\rm ref};\mathbb{R}^3)$. If $h_j \sim 2^{-j}$ denotes the step width of the Galerkin discretization, then the discretization error in our approximations of the forward solution operator $F_{\rm v}$ and its Fréchet derivative satisfy the estimates*

$$\big\|F_{\rm v}(\Psi) - F_{{\rm v},h_j}(\Psi)\big\|_{L^2(\Sigma)} = \mathcal{O}(h_j^2) \tag{A.6}$$

$$\big\|F_{\rm v}'[\Psi]\mathbf{V} - F'[\Psi]_{{\rm v},h_j}\mathbf{V}\big\|_{L^2(\Sigma)} = \mathcal{O}(h_j) \tag{A.7}$$

uniformly in j.

Proof. The energy space of the boundary integral operator $\mathcal{A}$ is the product space $H^{-1/2}(\Gamma)\times L^2(\Sigma)$. Denoting the exact solution of the first equation in (5.3) by a and its approximation, computed from the corresponding version of (5.6), by a_j, then, the assertion holds true if

$$\|a - a_j\|_{H^{-1/2}(\Gamma)\times L^2(\Sigma)} \lesssim h_j^2 \|a\|_{H^{3/2}(\Gamma)\times H^2(\Sigma)}, \tag{A.8}$$

which will now be proven.

We indicate the exact and the approximate right hand sides by $b := \mathcal{B}g\chi_\Sigma$ and $b_j = \mathcal{B}\Pi_j g\chi_\Sigma$. Similarly to the proof of Theorem 9, it follows that

$$\|b - b_j\|_{H^{1/2}(\Gamma)\times L^2(\Sigma)} \lesssim h_j^2 \|a\|_{H^{3/2}(\Gamma)\times H^2(\Sigma)}.$$

In view of Strang's first Lemma this leads to

$$\|a - a_j\|_{H^{-1/2}(\Gamma)\times L^2(\Sigma)} \lesssim h_j^2 \|a\|_{H^{3/2}(\Gamma)\times H^2(\Sigma)},$$

and thus, by straightforward modification of the arguments of the proof to Theorem 9 we derive the desired estimate (A.8), completing the proof of (A.6)

To establish (A.7), let $b := -(1/2+\mathcal{K})[(\mathbf{V}\cdot\mathbf{n})a\chi_\Gamma]$ denote the exact right hand side of the second equation in (5.3) and $b_j := -(1/2+\mathcal{K})\Pi_j[(\mathbf{V}\cdot\mathbf{n})a_j\chi_\Gamma]$ its approximation based on (5.6). By virtue of (A.8) we obtain

$$\|b - b_j\|_{H^{1/2}(\Gamma)\times L^2(\Sigma)} \lesssim h_j \|a\|_{H^{3/2}(\Gamma)\times H^2(\Sigma)}.$$

Hence, the First Strang Lemma leads to

$$\left\|\frac{\partial v_{\mathbf{V}}'}{\partial\mathbf{n}} - \left[\frac{\partial v_{\mathbf{V}}'}{\partial\mathbf{n}}\right]_{h_j}^{\rm app}\right\|_{H^{-1/2}(\Gamma)\times L^2(\Sigma)} \lesssim h_j \left\{\left\|\frac{\partial v_{\mathbf{V}}'}{\partial\mathbf{n}}\right\|_{H^1(\partial\Omega)} + \|a\|_{H^{3/2}(\Gamma)\times H^2(\Sigma)}\right\},$$

and thus (A.7). $\square$

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2009-01	H. Harbrecht, T. Hohage	A Newton method for reconstructing non star-shaped domains in electrical impedance tomography