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RIESZ BASES AND JORDAN FORM OF THE TRANSLATION OPERATOR IN SEMI-INFINITE PERIODIC WAVEGUIDES

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Abstract. We study the propagation of time-harmonic acoustic or transverse magnetic (TM) polarized electromagnetic waves in a periodic waveguide lying in the semi-strip $(0, \infty) \times (0, L)$. It is shown that there exists a Riesz basis of the space of solutions to the time-harmonic wave equation such that the translation operator shifting a function by one periodicity length to the left is represented by an infinite Jordan matrix which contains at most a finite number of Jordan blocks of size > 1. Moreover, the Dirichlet-, Neumann- and mixed traces of this Riesz basis on the left boundary also form a Riesz basis. Both the cases of frequencies in a band gap and frequencies in the spectrum and a variety of boundary conditions on the top and bottom are considered.

Key words. photonic crystal, photonic band gap, periodic dielectric medium, Floquet theory, analytic theory of operators

AMS subject classifications. 35P30, 78A40, 35Q60, 35B27, 35B30

1. Introduction.

Periodic media have received much attention in recent years since they can prohibit the propagation of electromagnetic and acoustic waves in some frequency ranges [24]. This localization property is a consequence of band structure of the spectrum of the underlying differential operator and of the presence of band gaps in this spectrum. Waves with frequencies in a band gap will decrease exponentially inside such media as a consequence of the exponential decay of the Green's kernel, see [6]. The band structure of the spectrum is explained by Floquet theory, see [9, 10, 16]. We refer the reader to [17] for a mathematical introduction to photonic crystals and to [15, 19] for a physical introduction.

The strong localization property can be used to construct devices that mould the flow of light or sound at a very small length scale. The simulation of such devices requires the numerical solution of differential equations in locally perturbed periodic media, which is a challenging task. The proof of existence and numerical computation of defect modes has been studied in [2, 4, 5, 6, 7, 8, 21]. Problems in locally perturbed infinite periodic media with a source term considered as part of the problem have been studied more recently in [11, 12, 13]. Here a natural approach consists in solving a boundary value problem on a compact set enclosing the perturbation and using a Dirichlet-to-Neumann or a related operator on the artificial boundary. Hence, the problem is decomposed into an interior and an exterior problem. The purpose of this paper is to contribute to the understanding of exterior boundary value problems in semi-infinite periodic waveguides. A summary of our results has already been given in the abstract, for a precise formulation we refer to the next section.

The study of wave propagation in doubly periodic half planes can be reduced via the Floquet transform to wave propagation in semi-infinite periodic waveguides with quasi-periodic boundary conditions on the lateral boundaries. Moreover, using a clever trick proposed by Fliss & Joly [12], boundary value problems in the doubly periodic exterior of a square can be reduced in some sense to boundary value problems

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in a doubly-periodic half plane. Therefore, the analysis of this paper is also relevant for these problems.

Our results prove two open conjectures formulated in connection with a numerical method to compute the Neumann-to-Dirichlet map proposed by Fliss, Joly and Li (see the discussion of Corollary 2.3). Actually, this study arose from attempts to justify an alternative numerical approach, which will be published elsewhere. Moreover, our results explain with a new approach the exponential decay of waves in periodic media with frequency in the band gap studied in [4, 5, 6, 7, 8] and even provides the optimal decay rate which corresponds to the one of the slowest decaying generalized Floquet mode, which may provide guidance for photonic crystal optimization by focusing on this generalized Floquet mode and trying to make its decay as fast as possible.

The paper is outlined as follows: in § 2 we introduce the problem and present the main theorem. Some prerequisites concerning function spaces and Floquet theory are collected in § 3. In § 4 we consider a holomorphic extension of the Floquet transformed differential operator and show that generalized Floquet modes correspond to characteristic values of this family of operators. This allows us to use a generalization of Rouché's theorem to operator valued mappings shown in [14], which will serve as an essential tool in the following analysis. The remaining part of the paper is dedicated to the proof of the main theorem. Finally, we show in an appendix that a uniqueness assumption in our main theorem is satisfied at least in an interesting special case.

2. Statement of the problem and the main results. The propagation of time-harmonic acoustic or transverse magnetic (TM) polarized electromagnetic waves in a 2-D waveguide lying in the semi-strip $S^+ = \mathbb{R}^+ \times (0, L)$ is described by the differential equation

$$\Delta v + \omega^2 \varepsilon_{\mathrm{p}} v = 0, \qquad \text{in } S^+. \tag{2.1a}$$

We assume that $\varepsilon_{\mathrm{p}} \in L^\infty(S^+)$ is periodic with period length 1 in first variable and bounded away from 0, i.e.

$$\begin{aligned} &\varepsilon_{\mathrm{p}}(x_1+1, x_2) = \varepsilon_{\mathrm{p}}(x_1, x_2), && \text{for all } (x_1, x_2) \in S^+, \\ &0 < \operatorname{essinf} \varepsilon_{\mathrm{p}} \le \varepsilon_{\mathrm{p}} \le \bar{\varepsilon} && \text{a.e. in } S^+ \end{aligned}$$

for some $\bar{\varepsilon} > 0$. On the top and bottom of S^+ we consider a boundary condition

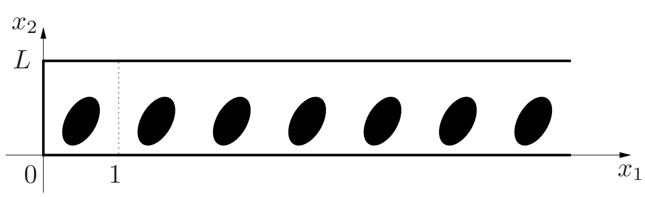

FIGURE 2.1. *Dielectric permittivity of the semi-infinite waveguide.*

$$\gamma v(x_1, \cdot) = (0, 0)^\top, \qquad x_1 > 0 \tag{2.1b}$$

with one of the following boundary value operators $\gamma : H^2((0,L)) \to \mathbb{R}^2$:

Dirichlet	$\gamma_{\mathrm{D}} v := (v(0), v(L))^\top$
Neumann	$\gamma_{\mathrm{N}} v := (v'(0), v'(L))^\top$
mixed	$\gamma_{\mathrm{DN}} v := (v(0), v'(L))^\top$
β-quasi-periodic	$\gamma_\beta v := (e^{i\beta} v(0) - v(L), e^{i\beta} v'(0) - v'(L))^\top$ with $\beta \in [0, 2\pi)$.

To treat β-quasi-periodic boundary conditions with $\beta = 0$ and $\beta = \pi$ we have to impose the symmetry condition $\varepsilon_{\mathrm{p}}(x_1, x_2) = \varepsilon_{\mathrm{p}}(x_1, L - x_2)$, $x \in (0,1) \times (0,L)$ for reasons explained in § 7.
Furthermore, we imposed a boundary condition on the left:

$$\natural\gamma\, v = f \qquad \text{with} \qquad \natural\gamma\, v := \theta_{\mathrm{D}} v(0,\cdot) + \theta_{\mathrm{N}} \frac{\partial v}{\partial x_1}(0,\cdot). \tag{2.1c}$$

Here $\theta_{\mathrm{D}}, \theta_{\mathrm{N}} \in \mathbb{C}$ with $|\theta_{\mathrm{D}}| + |\theta_{\mathrm{N}}| > 0$, and we consider $\natural\gamma$ as an operator with values in a space $H^{\natural\gamma}_\gamma$ which depends on θ_{N} and γ and will be defined in § 3.

To describe our condition on the behavior of the solution as $x_1 \to \infty$, which will complete the formulation of the boundary value problem, we need the following definition:

Definition 2.1. *A* generalized Floquet mode *is a nontrivial solution to* (2.1a)*,* (2.1b) *of the form*

$$\exp(i\xi x_1) \sum_{j=0}^{m} x_1^j u^{(j)}(x_1, x_2)$$

with $\xi \in \mathbb{C}$ and functions $u^{(j)}$ satisfying $u^{(j)}(x_1+1, x_2) = u^{(j)}(x_1, x_2)$ for all $(x_1, x_2) \in S^+$. We call ξ the quasi momentum *and m the* order *of the generalized Floquet mode, assuming that $u^{(m)} \neq 0$.*
Note that the quasi momentum is only determined up to an additive multiple of 2π. Typically we will choose $\Re\xi \in [-\pi, \pi)$.

It will be shown in Corollary 6.3 that the wave guide supports a finite even number $2\overline{n}$ of linearly independent generalized Floquet modes with real quasi momentum. Unless there are generalized Floquet modes with vanishing group velocity, precisely $\overline{n}$ of these $2\overline{n}$ generalized Floquet modes have positive group velocity, and these are labelled $v_1^+, \cdots, v_{\overline{n}}^+$ (see Remark 6.5). We require that

$$v \in H^{1,+}_\gamma(S^+) \quad \text{where} \quad H^{1,+}_\gamma(S^+) := H^1_\gamma(S^+) \oplus \operatorname{span}\{v_1^+, \cdots, v_{\overline{n}}^+\}. \tag{2.1d}$$

Each element $v \in H^{1,+}_\gamma(S^+)$ has a unique representation of the form $v = \tilde{v} + \sum_{n=1}^{\overline{n}} \alpha_n v_n^+$ with $\tilde{v} \in H^1_\gamma(S^+)$ and $\alpha_j \in \mathbb{C}$, and we introduce a norm on $H^{1,+}_\gamma(S^+)$ by $\|v\|^2_{H^{1,+}_\gamma(S^+)} = \|\tilde{v}\|^2_{H^1_\gamma(S^+)} + \sum_{n=1}^{\overline{n}} |\alpha_j|^2$.

We are now ready to formulate our main results:

Theorem 2.2. *Assume that the only solution to* (2.1) *with $f = 0$ is $v = 0$. Then:*

1. *Problem* (2.1) *is well posed in the sense that for all $f \in H^{\natural\gamma}_\gamma$ there exists a unique solution $u \in H^{1,+}_\gamma(S^+)$ to* (2.1)*, and u depends continuously on f.*
2. *There exist generalized Floquet modes v_n^+, $n \in \mathbb{N}$, which form a Riesz basis of the closed linear subspace $V \subset H^{1,+}_\gamma(S^+)$ of all weak solutions to* (2.1a) *and* (2.1b)*. Moreover, the traces $\{\natural\gamma\, v_n^+ : n \in \mathbb{N}\}$ form a Riesz basis of $H^{\natural\gamma}_\gamma$.*

3. *Suppose that $H^{1,+}(S^+)$ is invariant under the translation operator $(\mathcal{T}v)(x) := v(x_1+1, x_2)$. (This is always the case if $v_1^+, \dots, v_{\overline{n}}^+$ are of order 0.) Then the Riesz basis can be chosen such that $\mathcal{T} : V \to V$ is represented by an infinite Jordan matrix, which contains at most a finite number of Jordan blocks of size greater than 1 all of which are of finite size.*

Since according to the first part of this theorem, the operator $\natural_\gamma : V \to H_\gamma^{\natural}$ has a bounded inverse, we obtain the following corollary, which proves the conjectures in [13, Remark 5.1] and [11, Conjecture 3.2.52]. (In the latter case, to prove that $R|_{\mathrm{span}\{\natural_\gamma v_n^+ : n > \overline{n}\}}$ has spectral radius < 1, we also have to take into account Theorem 5.1.)

COROLLARY 2.3. *The operator $R := \natural_\gamma \mathcal{T} (\natural_\gamma|_V)^{-1} : H_\gamma^{\natural} \to H_\gamma^{\natural}$, which maps the trace of a solution to (2.1) to its trace at one periodicity length to the right, is represented by the same Jordan matrix as $\mathcal{T}$ with respect to the Riesz basis $\{\natural_\gamma v_n^+ : n \in \mathbb{N}\}$.*

To demonstrate that our formulation of the boundary value problem, and in particular the form radiation condition (2.1d) is reasonable, we show in Appendix A that the uniqueness assumption in Theorem 2.2 is satisfied for an interesting special class of problems.

3. Preliminaries.

3.1. Sobolev spaces on the cross section. For the boundary value operators $\gamma \in \{\gamma_{\mathrm{D}}, \gamma_{\mathrm{N}}, \gamma_{\mathrm{DN}}, \gamma_\beta\}$ defined in the introduction, we define the second derivative operators

$$\begin{aligned} &D_\gamma : \mathcal{D}(D_\gamma) \to L^2((0,L)), \qquad w \mapsto -w'' \\ &\text{with } \mathcal{D}(D_\gamma) := \{w \in H^2((0,L)) : \gamma w = (0,0)^\top\}. \end{aligned}$$

It is well known that these operators are positive and self-adjoint with compact resolvents. Moreover, complete orthonormal systems of eigenpairs $\left\{\left(\tilde{\psi}_k^{(\gamma)}, (\tilde{\kappa}_k^{(\gamma)})^2\right) : k \in \mathcal{I}\right\}$ are known explicitly:

boundary condition	$\tilde{\psi}_k^{(\gamma)}(t)$	$\tilde{\kappa}_k^{(\gamma)}$	$\mathcal{I}$
Dirichlet	$\sqrt{\frac{2}{L}}\sin\left(\frac{\pi k}{L}t\right)$	$\frac{\pi k}{L}$	$\mathbb{N}$
Neumann	$\sqrt{\frac{2}{L}}\cos\left(\frac{\pi k}{L}t\right)$	$\frac{\pi k}{L}$	$\mathbb{N} \cup \{0\}$
mixed	$\sqrt{\frac{2}{L}}\sin\left(\frac{\pi(2k-1)}{2L}t\right)$	$\frac{\pi(2k-1)}{2L}$	$\mathbb{N}$
β-quasi-periodic	$\sqrt{\frac{1}{L}}\exp\left(i\frac{\beta+2\pi k}{L}t\right)$	$\frac{\beta+2\pi k}{L}$	$\mathbb{Z}$

To simplify our notation we will often suppress the dependence of $\tilde{\psi}_k^{(\gamma)}$ and $\tilde{\kappa}_k^{(\gamma)}$ on γ in the following. Sobolev spaces corresponding a boundary value operator $\gamma \in \{\gamma_{\mathrm{D}}, \gamma_{\mathrm{N}}, \gamma_{\mathrm{DN}}, \gamma_\beta\}$ can then be defined by $H_\gamma^s((0,L)) = \mathcal{D}((I+D_\gamma)^{s/2})$ with norm $\|w\|_{H_\gamma^s} := \|(I+D_\gamma)^{s/2}w\|_{L^2}$ for $s \geq 0$, and for $s < 0$ the space $H_\gamma^s((0,L))$ can be defined as completion of $L^2((0,L))$ under the norm $\|\cdot\|_{H_\gamma^s}$. More explicitly,

$$\|w\|_{H_\gamma^s}^2 = \sum_{k \in \mathcal{I}} (1+\tilde{\kappa}_k^2)^s |\langle w, \tilde{\psi}_k\rangle|^2 \tag{3.1}$$

Remark 3.1. *It can be shown as in [18, Theorem 11.7] that*

$$
\begin{aligned}
H^{1/2}_{\gamma_{\mathrm{D}}}((0,L)) &= H^{\frac{1}{2}}_{00}((0,L)) = \{h \in H^{\frac{1}{2}}((0,L)) : \int_0^L \frac{|h(t)|^2}{t(L-t)}dt < \infty\} \\
H^{1/2}_{\gamma_{\mathrm{DN}}}((0,L)) &= \{h \in H^{\frac{1}{2}}((0,L)) : \int_0^L \frac{|h(t)|^2}{t}dt < \infty\} \\
H^{1/2}_{\gamma_\beta}((0,L)) &= \{h \in H^{\frac{1}{2}}((0,L)) : \int_0^L \frac{|e^{i\beta(L-t)}h(t) - h(L-t)|^2}{t}dt < \infty\}
\end{aligned}
$$

and $H^{1/2}_{\gamma_{\mathrm{N}}}((0,L)) = H^{1/2}((0,L))$, *but we will not use these facts.*

It will be convenient to renumber the square roots of the $\tilde{\kappa}_l$'s in increasing order such that

$$\sigma(D_\gamma) = \{\tilde{\kappa}_k^2 : k \in \mathcal{I}\} = \{\kappa_l^2 : l \in \mathbb{N}\}$$

and $0 \leq \kappa_1 \leq \kappa_2 \leq \dots$. This defines a bijective mapping $\mathbb{N} \to \mathcal{I}$, $k \mapsto l(k)$. Note that

$$\kappa_{2l+1} = \kappa_1 + \frac{2\pi}{L}l \qquad \text{and} \qquad \kappa_{2l+2} = \kappa_2 + \frac{2\pi}{L}l \tag{3.2}$$

for $l = 1, \dots$. Moreover, we will write $\psi_k = \tilde{\psi}_{l(k)}$ for $k \in \mathbb{N}$.

3.2. Sobolev spaces on the strip. Let $S := \mathbb{R} \times (0,L)$, and introduce the self-adjoint negative Laplace operators

$$
\begin{aligned}
&\tilde{D}_\gamma : \mathcal{D}(\tilde{D}_\gamma) \to L^2(S), \qquad v \mapsto -\Delta v, \\
&\text{with } \mathcal{D}(\tilde{D}_\gamma) := \{v \in H^2(S) : \forall x_1 \in \mathbb{R}\, \gamma v(x_1,\cdot) = (0,0)^\top\}
\end{aligned}
$$

for $\gamma \in \{\gamma_{\mathrm{D}}, \gamma_{\mathrm{N}}, \gamma_{\mathrm{DN}}, \gamma_\beta\}$. Then we define

$$H^s_\gamma(S) := \mathcal{D}\left((I + \tilde{D}_\gamma)^{s/2}\right), \qquad s \geq 0,$$

and $\|v\|_{H^s_\gamma} := \|(1+\tilde{D}_\gamma)^{s/2}v\|_{L^2}$. For $s < 0$, $H^s_\gamma(S)$ is defined as completion of $L^2(S)$ under the norm $\|\cdot\|_{H^s_\gamma}$. It can be shown that $H^1_{\gamma_{\mathrm{D}}}(S) = H^1_0(S)$, $H^1_{\gamma_{\mathrm{N}}}(S) = H^1(S)$, and $H^1_{\gamma_\beta}(S) = \{v \in H^1(S) : e^{i\beta}v(\cdot,0) = v(\cdot,L)\}$, and the norm $\|\cdot\|_{H^1_\gamma}$ is equivalent to $\|\cdot\|_{H^1}$ given by $\|v\|^2_{H^1} = \int_S(|v|^2 + |\nabla v|^2)\,dx$. $H^s_\gamma(S^+)$ can be defined as the set of all restrictions to S^+ of functions in $H^s_\gamma(S)$ with $\|v\|_{H^s_\gamma(S^+)} := \inf\{\|\tilde{v}\|_{H^s_\gamma(S)} : \tilde{v}|_{S^+} = v\}$. Moreover, the trace operators

$$
\begin{aligned}
&\overset{\vdash}{\gamma}_{\mathrm{D}} : H^1_\gamma(S^+) \to H^{1/2}_\gamma((0,L)), && v \mapsto v(0,\cdot), \\
&\overset{\vdash}{\gamma}_{\mathrm{N}} : H^1_\gamma(S^+;\Delta) \to H^{-1/2}_\gamma((0,L)), && v \mapsto \frac{\partial v}{\partial x_1}(0,\cdot),
\end{aligned}
$$

are well defined, continuous, and surjective, and have bounded right-inverses (see [18]). (Here $\|v\|^2_{H^1_\gamma(S^+;\Delta)} := \|v\|^2_{H^1_\gamma(S^+)} + \|\Delta v\|^2_{L^2(S^+)}$.) We choose $H^{\overset{\vdash}{\gamma}}_\gamma := H^{1/2}_\gamma((0,L))$ if $\theta_{\mathrm{N}} = 0$ and $H^{\overset{\vdash}{\gamma}}_\gamma := H^{-1/2}_\gamma((0,L))$ if $\theta_{\mathrm{N}} \neq 0$.

3.3. Floquet transform. The Floquet transform is defined by

$$\mathcal{F} : L^2(S) \to L^2\left((-\pi,\pi), L^2(\Omega_0)\right)$$
$$\mathcal{F}v(\alpha,x) := \frac{1}{\sqrt{2\pi}}\sum_{l\in\mathbb{Z}} v(x_1+l,x_2)e^{-i\alpha(x_1+l)}$$

where $\Omega_0 = \mathbb{R}/\mathbb{Z}\times(0,L)$. It is isometric with inverse $v(x) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\mathcal{F}v(\alpha,x)e^{i\alpha x_1}\,d\alpha$, $x\in S$.

We will frequently use the orthonormal bases $\{\varphi_l^{(\gamma)} : l\in\mathbb{Z}\times\mathbb{N}\}$ defined by

$$\varphi_l(x) := \exp(2\pi i l_1 x_1)\psi_{l_2}^{(\gamma)}(x_2), \qquad x\in\Omega_0. \tag{3.3}$$

If $v\in H^s_\gamma(S)$, then for all $\alpha\in[-\pi,\pi]$ the function $\mathcal{F}v(\alpha,\cdot)$ belongs to the Sobolev space $H^s_\gamma(\Omega_0)$ defined by $H^s_\gamma(\Omega_0) := \{u\in L^2(\Omega_0) : \|u\|_{H^s(\Omega_0)} < \infty\}$ with

$$\|u\|_{H^s_\gamma(\Omega_0)} := \Big(\sum_{l\in\mathbb{Z}\times\mathbb{N}}(1+|l|^2)^s|\langle\varphi_l^{(\gamma)},u\rangle|^2\Big)^{1/2}$$

for $s\geq 0$ and as completion of $L^2(\Omega_0)$ under this norm for $s<0$. For $\alpha\in[-\pi,\pi]$ the operator $\Delta_\alpha^{(\gamma)} : H^2_\gamma(\Omega_0)\to L^2(\Omega_0)$ uniquely defined by the property

$$\Delta_\alpha(\mathcal{F}v(\alpha,\cdot)) = (\mathcal{F}\Delta v)(\alpha,\cdot)$$

is given by

$$\Delta_\alpha = e^{-i\alpha x_1}\Delta e^{i\alpha x_1} = (\partial_{x_1}+i\alpha)^2 + \partial_{x_2}^2 = \Delta + 2i\alpha\partial_{x_1} - \alpha^2 \tag{3.4}$$

3.4. Band structure of the periodic waveguide spectrum. The study of solutions to the Helmholtz equation $\Delta v + \omega^2\varepsilon_{\mathrm{p}}v = 0$ in S with boundary conditions $\gamma v(x_1,\cdot) = (0,0)^\top$ for $x_1\in\mathbb{R}$ amounts to the study of spectral properties of the operator

$$A^{(\gamma)} := -\frac{1}{\varepsilon_{\mathrm{p}}}\Delta : \quad H^2_\gamma(S)\to L^2(S).$$

Due to the isometry of the Floquet transform, the spectrum of A_γ is the union of the spectra the operators defined by

$$A_\alpha^{(\gamma)} := -\frac{1}{\varepsilon_{\mathrm{p}}}\Delta_\alpha : \quad H^2_\gamma(\Omega_0)\to L^2(\Omega_0) \qquad \text{for } \alpha\in\mathbb{R}.$$

Since these operators are positive and self-adjoint in the weighted Hilbert space $L^2(\Omega_0,\varepsilon_{\mathrm{p}})$ with a compact resolvent, their spectra consist of a countable number of positive eigenvalues with finite multiplicities accumulating only at ∞:

$$\sigma(A_\alpha) = \{\tilde\lambda_m(\alpha) : n\in\mathbb{N}\}, \qquad \alpha\in[-\pi,\pi).$$

We assume that the $\tilde\lambda_m$ are arranged in increasing order. In this case they are even, periodic functions of α,

$$\begin{aligned}\tilde\lambda_m(\alpha) &= \tilde\lambda_m(-\alpha)\\ \tilde\lambda_m(\alpha+2\pi) &= \tilde\lambda_m(\alpha)\end{aligned} \qquad \alpha\in\mathbb{R}. \tag{3.5}$$

Moreover, the functions $\tilde{\lambda}_m$ are smooth, except at points where two or more of them cross. Alternatively, the eigenvalues of A_α^γ can be arranged such that they are analytic functions λ_m of α, which have holomorphic extensions to a complex neighborhood $\mathcal{U}$ of $[-\pi, \pi)$ ([20]). Furthermore, there exists a holomorphic family of eigenfunctions $\mathcal{U} \to L^2(\Omega_0)$, $\xi \mapsto w_{n,\xi}$:

$$\Delta_\xi w_{m,\xi} + \lambda_m(\xi)\varepsilon_{\rm p} w_{m,\xi} = 0, \qquad \xi \in \mathcal{U}, m \in \mathbb{N}. \tag{3.6}$$

However, if we insist on analyticity we have to sacrifice the properties (3.5).

The spectrum of $A^{(\gamma)}$ which we will also call the *spectrum of the periodic waveguide*, is given by the union of the spectral bands $I_m = [\min_{\alpha\in[-\pi,\pi]} \lambda_m(\alpha), \max_{\alpha\in[-\pi,\pi]} \lambda_m(\alpha)]$. Band gaps appear when two consecutive bands do not overlap. One of the most important features of periodic waveguides is that there cannot be guided waves for any frequency lying in a band gap.

4. Generalized Floquet modes and characteristic values of (B_ξ)**.** Although the Floquet transform of a function $v \in H^s_\gamma(S)$ is typically not smooth in α and does not possess an analytic extension, due to (3.4) the mapping $\alpha \mapsto \Delta_\alpha$ is a polynomial with coefficients in $\mathcal{L}(H^s(\Omega_0), H^{s-2}(\Omega_0))$ for all s, so in particular, it has a holomorphic extension denoted by

$$\Delta_\xi := \Delta + 2i\xi\partial_{x_1} - \xi^2, \qquad \xi \in \mathbb{C}. \tag{4.1}$$

Moreover, let us introduce the operators

$$B_\xi : H^s_\gamma(\Omega_0) \to H^{s-2}_\gamma(\Omega_0) \qquad B_\xi u := \Delta_\xi u + \omega^2 \varepsilon_{\rm p} u$$

for $\xi \in \mathbb{C}$, which are well defined for any $s \in [0, 2]$. For a review of the properties of holomorphic extensions of Floquet transformed periodic differential operators in a much greater generality we refer to [16].

$\xi_0 \in \mathbb{C}$ is called a *characteristic value* of $\xi \mapsto B_\xi$ if B_{ξ_0} is not injective. If ξ_0 is a characteristic value of B_ξ for some choice of $s \in [0, 2]$, and $B_{\xi_0} u_0 = 0$ for $u_0 \in H^s_\gamma(\Omega_0) \setminus \{0\}$, then $u_0 \in H^2_\gamma(\Omega_0)$ by elliptic regularity results, and ξ_0 is a characteristic value of B_ξ for any parameter $s \in [0, 2]$. Therefore, the set of characteristic values does not depend on the parameter s, and for studying this set we may choose s at our convenience. The operators B_ξ are defined such that (with $s = 2$):

Remark 4.1. *The following statements are equivalent for $\xi_0 \in \mathbb{C}$:*

- *ξ_0 is a characteristic value of (B_ξ).*
- *There exists a generalized Floquet mode $v(x) = e^{i\xi_0 x_1} u(x)$ with $u \in H^2_\gamma(\Omega_0)$.*

It follows from the second statement that:

Remark 4.2. *If $\xi \in \mathbb{C}$ is a characteristic value of $\xi \mapsto B_\xi$, then $\xi + 2\pi l$ is a characteristic value for all $l \in \mathbb{Z}$.*

To formulate some results by Gohberg and Sigal [14] in Theorem 4.4 below, which will serve as an essential tool in the following analysis, we first have to recall the definition of the multiplicity of characteristic values. Note that for all ξ the operator B_ξ is a compact perturbation of the operator $\Delta - 1$ in $\mathcal{L}(H^2_\gamma(\Omega_0), L^2(\Omega_0))$, and that $\Delta - 1$ has a bounded inverse. Therefore, by analytic Fredholm theory [22] the set of characteristic values of B_ξ is discrete. For the special case at hand the definitions in [14] simplify as follows:

Definition 4.3. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces and $\xi \mapsto B_\xi$ a holomorphic mapping defined on a domain in $\mathbb{C}$ with values in $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$. Moreover, let $B_\xi = B + K_\xi$ where B has a bounded inverse in $\mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ and K_ξ is compact for all ξ.*

The point $\xi_0 \in \mathbb{C}$ is called a characteristic value *of (B_ξ) if there exists a holomorphic function $\xi \mapsto u_\xi$ (called* root function*) with values in $\mathcal{B}_1$ such that $v(\xi_0) \neq 0$ and $B_{\xi_0} u_{\xi_0} = 0$. The* multiplicity *of a root function (u_ξ) is the order of ξ_0 as a root of $\xi \mapsto B_\xi u_\xi$. $\overline{u} \in \mathcal{B}_1$ is called an* eigenvector *of (B_ξ) corresponding to ξ_0 if $\overline{u} = u_{\xi_0}$ for some root function (u_ξ) of (B_ξ) corresponding to ξ_0. The* rank $\mathrm{rank}(\overline{u})$ *of an eigenvector $\overline{u}$ is defined as the maximum of all the multiplicities of root functions (u_ξ) with $\overline{u} = u_{\xi_0}$. (Under the given assumptions the geometric multiplicity $\alpha := \dim \ker B_{\xi_0}$ of ξ_0 and the ranks of all eigenvectors $\overline{u} \in \ker B_{\xi_0}$ are finite, see [14, Lemma 2.1].)*

A canonical system of eigenvectors *of (B_ξ) corresponding to ξ_0 is defined as a basis $\{u^{(1)},...,u^{(\alpha)}\}$ of $\ker B_{\xi_0}$ with the following properties:* $\mathrm{rank}\, u^{(1)}$ *is the maximum of the ranks of all eigenvectors corresponding to ξ_0, and* $\mathrm{rank}\, u^{(j)}$ *for $j = 2, ..., \alpha$ is the maximum of the ranks of all eigenvectors in some direct complement of* $\mathrm{span}\{u_0^{(1)}, ..., u_0^{(j-1)}\}$ *in $\ker B_{\xi_0}$. The numbers $r_j = \mathrm{rank}\, u_0^{(j)}$ $(j = 1, ..., \alpha)$ are called the* partial null multiplicities *of the characteristic value ξ_0, and $\mathfrak{n}((B_\xi); \xi_0) = (r_1, r_2, ..., r_\alpha)$ the α-tuple of partial null multiplicities. Finally, we call $N((B_\xi); \xi_0) = r_1 + r_2 + ... + r_\alpha$ the (total) null multiplicity of the characteristic value ξ_0 of (B_ξ).*

If Γ is a simple, closed, rectifiable contour contained in the domain of analyticity of (B_ξ) and of $(B_\xi)^{-1}$ and if $\xi_1, \xi_2, ..., \xi_n$ are the characteristic values of (B_ξ) enclosed by Γ, we set

$$N((B_\xi); \Gamma) := \sum_{j=1}^{n} N((B_\xi); \xi_j). \tag{4.2}$$

We cite the following results:

THEOREM 4.4 ([14, Theorems 2.1 and 2.2]). *Assume that (A_ξ) satisfies the assumptions of Definition 4.3 and let Γ be a simple, closed, rectifiable contour in the domain of analyticity of (A_ξ) and of $(A_\xi)^{-1}$. If $\xi \mapsto S_\xi$ is another holomorphic function defined on the same domain as (A_ξ) with values in $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ and if*

$$\|A_\xi^{-1} S_\xi\| < 1 \qquad \textit{for all } \xi \in \Gamma,$$

then $(A_\xi + S_\xi)^{-1}$ is analytic in some neighborhood of Γ and

$$N((A_\xi + S_\xi); \Gamma) = N((A_\xi); \Gamma) = \frac{1}{2\pi i} \mathrm{tr} \oint_\Gamma \frac{\partial A_\xi}{\partial \xi} A_\xi^{-1}\, d\xi. \tag{4.3}$$

Note that the first equation in (4.3) generalizes Rouché's theorem, and the second equation generalizes the logarithmic residue theorem.

We can now state the following generalization of Remark 4.1:

THEOREM 4.5. *The following statements are equivalent for $\xi_0 \in \mathbb{C}$:*

1. *ξ_0 is a characteristic value of (B_ξ) with partial null multiplicities $\mathfrak{n}((B_\xi); \xi_0) = (r_1, r_2, ..., r_\alpha)$.*
2. *The vector space $\mathcal{V}_{\xi_0}$ of all generalized Floquet modes with quasi momentum ξ_0 is a direct sum of $\mathcal{T}$-invariant subspaces $\mathcal{V}_{\xi_0,l}$ of dimensions $r_l \in \mathbb{N}$, $l = 1, \ldots, \alpha$, and none of the subspaces $\mathcal{V}_{\xi_0,l}$ can be decomposed into smaller $\mathcal{T}$-invariant subspaces.*

In this case there exists a basis of $\mathcal{V}_{\xi_0}$ with respect to which the linear mapping $\mathcal{T} : \mathcal{V}_{\xi_0} \to \mathcal{V}_{\xi_0}$ is represented by a Jordan matrix consisting of Jordan blocks of sizes $r_1, \ldots, r_\alpha$ corresponding to the eigenvalue $\lambda = \exp(i\xi_0)$.

The proof of this theorem is based on the following lemma:

LEMMA 4.6. *Let $\xi_0 \in \mathbb{C}$ and $\lambda := \exp(i\xi_0)$.*

1. *Suppose* (u_ξ) *is a root function of multiplicity* m *of* (B_ξ) *corresponding to* ξ_0 *and define*

$$v_\xi^{(j)}(x) := i^{-j}\frac{\partial^j}{\partial\xi^j}\left(e^{i\xi x_1}u_\xi(x)\right), \qquad j=0,\dots,m-1, \quad x\in S^+. \tag{4.4}$$

Then the functions $v_j := v_{\xi_0}^{(j)}$, $j=0,\dots,m-1$ *are linearly independent generalized Floquet modes of order* j *with quasi momentum* ξ_0*, and for all* $\underline{\alpha}\in\mathbb{C}^m$ *we have*

$$\sum_{j=0}^{m-1}\underline{\alpha}_j\mathcal{T}v_j = \lambda\sum_{k=0}^{m-1}(C^{(m)}\underline{\alpha})_k v_k \quad \textit{with} \quad C_{kj}^{(m)} := \begin{cases}\begin{pmatrix} j\\ k\end{pmatrix} & \textit{if } j-k\geq 0,\\ 0 & \textit{else.}\end{cases} \tag{4.5}$$

2. *Assume that* v_j*,* $j=0,\dots,m-1$ *are linearly independent generalized Floquet modes with quasi momentum* ξ_0 *satisfying* (4.5)*. Then there exists a root function* (u_ξ) *with the properties described in the first part.*

Proof. (ad 1): By assumption the function $\xi\mapsto B_\xi u_\xi$ has a root of order m at $\xi=\xi_0$. Therefore, using (3.4),

$$F(\xi) := e^{i\xi x_1}B_\xi u_\xi = e^{i\xi x_1}(\Delta_\xi+\omega^2\varepsilon_{\mathrm{p}})u_\xi = (\Delta+\omega^2\varepsilon_{\mathrm{p}})e^{i\xi x_1}u_\xi \tag{4.6}$$

also has a root of order m at $\xi=\xi_0$, i.e.

$$0 = i^{-j}F^{(j)}(\xi_0) = (\Delta+\omega^2\varepsilon_{\mathrm{p}})v_j, \qquad j=0,\dots,m-1, \tag{4.7}$$

and the functions v_j also satisfy the boundary condition $\gamma v_j(x_1,\cdot)=(0,0)^\top$. By the Leibniz rule

$$v_j(x) = e^{i\xi_0 x_1}\sum_{k=0}^{j}\begin{pmatrix} j\\ k\end{pmatrix}x_1^{j-k}u_k(x) \qquad j=0,\dots,m-1, \tag{4.8}$$

where $u_k := i^{-k}\frac{\partial^k u_\xi}{\partial\xi^k}|_{\xi=\xi_0}$. We compute

$$e^{-i\xi_0 x_1}(\mathcal{T}v_j)(x) = \lambda\sum_{k=0}^{j}\begin{pmatrix} j\\ k\end{pmatrix}(x_1+1)^{j-k}(\mathcal{T}u_k)(x), \tag{4.9a}$$

$$\begin{aligned} e^{-i\xi_0 x_1}\lambda\sum_{q=0}^{j}\begin{pmatrix} j\\ q\end{pmatrix}v_q(x) &= \lambda\sum_{q=0}^{j}\begin{pmatrix} j\\ q\end{pmatrix}\sum_{k=0}^{q}\begin{pmatrix} q\\ k\end{pmatrix}x_1^{q-k}u_k(x)\\ &= \lambda\sum_{k=0}^{j}\sum_{q=k}^{j}\begin{pmatrix} j\\ q\end{pmatrix}\begin{pmatrix} q\\ k\end{pmatrix}x_1^{q-k}u_k(x) = \lambda\sum_{k=0}^{j}\sum_{l=0}^{j-k}\begin{pmatrix} j\\ l+k\end{pmatrix}\begin{pmatrix} l+k\\ k\end{pmatrix}x_1^{l}u_k(x)\\ &= \lambda\sum_{k=0}^{j}\sum_{l=0}^{j-k}\begin{pmatrix} j\\ k\end{pmatrix}\begin{pmatrix} j-k\\ l\end{pmatrix}x_1^{l}u_k(x) = \lambda\sum_{k=0}^{j}\begin{pmatrix} j\\ k\end{pmatrix}(x_1+1)^{j-k}u_k(x) \end{aligned} \tag{4.9b}$$

using the identity $\begin{pmatrix} j\\ k\end{pmatrix}\begin{pmatrix} j-k\\ l\end{pmatrix} = \frac{j!}{(j-l-k)!k!l!} = \begin{pmatrix} j\\ l+k\end{pmatrix}\begin{pmatrix} l+k\\ k\end{pmatrix}$. Since $\mathcal{T}u_k = u_k$, we obtain that $(\mathcal{T}v_j)(x) = \lambda\sum_{q=0}^{j}\begin{pmatrix} j\\ q\end{pmatrix}v_q(x)$ for $j=0,\dots,m-1$, which implies (4.5).

To show linear independence of $v_0, \dots, v_{m-1}$ we assume that $\sum_{j=0}^{m-1} \alpha_j v_j = 0$ and show by induction in m that $\alpha_0 = \dots = \alpha_{m-1} = 0$. For $m = 1$ the assertion follows from $u_0 \neq 0$. Suppose the assertion holds for $m - 1$. According to (4.4) we have $0 = \sum_{j=0}^{m-1} \alpha_j i^{-j} \partial_\xi^j (e^{i\xi x_1} u_\xi(x))|_{\xi=\xi_0}$. It follows that

$$0 = \alpha_{m-1} x_1^{m-1} u_0(x) + \sum_{k=0}^{m-2} p_k(x_1) u_k(x), \tag{4.10}$$

where p_k are polynomials of degree $\leq m - 2$. Since the functions u_k are periodic and $u_0 \neq 0$, it follows that $\alpha_{m-1} = 0$. Therefore, by the induction hypothesis also $\alpha_{m-2} = \dots = \alpha_0 = 0$.
(ad 2): Let $v_0, \dots, v_{m-1}$ satisfy the assumptions of the second statement. For each $x \in S^+$ the system of equations (4.8) can be solved for the the unknowns $u_k(x)$, $k = 0, \dots, m-1$. As linear combinations of the v_j with smooth x_1-dependent coefficients, they have the same smoothness and satisfy the boundary conditions $\gamma u_k(x_1, \cdot) = (0, 0)^\top$. Moreover, it follows from (4.9) and the assumption (4.5) that

$$\sum_{k=0}^{j} \binom{j}{k} (x_1 + 1)^{j-k} [(\mathcal{T} u_k)(x) - u_k(x)] = 0, \qquad \text{for } j = 0, \dots, m-1, x \in S^+,$$

which implies that $\mathcal{T} u_k = u_k$ for all $k = 0, \dots, m - 1$. Introducing the family of functions $u_\xi(x) := \sum_{j=0}^{m-1} \frac{(i\xi - i\xi_0)^j}{j!} u_j(x)$, the relation (4.4) holds due to our definition of u_j. Therefore, the identities (4.6) and (4.7) hold true, showing that (u_ξ) is a root function of multiplicity m of (B_ξ). □

Proof. [Proof of Theorem 4.5]

1. Since the matrix $C^{(m)}$ in (4.5) is upper triangular with 1s on the diagonal, 1 is the only eigenvalue of $C^{(m)}$. Moreover, it is easy to see that for any $l \geq 0$ the entries of $D^{(l)} := (C^{(m)} - I)^l$ are $D_{kj}^{(l)} = 0$ if $j - k - l + 1 \geq 0$ and positive integers else. In particular, $(C^{(m)} - I)^l = 0$ if and only if $l \geq m$. Therefore, $C^{(m)}$ is similar to an $m \times m$ Jordan block with 1s on the diagonal.

2. Assume that $u^{(1)}, \dots, u^{(\alpha)}$ are linearly independent eigenvectors of (B_ξ) corresponding to the characteristic value ξ_0 with $\operatorname{rank}(u^{(l)}) = r_l$. For each $l = 1, \dots, \alpha$ Lemma 4.6 yields an r_l-dimensional $\mathcal{T}$-invariant vector space $\mathcal{V}_{\xi_0,l}$. Moreover, together with the first part of the proof, it follows that there exists a basis of $\mathcal{V}_{\xi_0,l}$ with respect to which $\mathcal{T}$ is represented by a Jordan block of size r_l corresponding to the eigenvalue λ. To show that $\mathcal{V}'_{\xi_0} := \bigoplus_{l=1}^{\alpha} \mathcal{V}_{\xi_0,l}$ is a direct sum, i.e. $\dim \mathcal{V}'_{\xi_0} = \sum_{l=1}^{\alpha} r_l$, we repeat the argument of the proof of Lemma 4.6, replacing u_0 in (4.10) by a linear combination of $u^{(1)}, \dots, u^{(\alpha)}$. To complete the proof of the implication $1 \Rightarrow 2$ and the last statement of the theorem it only remains to show that $\mathcal{V}'_{\xi_0} = \mathcal{V}_{\xi_0}$ if $\{u^{(1)}, \dots, u^{(\alpha)}\}$ is a canonical system.

3. Let $\mathcal{V}_{\xi_0,l}$, $l = 1, \dots, \alpha$ satisfy the assumptions of the second assertion. By assumption, each $\mathcal{V}_{\xi_0,l}$ has one linearly independent eigenvector $v^{(l)}$ corresponding to some eigenvalue $\lambda^{(l)}$, which we write as $\lambda^{(l)} = \exp(i\xi^{(l)})$ with $\xi^{(l)} \in \mathbb{C}$. Note that $v^{(l)}(x + 1) = \lambda v^{(l)}(x)$ implies that $u^{(l)}(x) := \exp(-i\xi^{(l)} x_1) v^{(l)}(x)$ is periodic. Therefore, $v^{(l)}$ is a Floquet mode with quasi momentum $\xi^{(l)}$. Since the quasi momentum is uniquely determined up to additive multiples of 2π and $v^{(l)} \in \mathcal{V}_{\xi_0}$, it follows that $\lambda^{(l)} = \exp(i\xi^{(l)}) = \exp(i\xi_0)$. Moreover, the fact that $\mathcal{V}_{\xi_0,l}$ cannot be decomposed into smaller invariant subspaces together with the first part of the proof implies that there exists a basis $\{v_1^{(l)}, \dots, v_{r_l}^{(l)}\}$ of $\mathcal{V}_{\xi_0,l}$ with respect to which $\mathcal{T}$ is represented by

the matrix $\lambda C^{(r_l)}$. Now Lemma 4.6, part 2 yields corresponding linearly independent eigenvectors $u^{(l)} \in \ker B_{\xi_0}$ with $\operatorname{rank}(u^{(l)}) \geq r_l$. Part 2 of the proof implies that $\operatorname{rank}(u^{(l)}) = r_l$, which completes the proof of the implication $2 \Rightarrow 1$.

4. Now suppose we start with a canonical system $u^{(1)}, \ldots, u^{(\alpha)}$ in part 2, and $\dim(\mathcal{V}_{\xi_0}) > \dim(\mathcal{V}'_{\xi_0})$. Since $\mathcal{V}_{\xi_0}$ is $\mathcal{T}$-invariant, part 3 of the proof yields a system of linearly independent eigenvectors $\tilde{u}^{(l)}$ with $\sum_{l=1}^{\alpha'} \operatorname{rank}(\tilde{u}^{(l)}) > \sum_{l=1}^{\alpha} r_l$, a contradiction. This proves the implication $1 \Rightarrow 2$. ◻

5. Estimates of rapidly decaying generalized Floquet modes. Recall that $-\Delta\varphi_l = ((2\pi l_1)^2 + \kappa_{l_2}^2)\varphi_l = |\sigma_l|^2\varphi_l$ for $l \in \mathbb{Z} \times \mathbb{N}$ if we set

$$\sigma_l := 2\pi l_1 + i\kappa_{l_2}.$$

Define $\widehat{u}(l) := \langle u, \varphi_l^{(\gamma)} \rangle$. Due to (4.1) we have

$$\begin{aligned} -\widehat{(\Delta_\xi u)}(l) &= \left((2\pi l_1 + \Re\xi)^2 - \Im\xi^2 + \kappa_{l_2}^2 + 2i\Im\xi(2\pi l_1 + \Re\xi)\right) \widehat{u}(l) \\ &= (\xi + \sigma_l)(\xi + \overline{\sigma_l})\, \widehat{u}(l). \end{aligned} \tag{5.1}$$

In particular, the characteristic values of (Δ_ξ) are precisely the numbers σ_l and $\overline{\sigma_l}$.

Define the set

$$\mathcal{S} := \bigcup_{l_2 \geq N} \left\{ z \in \mathbb{C} : |z - i\kappa_{l_2}| < \frac{\omega^2 \bar{\varepsilon}}{\kappa_{l_2}} \right\}, \qquad N > \min\left\{ l_2 \in \mathbb{N} : \kappa_{l_2} > \frac{\omega^2 \bar{\varepsilon} \max(1, L)}{\pi} \right\} \tag{5.2}$$

(see Fig. 5.1). Note that N is chosen such that all circles are contained in the strip $\{z \in \mathbb{C} : |\Im z| < \pi\}$ and each circle overlaps with at most one neighboring circle since each κ_{l_2} has a distance $< \pi/L$ to at most one other $\kappa_{l_2'}$ (see (3.2)). Moreover, let $2\delta_\gamma$ be the smallest distance of neighboring, but different κ_{l_2}, i.e. $\delta_\gamma = \pi/(2L)$ for $\gamma \in \{\gamma_{\mathrm{D}}, \gamma_{\mathrm{N}}, \gamma_{\mathrm{DN}}\}$, $\delta_{\gamma_\beta} = \min_{l \in \mathbb{N}} |l\pi - \beta|/L$ if $\beta \notin \{0, \pi\}$, and $\delta_{\gamma_\beta} = \pi/L$ for $\beta \in \{0, \pi\}$ (see Fig. 5.1).

Theorem 5.1. *Let Γ denote the contour of the boundary of a connected component $\tilde{\mathcal{S}}$ of $\mathcal{S}$. Then the following holds true:*

1. *We have*

$$N((B_\xi); \Gamma) = N((\Delta_\xi); \Gamma) = \#\{l \in \mathbb{Z} \times \mathbb{N} : \sigma_l \in \tilde{\mathcal{S}}\}, \tag{5.3}$$

and all characteristic values of (B_ξ) in $\{z \in \mathbb{C} : \Re z \in (-\pi, \pi), \Im z \geq \kappa_N\}$ are contained in $\mathcal{S}$.

2. *Define $I_{\tilde{\mathcal{S}}} := \{l \in \mathbb{Z} \times \mathbb{N} : \sigma_l \in \tilde{\mathcal{S}}\}$, $\kappa_{\tilde{\mathcal{S}}} := \inf_{\xi \in \tilde{\mathcal{S}}} \Im\xi$ and the L^2-orthogonal projection $P_{\tilde{\mathcal{S}}} u := \sum_{l \in I_{\tilde{\mathcal{S}}}} \langle u, \varphi_l \rangle \varphi_l$. Then all eigenvectors u^* corresponding to characteristic values in $\tilde{\mathcal{S}}$ satisfy the estimates*

$$\|u^* - P_{\tilde{\mathcal{S}}} u^*\|_{L^2(\Omega_0)} \leq \frac{\omega^2 \bar{\varepsilon}}{\min(\delta_\gamma, \pi)\kappa_{\tilde{\mathcal{S}}}} \|u^*\|_{L^2(\Omega_0)}, \tag{5.4a}$$

$$\|\partial_{x_1} u^*\|_{L^2(\Omega_0)} \leq \frac{2\omega^2 \bar{\varepsilon}}{\kappa_{\tilde{\mathcal{S}}}} \|u^*\|_{L^2(\Omega_0)}, \tag{5.4b}$$

$$\|(u^* - P_{\tilde{\mathcal{S}}} u^*)(0, \cdot)\|_{L^2((0,L))} \leq C \frac{\omega^2 \bar{\varepsilon}}{\kappa_{\tilde{\mathcal{S}}}} \|u^*\|_{L^2(\Omega_0)}, \tag{5.4c}$$

$$\left\| \frac{\partial u^*}{\partial x_1}(0, \cdot) \right\|_{L^2((0,L))} \leq \frac{2\omega^2 \bar{\varepsilon}}{\sqrt{3}} \|u^*\|_{L^2(\Omega_0)}. \tag{5.4d}$$

with a constant C independent of ε, ω, and $\tilde{\mathcal{S}}$.

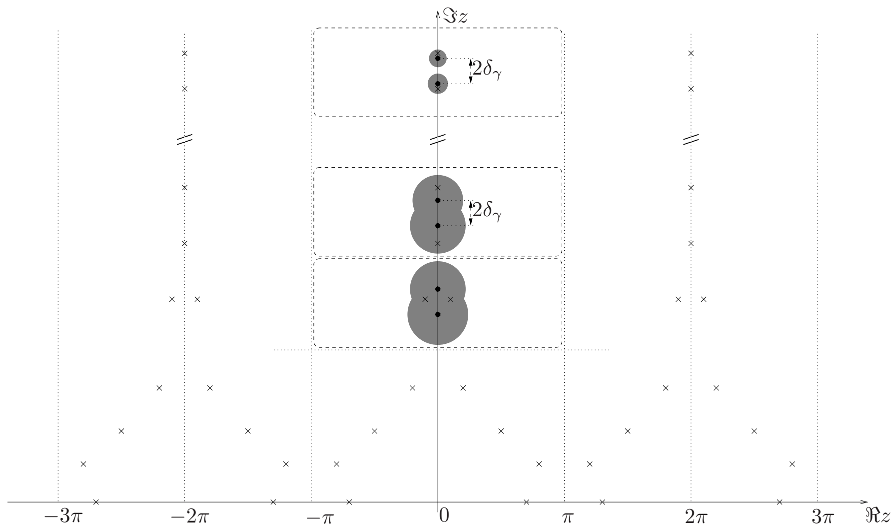

FIGURE 5.1. *The set $\mathcal{S}$ defined in (5.2) for the case of quasi-periodic boundary conditions. The dots indicate the characteristic values (σ_l) of (Δ_ξ), and the crosses indicate the characteristic values (ξ_n) of (B_ξ). The dashed lines show integration contours in the proof of Theorem 5.1, Part 1.*

Proof. (ad 1): We apply the generalized Rouché Theorem 4.4 with $A_\xi = \Delta_\xi \in \mathcal{L}(L^2(\Omega_0), H_\gamma^{-2}(\Omega_0))$. To this end we have to show that

$$\|\Delta_\xi^{-1}\omega^2\varepsilon_p\|_{\mathcal{L}(L^2(\Omega_0))} < 1, \qquad \xi \in \Gamma$$

Since $\|\omega^2\varepsilon_p\|_{\mathcal{L}(L^2(\Omega_0))} = \omega^2\bar{\varepsilon}$, it suffices to show that

$$\|\Delta_\xi^{-1}\|_{\mathcal{L}(L^2(\Omega_0))} < \frac{1}{\omega^2\bar{\varepsilon}}, \qquad \xi \in \Gamma.$$

By virtue of (5.1) this is equivalent to

$$\sup\left\{\frac{1}{|\xi+\sigma_l|\,|\xi+\overline{\sigma_l}|} : \xi \in \Gamma, l \in \mathbb{Z}\times\mathbb{N}\right\} < \frac{1}{\omega^2\bar{\varepsilon}}, \qquad \xi \in \Gamma,$$

which follows from the estimate

$$|\xi+\sigma_l|\,|\xi+\overline{\sigma_l}| > \kappa_{l_2}|\xi - i\kappa_{l_2}| \geq \omega^2\bar{\varepsilon}.$$

To prove the last statement, we extend each connected component of $\mathcal{S}$ to a rectangle such that the rectangles cover $\{z \in \mathbb{C} : \Re z \in (-\pi,\pi), \Im z > N\}$ (see Fig. 5.1) and apply Theorem 4.4 again.

(ad 2): Let $\xi \in \tilde{\mathcal{S}}$ be a characteristic value of (B_ξ) with eigenvector u^*. It follows from (5.1) that $(\xi+\sigma_l)(\xi+\overline{\sigma_l})\widehat{u^*}(l) = \omega^2\widehat{(\varepsilon_{\mathrm{p}}u^*)}(l)$, or

$$\widehat{u^*}(l) = \frac{\omega^2\widehat{(\varepsilon_{\mathrm{p}}u^*)}(l)}{(\xi+\sigma_l)(\xi+\overline{\sigma_l})}, \qquad l \in \mathbb{Z}\times\mathbb{N}. \tag{5.5}$$

This identity will be used extensively together with the following lower bounds, which hold for all $\xi \in \tilde{\mathcal{S}}$ and $l \in (\mathbb{Z}\times\mathbb{N}) \setminus I_{\tilde{\mathcal{S}}}$:

$$|\xi + \overline{\sigma_l}|^2 = (\Re\xi + 2\pi l_1)^2 + (\Im\xi - \kappa_{l_2})^2 > \begin{cases} (\pi l_1)^2, & l_2 \in I'_{\tilde{\mathcal{S}}} \\ (\pi l_1)^2 + \delta_\gamma^2, & l_2 \notin I'_{\tilde{\mathcal{S}}} \end{cases} \tag{5.6a}$$

$$|\xi + \sigma_l|^2 = (\Re\xi + 2\pi l_1)^2 + (\Im\xi + \kappa_{l_2})^2 > (\pi l_1)^2 + (\kappa_{\tilde{\mathcal{S}}} + \kappa_{l_2})^2 \tag{5.6b}$$

Here $I'_{\tilde{\mathcal{S}}} := \{l_2 \in \mathbb{N} : (0, l_2) \in I_{\tilde{\mathcal{S}}}\}$. In all estimates we use the inequality $|\Re\xi| < \pi$, in (5.6a) additionally the definition of δ_γ, and in (5.6b) $\kappa_{l_2} \geq 0$. Estimating $|(\xi + \overline{\sigma_l})(\xi+\sigma_l)| \geq \min(\delta_\gamma, \pi)\kappa_{\tilde{\mathcal{S}}}$ for $l \in \mathbb{Z}\times\mathbb{N}\setminus I_{\tilde{\mathcal{S}}}$ we obtain

$$\begin{aligned}\|u^* - P_{\tilde{\mathcal{S}}}u^*\|_{L^2}^2 &= \Big\| \sum_{l\in\mathbb{Z}\times\mathbb{N}\setminus I_{\tilde{\mathcal{S}}}} \widehat{u^*}(l)\varphi_l \Big\|_{L^2}^2 = \sum_{l\in\mathbb{Z}\times\mathbb{N}\setminus I_{\tilde{\mathcal{S}}}} \Big| \frac{\omega^2 \widehat{(\varepsilon_{\mathrm{p}} u^*)}(l)}{(\xi+\overline{\sigma_l})(\xi+\sigma_l)} \Big|^2 \\ &\leq \left(\frac{\omega^2}{\min(\delta_\gamma,\pi)\kappa_{\tilde{\mathcal{S}}}}\right)^2 \sum_{l\in\mathbb{Z}\times\mathbb{N}} |\widehat{(\varepsilon_{\mathrm{p}} u^*)}(l)|^2 \leq \left(\frac{\omega^2\bar{\varepsilon}}{\min(\delta_\gamma,\pi)\kappa_{\tilde{\mathcal{S}}}}\right)^2 \|u^*\|_{L^2}^2.\end{aligned}$$

To prove (5.4b) we use that $|\xi+\sigma_l|\,|\xi+\overline{\sigma_l}| \geq \kappa_{\tilde{\mathcal{S}}}\pi|l_1|$ and $\partial_{x_1}\varphi_{(0,l_2)} = 0$ to obtain

$$\|\partial_{x_1}u^*\|_{L^2}^2 = \Big\| \sum_{l\in(\mathbb{Z}\setminus\{0\})\times\mathbb{N}} \widehat{u^*}(l)\partial_{x_1}\varphi_l \Big\|^2 = \sum_{l\in(\mathbb{Z}\setminus\{0\})\times\mathbb{N}} \Big| \frac{(2\pi l_1)\omega^2\widehat{(\varepsilon_{\mathrm{p}}u^*)}(l)}{(\xi+\overline{\sigma_l})(\xi+\sigma_l)}\Big|^2 \leq \left(\frac{2\omega^2\bar{\varepsilon}}{\kappa_{\tilde{\mathcal{S}}}}\right)^2 \|u^*\|_{L^2}^2.$$

Since

$$u^*(0,\cdot) = \sum_{l\in\mathbb{Z}\times\mathbb{N}} \widehat{u^*}(l)\varphi_l(0,\cdot) = \sum_{l_2\in\mathbb{N}} \left(\sum_{l_1\in\mathbb{Z}} \widehat{u^*}(l_1,l_2)\right)\psi_{l_2},$$

we have

$$\|(u^* - P_{\tilde{\mathcal{S}}}u^*)(0,\cdot)\|_{L^2((0,L))}^2 = \sum_{l_2\in\mathbb{N}\setminus I'_{\tilde{\mathcal{S}}}} \Big|\sum_{l_1\in\mathbb{Z}} \widehat{u^*}(l_1,l_2)\Big|^2 + \sum_{l_2\in I'_{\tilde{\mathcal{S}}}} \Big| \sum_{l_1\in\mathbb{Z}\setminus\{0\}} \widehat{u^*}(l_1,l_2)\Big|^2. \tag{5.7}$$

Using the Cauchy-Schwarz inequality, (5.5), and the lower bounds $|\xi+\sigma_l|^2 \geq \pi^2 l_1^2 + \delta_\gamma^2$ and $|\xi+\overline{\sigma_l}|^2 \geq \kappa_{\tilde{\mathcal{S}}}^2$ (see (5.6)), the first term in (5.7) can be bounded by

$$\begin{aligned}\sum_{l_2\in\mathbb{N}\setminus I'_{\tilde{\mathcal{S}}}} \Big|\sum_{l_1\in\mathbb{Z}} \widehat{u^*}(l_1,l_2)\Big|^2 &\leq \left(\sum_{l_1\in\mathbb{Z}} \frac{1}{\pi^2 l_1^2 + \delta_\gamma^2}\right) \sum_{l_1\in\mathbb{Z}, l_2\in\mathbb{N}\setminus I'_{\tilde{\mathcal{S}}}} (\pi^2 l_1^2 + \delta_\gamma^2)|\widehat{u^*}(l_1,l_2)|^2 \\ &= C' \sum_{l_1\in\mathbb{Z}, l_2\in\mathbb{N}\setminus I'_{\tilde{\mathcal{S}}}} \frac{\omega^4|\widehat{(\varepsilon_{\mathrm{p}}u^*)}(l_1,l_2)|^2}{|\xi+\overline{\sigma_l}|^2} \leq C'\frac{(\omega^2\bar{\varepsilon})^2}{\kappa_{\tilde{\mathcal{S}}}^2}\|u^*\|_{L^2}^2\end{aligned}$$

with $C' := \sum_{l_1\in\mathbb{Z}} \frac{1}{\pi^2 l_1^2+\delta_\gamma^2}$. Using (5.4b) the second term in (5.7) can be estimated by

$$\begin{aligned}\sum_{l_2\in I'_{\tilde{\mathcal{S}}}} \Big| \sum_{l_1\in\mathbb{Z}\setminus\{0\}} \widehat{u^*}(l_1,l_2)\Big|^2 &\leq \left(\sum_{l_1\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi l_1)^2}\right)\left(\sum_{l_1\in\mathbb{Z}\setminus\{0\}, l_2\in I'_{\tilde{\mathcal{S}}}} (2\pi l_1)^2|\widehat{u^*}(l_1,l_2)|^2\right) \\ &\leq \frac{C'}{4}\|\partial_{x_1}u^*\|_{L^2}^2 \leq C'\frac{(\omega^2\bar{\varepsilon})^2}{\kappa_{\tilde{\mathcal{S}}}^2}\|u^*\|_{L^2}^2\end{aligned}$$

completing the proof of (5.4c) with $C = \sqrt{2C'}$.

To prove (5.4d), first note that

$$\partial_{x_1} u^*(0,\cdot) = \sum_{l\in\mathbb{Z}\times\mathbb{N}} \widehat{u^*}(l)\partial_{x_1}\varphi_l(0,\cdot) = \sum_{l_2\in\mathbb{N}} \left(\sum_{l_1\in\mathbb{Z}\setminus\{0\}} (2\pi i l_1)\widehat{u^*}(l_1,l_2) \right) \psi_{l_2}.$$

It follows from (5.5), the Cauchy-Schwarz inequality, the estimate $|\xi+\sigma_l|\,|\xi+\overline{\sigma_l}| \geq (\pi l_1)^2$ (see (5.6)), and the identity $\sum_{l_1\in\mathbb{N}} l_1^{-2} = \frac{\pi^2}{6}$ that

$$\begin{aligned}\|\partial_{x_1} u^*(0,\cdot)\|_{L^2}^2 &= \sum_{l_2\in\mathbb{N}} \left| \sum_{l_1\in\mathbb{Z}\setminus\{0\}} \frac{\omega^2(2\pi i l_1)\widehat{(\varepsilon_p u^*)}(l)}{(\xi+\sigma_l)(\xi+\overline{\sigma_l})} \right|^2 \\ &\leq \left(2\sum_{l_1\in\mathbb{N}} \frac{1}{l_1^2}\right) \sum_{l\in\mathbb{Z}\setminus\{0\}\times\mathbb{N}} \frac{(2\pi\omega^2 l_1^2)^2|\widehat{(\varepsilon_p u^*)}|^2}{(\pi l_1)^4} \leq \frac{4}{3}(\omega^2\bar{\varepsilon})^2\|u^*\|_{L^2}^2,\end{aligned}$$

which proves (5.4d). ◻

6. Propagating and slowly decaying Floquet modes. We first show that the set of characteristic values of (B_ξ) is symmetric both with respect to the real axis and the imaginary axis:

LEMMA 6.1. *If ξ_0 is a characteristic value of (B_ξ), then both $\overline{\xi_0}$ and $-\overline{\xi_0}$ are characteristic values of (B_ξ) with the same total null multiplicities.*

Proof. Let (u_ξ) be a root function of (B_ξ) corresponding to ξ_0. Taking the complex conjugate of $B_\xi u_\xi = 0$ yields $B_{-\overline{\xi}}\overline{u_\xi} = 0$ and shows that $-\overline{\xi_0}$ is a characteristic value with the same multiplicity.

To show that $\overline{\xi_0}$ is also a characteristic value of (B_ξ), we note that the adjoint of B_ξ with respect to the dual pairing of $H^1_\gamma(\Omega_0)$ and $H^{-1}_\gamma(\Omega_0)$ is given by $B_\xi^* = B_{\overline{\xi}}$. Since B_ξ is Fredholm with index 0, we have $\dim\ker B_\xi = \dim\ker B_\xi^*$. Therefore, $\overline{\xi_0}$ is a characteristic value of (B_ξ) with the same geometric multiplicity. To show that the total null multiplicities also coincide, choose $\epsilon > 0$ sufficiently small such that $\Gamma : [0,2\pi] \to \mathbb{C}$, $t \mapsto \xi_0 + \epsilon e^{it}$ encloses only the characteristic value ξ_0. Then it follows from Theorem 4.4 that

$$\begin{aligned} N((B_\xi);\xi_0) =& \frac{1}{2\pi i}\mathrm{tr}\oint_\Gamma \frac{\partial B_\xi}{\partial \xi} B_\xi^{-1}\,d\xi = \frac{1}{2\pi i}\overline{\mathrm{tr}\left(\oint_\Gamma \frac{\partial B_\xi}{\partial \xi}B_\xi^{-1}\,d\xi\right)^*} \\ =& -\frac{1}{2\pi i}\overline{\mathrm{tr}\oint_{\bar\Gamma} B_\xi^{-1}\frac{\partial B_\xi}{\partial \xi}\,d\xi} = \overline{N((B_\xi);\overline{\xi_0})} = N((B_\xi);\overline{\xi_0}). \end{aligned}$$

◻

Next we will determine the number of characteristic values with "small" imaginary part. To this end we have to define a contour containing no characteristic values. By Theorem 5.1, Part 1, the segment $[z_N, z_N+2\pi]$ with $z_N := -\pi + \frac{i}{2}(\kappa_N+\kappa_{N+1})$ contains no characteristic value if N satisfies the bound in (5.2) and $|\kappa_{N+1}-\kappa_N| \geq \pi/L$.

PROPOSITION 6.2. *Choose z_N as above and any complex path P from z_N to $\bar{z}_N$ such that $\Gamma := [z_N, z_N+2\pi] \cup (P+2\pi) \cup [\bar{z}_N+2\pi, \bar{z}_N] \cup P$ encloses precisely all characteristic values of (B_ξ) in the rectangle $\{z \in \mathbb{C} : \Re z \in [-\pi,\pi), |\Im z| < |\Im z_N|\}$. Then*

$$N((B_\xi);\Gamma) = 2N.$$

Proof. Let us define $B(\xi,\mu)$ by

$$B(\xi,\mu) = \Delta_\xi + \mu\varepsilon_{\mathrm{p}}.$$

The characteristic values of $(B(\xi,\mu))$ are continuous in μ when repeated with respect to their algebraic multiplicities. This is also the case if we consider these characteristic values modulo 2π because of Remark 4.2.

From Theorem 5.1, Part 1, we can deduce that the segments $[z_N; z_N + 2\pi]$ and $[\bar{z}_N + 2\pi, \bar{z}_N]$ contain no characteristic values of $(B(\xi,\mu))$ for any $\mu \in [0,\omega^2]$. This means that the characteristic values of $B(\xi,\mu)$, when considered modulo 2π, stay at a positive distance from the boundary of the domain $\big(\mathbb{R}/2\pi\big) + i[-\frac{\kappa_N+\kappa_{N+1}}{2}, \frac{\kappa_N+\kappa_{N+1}}{2}]$ for any $\mu \in [0,\omega^2]$. Since they are continuous, the number of characteristic values located inside this domain does not depend on $\mu \in [0,\omega^2]$.

This means that for any contour Γ as defined in the proposition we have

$$N\Big((B(\xi,\omega^2));\Gamma\Big) = N\Big((B(\xi,0));\Gamma\Big).$$

Since the set of characteristic values of $B(\xi,0)$ is $\{2\pi l_1 \pm i\kappa_{l_2} : (l_1,l_2) \in \mathbb{Z}\times\mathbb{N})\}$, we have $N\Big((B(\xi,0));\Gamma\Big) = 2N$. ∎

COROLLARY 6.3. *The dimension of the space $\mathcal{V}_{\mathrm{p}}$ spanned by generalized Floquet modes with real quasi momentum is finite and even.*

Proof. It follows from Lemma 6.1. that out of the $2N$ characteristic values enclosed by the contour Γ in Proposition 6.2 the same number $\overline{m}$ has positive imaginary part and negative imaginary part. Therefore, the number of characteristic values on the interval $[-\pi,\pi)$ is $2(N-\overline{m})$, which is even and finite. According to Theorem 4.5 it equals the dimension of $\mathcal{V}_{\mathrm{p}}$. ∎

If $\dim \mathcal{V}_{\mathrm{p}} = 2\overline{n}$ we have to pick a basis $v_1^+,\ldots,v_{\overline{n}}^+$ of the subspace of "physical" generalized Floquet modes, which appear in (2.1d). We expect that these modes "propagate to the right". However, the "direction of propagation" is not obvious since the signs of group and quasi momentum do not necessarily coincide. Recall that the group velocity is given by

$$\frac{d\omega}{d\xi} = \frac{d\sqrt{\lambda(\xi)}}{d\xi} = \frac{\lambda'(\xi)}{2\sqrt{\lambda(\xi)}}. \tag{6.1}$$

It has been shown (see Fliss [11]) that Floquet modes $w_{m,\xi^*}(x)e^{i\xi^* x_1}$ with positive group velocity, i.e. $\lambda_m'(\xi^*) > 0$ satisfy the limiting absorption principle. This indicates that they correspond to physical solutions.

Recall from § 3.4 that if $\xi^* \in [-\pi,\pi)$ is a characteristic value of (B_ξ), then

$$\ker B_{\xi^*} = \ker(A_\xi^{(\gamma)} - \omega^2 I) = \mathrm{span}\{w_{m,\xi^*} : \exists m \in \mathbb{N}\ \lambda_m(\xi^*) = \omega^2\}. \tag{6.2}$$

PROPOSITION 6.4.

1. *$\xi^* \in [-\pi,\pi)$ is a characteristic value of (B_ξ) with a partial null multiplicity greater than 1 if and only if there exists $m \in \mathbb{N}$ such that $\lambda_m(\xi^*) = \omega^2$ and $\lambda_m'(\xi^*) = 0$.*

2. *Let* $\Xi(\omega^2) := \{(\xi, m) \in [-\pi, \pi) \times \mathbb{N} : \lambda_m(\xi) = \omega^2\}$*, and consider the basis* $\{\exp(i\xi^* x_1) w_{m,\xi^*}(x) : (\xi^*, m) \in \Xi(\omega^2)\}$ *of the space spanned by Floquet modes with real quasi momentum of order* 0*. Then this basis contains the same number of modes with positive and negative group velocity, i.e.* $\#\{(\xi^*, m) \in \Xi(\omega^2) : \lambda'_m(\xi^*) > 0\} = \#\{(\xi^*, m) \in \Xi(\omega^2) : \lambda'_m(\xi^*) < 0\}$*.*

Proof. (ad 1): Suppose that $\omega^2 = \lambda_m(\xi^*)$ and $\lambda'_m(\xi^*) = 0$. Taking the derivative of (3.6) with respect to ξ, which will be indicated by a prime in the rest of this proof, yields

$$(\Delta_\xi + \lambda_m(\xi)\varepsilon_{\mathrm{p}})\, w'_{m,\xi} + 2i(\partial_{x_1} + i\xi) w_{m,\xi} + \lambda'_m(\xi)\varepsilon_{\mathrm{p}} w_{m,\xi} = 0. \tag{6.3}$$

Since $\lambda'_m(\xi^*) = 0$, we have

$$\frac{dB_\xi w_{m,\xi}}{d\xi}\Big|_{\xi=\xi^*} = 0 \qquad \text{and} \qquad B_{\xi^*} w_{m,\xi^*} = 0$$

Therefore, $(w_{m,\xi})$ is a root function of (B_ξ) corresponding to ξ^* with a partial null multiplicity greater than 1.

Conversely, assume that ξ^* is a characteristic value of B_ξ with a partial null multiplicity greater than 1. Then, there exists a root function (u_ξ) such that $B_{\xi^*} u_{\xi^*} = 0$ and

$$B_{\xi^*} u'_{\xi^*} + B'_{\xi^*} u_{\xi^*} = 0. \tag{6.4}$$

Let $\Xi(\omega^2, \xi^*) := \{m \in \mathbb{N} : \lambda_m(\xi^*) = \omega^2\}$. Due to (6.2) there exist coefficients $\nu_m \in \mathbb{C}$ such that $w_{\xi^*} = u_{\xi^*}$ with $w_\xi := \sum_{m\in\Xi(\omega^2,\xi^*)} \nu_m w_{m,\xi}$. Taking a linear combination of the equations (6.3) and subtracting (6.4), we get

$$B_{\xi^*}\left(w'_{\xi^*} - u'_{\xi^*}\right) = -\sum_{m\in\Xi(\omega^2,\xi^*)} \lambda'_m(\xi^*)\nu_m \varepsilon_{\mathrm{p}} w_{m,\xi^*} \tag{6.5}$$

As $B_{\xi^*} w_{m,\xi^*} = 0$ the right hand side of (6.5) belongs to $\mathrm{ran}(B_{\xi^*}) \cap \ker(B_{\xi^*})$. Since $\xi^* \in \mathbb{R}$, B_{ξ^*} is self-adjoint, and hence $\ker(B_{\xi^*}) = \mathrm{ran}(B_{\xi^*})^\perp$. Therefore,

$$\sum_{m\in\Xi(\omega^2,\xi^*)} \lambda'_m(\xi^*)\nu_m w_{m,\xi^*} = 0.$$

As the functions w_{m,ξ^*} are linearly independent, it follows that $\lambda'_m(\xi^*)\nu_m = 0$ for all $m \in \Xi(\omega^2, \xi^*)$. Since not all ν_m vanish, we obtain that $\lambda'_m(\xi^*) = 0$ for some m.

(ad 2): We divide the set of characteristic value of (B_ξ) in $[-\pi, \pi)$ into subsets of the form $\{\xi^*, -\xi^*\}$ with $\xi^* \in (0, \pi)$, $\{0\}$ and $\{-\pi\}$, which are discussed separately. Choose a basis of $\ker B_{\xi^*}$ of the form $\{w_{m_1,\xi^*}, \ldots, w_{m_l,\xi^*}\}$ as in (6.2) and pick $m \in \{m_1, \ldots, m_l\}$. Due to the analyticity of the λ-functions, there exists a unique n such that $\lambda_m(\xi) = \tilde{\lambda}_n(\xi)$ for all $\xi \in [\xi, \xi + \epsilon)$ and some $\epsilon > 0$.

If $\xi^* \in (0, \pi)$ or $\xi^* = 0$, then due to (3.5) there exists a unique m' such that $\lambda_{m'}(-\xi^*) = \tilde{\lambda}_n(-\xi^*)$ for all $\xi \in [\xi, \xi + \epsilon)$. Since $\lambda'_{m'}(-\xi^*) = -\lambda'_m(\xi^*)$, precisely one of the Floquet modes $\exp(i\xi^* x_1) w_{m,\xi^*}(x)$ and $\exp(-i\xi^* x_1) w_{m',-\xi^*}(x)$ has positive group velocity.

If $\xi = -\pi$ we choose m' as follows: Note that since $\tilde{\lambda}_n$ is periodic and satisfies (3.5), it is also symmetric with respect to $-\pi$, i.e. $\tilde{\lambda}_n(-\pi+\xi) = \tilde{\lambda}_n(-\pi-\xi)$. We choose m' such that $\lambda_{m'}(\xi) = \tilde{\lambda}_n(\xi)$ for $\xi \in (-\pi - \epsilon, -\pi]$, and again $\lambda'_{m'}(-\xi^*) = -\lambda'_m(\xi^*)$. ∎

Remark 6.5. *If no Floquet modes with vanishing group velocity occur for the frequency* ω*, then Proposition 6.4 completely describes the choice of physical Floquet* *modes $\{v_1^+, \dots, v_{\overline{n}}^+\}$ with positive group velocity, and we also get a corresponding set $\{v_1^-, \dots, v_{\overline{n}}^-\}$ of Floquet modes with negative group velocity, which together form a basis of $\mathcal{V}_{\mathrm{p}}$. Note that due to the analyticity of the λ functions this is the case for almost all ω.*

For Floquet modes with vanishing group velocity, we do not know which are the physical solutions. However, our results are independent of the answer to this question as long as the uniqueness assumption in Theorem 2.2 is satisfied.

The characteristic values with non vanishing imaginary part are divided into the $+$ and $-$ group according to the sign of their imaginary parts, i.e. the generalized Floquet modes v_n^+ with $n > \overline{n}$ are exponentially decreasing. For multiple characteristic values we choose the corresponding v_n^+ as in Theorem 4.5. Moreover, we order the characteristic values with increasing imaginary part, i.e. $0 = \Im\xi_1^+ = \cdots = \Im\xi_{\overline{n}}^+ < \Im\xi_{\overline{n}+1}^+ \le \Im\xi_{\overline{n}+2}^+ \le \cdots$.

7. Proof of the Main Theorem. To simplify our notation, we will drop the $+$-indices in the following and write $\xi_n := \xi_n^+$ and $v_n := v_n^+$. If these generalized Floquet modes are given by

$$v_n(x) = \exp(i\xi_n x_1) \sum_{j=0}^{m_n} x_1^j u_n^{(j)}(x), \tag{7.1}$$

we set $u_n := u_n^{(m_n)}$ and assume that v_n is normalized such that $\|u_n\|^2_{L^2(\Omega_0)} = (1 + \kappa_n^2)^{-1/2}$ for all $n \in \mathbb{N}$. We will study the operator defined by

$$T : l^2(\mathbb{N}) \to H^{1,+}_\gamma(S^+), \qquad (a_n) \mapsto \sum_{n=1}^\infty a_n v_n$$

and show that is a compact perturbation of an operator

$$T_0 : l^2(\mathbb{N}) \to H^{1,+}_\gamma(S^+), \qquad (a_n) \mapsto \sum_{n=1}^\infty a_n w_n,$$

which is easier to analyze. To define the functions w_n, first note from the explicit form of κ_n given in § 3 that for all boundary conditions except quasi-periodic with $\beta \in \{0, \pi\}$ there exists $\tilde{N} \in \mathbb{N}$ such that for all $n > \tilde{N}$ the connected component $\tilde{\mathcal{S}}_n$ of $\mathcal{S}$ containing $i\kappa_n$ is a disk containing precisely one characteristic value. For quasi-periodic boundary conditions with $\beta \in \{0, \pi\}$ we set $\tilde{N} := N$. In this case all $\tilde{\mathcal{S}}_n$ with $n > \tilde{N}$ contain precisely two characteristic values. Now we define w_n as follows:

$$w_n(x) := \begin{cases} (1+\kappa_n^2)^{-1/4} e^{-\kappa_n x_1} \psi_n(x_2), & 1 \le n \le \tilde{N}, \\ e^{-\kappa_n x_1} (P_{\tilde{\mathcal{S}}_n} u_n)(x), & \tilde{N} < n < \infty. \end{cases}$$

LEMMA 7.1.

1. *There exist a constant $C > 0$ depending only on $\omega^2\bar{\varepsilon}$ and γ such that for all $n > \tilde{N}$*

$$\kappa_n^{3/2} \left\| u_n - P_{\tilde{\mathcal{S}}_n} u_n \right\|_{L^2(\Omega_0)} \le C, \tag{7.2a}$$

$$\kappa_n^{3/2} \left\| (u_n - P_{\tilde{\mathcal{S}}_n} u_n)(0,\cdot) \right\|_{L^2((0,L))} \le C, \tag{7.2b}$$

$$\kappa_n^{1/2} \left\| \partial_{x_1} u_n|_{x_1=0} \right\|_{L^2((0,L))} \le C. \tag{7.2c}$$

2. T_0 is bounded and bounded from below, i.e. there exist constants $c, C > 0$ such that $c\|(a_n)\| \leq \|T_0(a_n)\|_{H^{1,+}_\gamma} \leq C\|(a_n)\|$ for all $(a_n) \in l^2(\mathbb{N})$.

3. $\ulcorner\!\gamma T_0 : l^2(\mathbb{N}) \to H^{\ulcorner\!\gamma}_\gamma$ is a Fredholm operator with index 0.

Proof. (ad 1): Since according to Proposition 6.2 precisely N characteristic values with a +-index have imaginary part $\leq \frac{1}{2}(\kappa_N + \kappa_{N+1})$, it follows from Theorem 5.1, Part 1 that ξ_n is contained in $\tilde{\mathcal{S}}_n$ for all $n > \tilde{N}$. Therefore, the estimates (7.2) follow from Part 2 of Theorem 5.1 (taking into account the scaling of u_n) if we can show that none of the characteristic values ξ_n with $n > \tilde{N}$ has a partial null multiplicities ≥ 2.

This is obvious for all γ except $\gamma = \gamma_\beta$ with $\beta \in \{0, \pi\}$ since by the definition of $\tilde{N}$, all $\tilde{\mathcal{S}}_n$ with $n > \tilde{N}$ contain precisely one characteristic value counted with its multiplicity. For $\gamma \in \{\gamma_0, \gamma_\pi\}$ we cannot exclude the possibility of infinitely many multiple characteristic values in general. Therefore, we have imposed the additional assumption that $\varepsilon_{\mathrm{p}}(x_1, x_2) = \varepsilon_{\mathrm{p}}(x_1, L - x_2)$ in this case. Then by a symmetry argument we can split the problem into two problems for wave guides of width $L/2$: For $\beta = 0$ we either impose Dirichlet conditions at $\{x : x_2 \in \{0, L/2, L\}\}$ or Neumann conditions at $\{x : x_2 \in \{0, L/2, L\}\}$, and for $\beta = \pi$ we impose Dirichlet conditions at $x_2 = 0$, $x_2 = L$ and Neumann conditions at $x_2 = L/2$, or Neumann conditions at $x_2 = 0$, $x_2 = L$ and Dirichlet conditions at $x_2 = L/2$. For each of the subproblems for waveguides of width $L/2$ we can exclude that possibility of infinitely many multiple characteristic values as above.

(ad 2): Note that $P_{\tilde{\mathcal{S}}_n} u_n$ is constant in the first variable, i.e. $(P_{\tilde{\mathcal{S}}_n} u_n)(x) = \tilde{\psi}_n(x_2)$, $x \in \Omega_0$ for some function $\tilde{\psi}_n$. For $\gamma \notin \{\gamma_0, \gamma_\pi\}$ we have $\tilde{\psi}_n = \theta_n \psi_n$ for some $\theta_n \in \mathbb{C}$, and for $\gamma \in \{\gamma_0, \gamma_\pi\}$ it follows that $\tilde{\psi}_n \in \mathrm{span}\{\psi_n, \psi_{n\pm1}\}$. Moreover,

$$\|\tilde{\psi}_n\|^2_{L^2((0,L))} = \|P_{\tilde{\mathcal{S}}_n} u_n\|^2_{L^2(\Omega_0)} \overset{(7.2\mathrm{a})}{\sim} \|u_n\|^2_{L^2(\Omega_0)} = (1 + \kappa_n^2)^{-1/2}. \tag{7.3}$$

Set $\tilde{\psi}_n := (1 + \kappa_n^2)^{-1/4} \psi_n$ for $1 \leq n \leq \tilde{N}$. It is easy to see that $\{\tilde{\psi}_n : n \in \mathbb{N}\}$ is an orthogonal system for all γ, both in $L^2((0, L))$ and in $H^1((0, L))$. This implies that $\{w_n : n \in \mathbb{N}\}$ is an orthogonal system in $H^{1,+}_\gamma(S^+)$. Hence, the assertion follows from the identity

$$\|w_n\|^2_{H^1(S^+)} = \left(\kappa_n + \frac{1}{2\kappa_n}\right) \|\tilde{\psi}_n\|^2_{L^2((0,L))}, \qquad n \in \mathbb{N},$$

where we have used the fact that since $\tilde{\psi}_n \in \mathrm{span}\{\psi_m : m \in I_{\tilde{\mathcal{S}}_n}\}$ and since $\{\kappa_m : \kappa_m \in \tilde{\mathcal{S}}_n\} = \{\kappa_n\}$, we have $\tilde{\psi}_n'' = -\kappa_n^2 \tilde{\psi}_n$.

(ad 3): Note that $(\ulcorner\!\gamma T_0)(a_n) = \sum_{n=1}^\infty (\theta_{\mathrm{D}} - \kappa_n \theta_{\mathrm{N}}) a_n \tilde{\psi}_n$. If $\theta_{\mathrm{N}} = 0$, the assertion follows from the fact that $\{\tilde{\psi}_n : n \in \mathbb{N}\}$ is a complete orthogonal system in $H^{\ulcorner\!\gamma}_\gamma = H^{1/2}_\gamma((0, L))$ and $\sup_{n\in\mathbb{N}} \|\tilde{\psi}_n\|_{H^{1/2}} / \inf_{n\in\mathbb{N}} \|\tilde{\psi}_n\|_{H^{1/2}} < \infty$ (see (7.3)). To treat the case $\theta_{\mathrm{N}} \neq 0$ note that $\{\tilde{\psi}_n : n \in \mathbb{N}\}$ is also a complete orthogonal system in $H^{\ulcorner\!\gamma}_\gamma = H^{-1/2}_\gamma((0, L))$ and that $\sup_{n\in\mathbb{N}}((1 + \kappa_n^2)^{1/2} \|\tilde{\psi}_n\|_{H^{-1/2}}) / \inf_{n\in\mathbb{N}}((1 + \kappa_n^2)^{1/2} \|\tilde{\psi}_n\|_{H^{-1/2}}) < \infty$. If none of the coefficients $\theta_{\mathrm{D}} - \kappa_n \theta_{\mathrm{N}}$, $n \in \mathbb{N}$ vanishes, this immediately implies that $\ulcorner\!\gamma T_0 : l^2(\mathbb{N}) \to H^{-1/2}_\gamma((0, L))$ is bounded and boundedly invertible. If $\theta_{\mathrm{D}} - \kappa_n \theta_{\mathrm{N}} = 0$ precisely for $n = n_0$, then the operator $(a_n) \mapsto (\ulcorner\!\gamma T_0)(a_n) + a_{n_0} \psi_{n_0}$, which is a rank-1 perturbation of $\ulcorner\!\gamma T_0$, is boundedly invertible and hence $\ulcorner\!\gamma T_0$ is Fredholm with index 0. The case that two of the coefficients

$\theta_{\mathrm{D}} - \kappa_n\theta_{\mathrm{N}}$ vanish can be treated analogously, and because of the form (3.2) of the κ_n no more than two coefficients can vanish. ∎

PROPOSITION 7.2.

1. *The operator T is well-defined and bounded, and $T - T_0$ is compact.*
2. *$T(a_n)$ satisfies* (2.1a) *and* (2.1b) *for all $(a_n) \in l^2(\mathbb{N})$.*

Proof. (ad 1): In the following C denotes a generic constant depending only on $\omega^2\bar{\varepsilon}$ and γ. Inserting the term $\pm e^{i\xi_n x_1}\tilde{\psi}(x_2)$ in $\|v_n - w_m\|_{L^2}$ and applying the triangle inequality yields the estimate

$$\begin{aligned}\|v_n - w_n\|_{L^2(S^+)} &\leq \left\|e^{i\xi_n x_1}\left(u_n - P_{\tilde{\mathcal{S}}_n} u_n\right)\right\|_{L^2(S^+)} + \left\|e^{i\xi_n x_1} - e^{-\kappa_n x_1}\right\|_{L^2} \|\tilde{\psi}_n\|_{L^2} \\ &\leq \frac{\left\|u_n - P_{\tilde{\mathcal{S}}_n} u_n\right\|_{L^2(\Omega_0)}}{1 - e^{-\Im\xi_{\tilde{N}}}} + \left(\int_0^{+\infty} \left|e^{i\xi_n x_1} - e^{-\kappa_n x_1}\right|^2\right)^{\frac{1}{2}} \|\tilde{\psi}_n\|_{L^2} \\ &\leq \frac{C}{\kappa_n^{3/2}}, \qquad n > \tilde{N}. \end{aligned} \tag{7.4}$$

In the second line we have used that $0 < \Im\xi_{\tilde{N}} \leq \Im\xi_n$ for all $n > \tilde{N}$ and $\sum_{k=0}^{\infty} e^{-k\Im\xi_n} = (1 - \Im\xi_n)^{-1} \leq (1 - \Im\xi_{\tilde{N}})^{-1}$, and in the third line (7.2a) was applied together with the identity

$$\int_0^{+\infty} \left|e^{i\xi_n x_1} - e^{-\kappa_n x_1}\right|^2 dx_1 = \frac{\xi_n - i\kappa_n}{2}\left(\frac{1}{\Im\xi_n(\bar{\xi}_n - i\kappa_n)} + \frac{1}{\kappa_n(\xi_n + i\kappa_n)}\right),$$

and Theorem 5.1, Part 1.

To obtain an identity for the L^2-distance of the gradients, we apply Green's first theorem in $(0,l) \times (0,L)$, use the identity $\Delta(v_n - w_n) = -\omega^2\varepsilon_{\mathrm{p}} v_n$, and let $l \to \infty$:

$$\begin{aligned}\|\nabla(v_n - w_n)\|^2_{L^2(S^+)} =& \omega^2 \int_{S^+} \varepsilon_{\mathrm{p}} v_n(\overline{v_n} - \overline{w_n}) dx \\ &- \int_0^L \frac{\partial(v_n - w_n)}{\partial x_1}(0, x_2)(\overline{v_n} - \overline{w_n})(0, x_2)\, dx_2\end{aligned}$$

Since $\frac{\partial(v_n - w_n)}{\partial x_1}(0, x_2) = i\xi_n u_n(0, x_2) + \kappa_n\tilde{\psi}_n + \frac{\partial u_n}{\partial x_1}(0, x_2)$, it follows after adding $\pm i\xi_n\tilde{\psi}_n$ and using (7.2b), (7.2c), and Theorem 5.1, Part 1 that $\|\frac{\partial(v_n - w_n)}{\partial x_1}(0, \cdot)\|_{L^2} \leq C/\sqrt{\kappa_n}$. Using Cauchy's inequality, (7.4), and (7.2b) yields

$$\|\nabla(v_n - w_n)\|^2_{L^2(S^+)} \leq \frac{C}{\kappa_n^2}, \qquad n > \tilde{N}. \tag{7.5}$$

Define $K_j : V \to H^{1,+}_{\gamma}(S^+)$ by $K_j(a_n) := \sum_{n=1}^{j} a_n(v_n - w_n)$. Combining (7.4) and (7.5) and using Cauchy's inequality, we deduce that

$$\|(K_{m_2} - K_{m_1})(a_n)\|^2_{H^1(S^+)} \leq C\left(\sum_{n=m_1+1}^{m_2} \frac{1}{\kappa_n^3} + \sum_{n=m_1+1}^{m_2} \frac{1}{\kappa_n^2}\right) \|(a_n)\|^2,$$

which implies together with (3.2) that (K_j) is a Cauchy sequence with respect to the operator norm. Therefore, $K = \lim_{j\to\infty} K_j$ is well defined, and since the range of the operators K_j is finite dimensional, K is compact. Moreover, $T = T_0 + K$ is well-defined and bounded.

(ad 2): Since the differential operator $\Delta + \omega^2\varepsilon_{\mathrm{p}}$ is continuous from $H^1_\gamma(S^+)$ to $H^{-1}_\gamma(S^+)$, we can interchange its application with summation to show that w satisfies (2.1a). Analogously, it follows from the continuity of the trace operators that w satisfies (2.1b). ◻

PROPOSITION 7.3. *The operator T is injective.*

Proof. Let $v := \sum_{n=1}^{\infty} a_n v_n$ for some sequence $(a_n) \in l^2(\mathbb{N})$ and assume that $v \equiv 0$. We have to show that $a_n = 0$ for all $n \in \mathbb{N}$. Let $\{\nu_1, \nu_2, \dots\} = \{\Im\xi_n : n \in \mathbb{N}\}$ with $0 \le \nu_1 < \nu_2 < \cdots$. We show by induction in $l \in \mathbb{N}$ that $a_n = 0$ for all $n \in \mathbb{N}$ satisfying $\Im\xi_n \le \nu_l$: Formally adding $\nu_0 := -1$, the induction base is trivial. For the induction step assume that the statement holds true for $l-1$. Suppose that the ξ_n satisfying $\Im\xi_n = \nu_l$ are ordered with increasing real part and let $k_1 < \cdots < k_M < k_{M+1}$ denote the indices such that

$$\xi_{k_l} = \xi_{k_l+1} = \cdots = \xi_{k_{l+1}-1}, \quad l = 1,\dots,M, \qquad \text{and} \qquad \xi_{k_l} \ne \xi_{k_m} \text{ for } l \ne m, \tag{7.6}$$

$k_1 = \min\{n : \Im\xi_n = \nu_l\}$ and $k_{M+1} - 1 = \max\{n : \Im\xi_n = \nu_l\}$. It follows from the induction hypothesis and $v \equiv 0$ that

$$\exp(\nu_l x_1) \sum_{n=k_1}^{k_{M+1}-1} a_n v_n(x) = O(e^{-(\nu_{l+1}-\nu_l-\epsilon)x_1}), \qquad x_1 \to \infty \tag{7.7}$$

for any $\epsilon > 0$. For simplicity, first assume that the order m_n of all generalized Floquet modes v_n in (7.7) is 0. Then $u_n(x) := \exp(-i\xi_n x_1)v_n(x)$ is periodic, and setting $g_l := \sum_{n=k_l}^{k_{l+1}-1} a_n u_n(x)$ for some $x \in \Omega_0$ we find that

$$\sum_{l=1}^{M} \exp(i\Re\xi_{k_l})^p g_l \to 0, \qquad p \to \infty. \tag{7.8}$$

Since the matrices $A^{(p)} \in \mathbb{C}^{M\times M}$ given by $A^{(p)}_{jl} = \exp(i\Re\xi_{k_l})^{p+j}$, $j, l = 1,\dots,M$ have a factorization $A^{(p)} = \operatorname{diag}(\exp(i\xi_1)^{-p-1},\dots,\exp(i\xi_{k_M})^{-p-1})A^{(0)}$ and $A^{(0)}$ is a Vandermonde matrix, it follows that $\|[A^{(p)}]^{-1}\|$ is independent of p. Therefore, $g_1 = \cdots = g_M = 0$ for all $x \in \Omega_0$, i.e. $\sum_{n=k_l}^{k_{l+1}-1} a_n v_n \equiv 0$ for $l = 1,\dots,M$. Since we already know from Theorem 4.5 that $v_{k_l},\dots,v_{k_{l+1}-1}$ are linearly independent, it follows that $a_{k_1} = \cdots = a_{k_{M+1}-1} = 0$.

Now recall the notation in (7.1) and assume that $\overline{m} := \max_{n=k_1,\dots,k_{M+1}-1} m_n$ is positive. Setting $\mathcal{I}_{\overline{m}} := \{n : k_1 \le n < k_{M+1} - 1 \wedge m_n = \overline{m}\}$ and multiplying (7.7) by $x_1^{-\overline{m}}$ it follows that (7.8) holds with $g_l := \exp(-i\Re\xi_l x_1)\sum_{n\in\mathcal{I}_{\overline{m}}\wedge k_l \le n < k_{l+1}} a_n u_{n,\overline{m}}(x)$ (with the convention that an empty sum is 0), and it follows as above that $g_1 = \cdots = g_M = 0$ for all $x \in \Omega_0$. Recall from (4.4) that the functions $u_{n,\overline{m}}$ with $k_l \le n < k_{l+1}$ and $n \in \mathcal{I}_{\overline{m}}$ are elements of a basis of $\ker B_{\xi_l}$ and hence linearly independent. Therefore, $a_n = 0$ for all $n \in \mathcal{I}_{\overline{m}}$. In a second step we can show analogously that a_n for all n with $m_n = \overline{m} - 1$, and so on. Finally, we obtain $a_{k_1} = \cdots = a_{k_{M+1}-1} = 0$. ◻

With these preparations the proof of the main theorem is now simple:

Proof. [Proof of Theorem 2.2] (ad 1): The operator $F := \natural_\gamma T$ is Fredholm with index 0 since $F = \natural_\gamma T_0 + \natural_\gamma (T - T_0)$, and $\natural_\gamma T_0$ is Fredholm with index 0 by Lemma 7.1, and $\natural_\gamma (T - T_0)$ is compact by Proposition 7.2. Assume that $F(a_n) = 0$ and set $v := T(a_n)$. Then v is a solution to (2.1) with $f = 0$, and hence $v = 0$ by the assumption of the theorem. Using Proposition 7.3 we conclude that $(a_n) = 0$, i.e. F is injective. Since we have already seen that F is a Fredholm operator with index

0, it follows that F has a bounded inverse. Using Proposition 7.2, it follows that $v := TF^{-1}f$ is a solution of (2.1), which depends continuously on f.

(ad 2): The fact that F is bounded and boundedly invertible is equivalent to the fact that $\{\gamma v_n : n \in \mathbb{N}\}$ is a Riesz basis of H_γ^{γ}. Since T has the bounded left inverse $F^{-1}\gamma$, $\{v_n : n \in \mathbb{N}\}$ is a Riesz basis of $\mathrm{ran}(T) = V$, and V is closed.

(ad 3): Since the functions v_n^+ are chosen as in Theorem 4.5, the matrix representing $\mathcal{T}$ consists of Jordan blocks. Because of Theorem 5.1 at most a finite number of these Jordan blocks has size > 1. ☐

Appendix A. a uniqueness result. In this appendix we prove the uniqueness assumption in Theorem 2.2 for an interesting special class of problems:

THEOREM A.1. *We make the following assumptions:*

1. ε_{p} *satisfies the symmetry condition*

$$\varepsilon_{\mathrm{p}}(1 - x_1, x_2) = \varepsilon_{\mathrm{p}}(x_1, x_2), \qquad x_1 \in (0,1), x_2 \in (0, L). \tag{A.1}$$

2. γ *is the Dirichlet trace, i.e.* $\theta_{\mathrm{N}} = 0$ *and* $\theta_{\mathrm{D}} = 1$.
3. *Floquet modes with real quasi momentum and vanishing group velocity do not exist for the given* ω *(see Remark 6.5).*
4. ε_{p} *is analytic in* $[0, \delta) \times [l_1, l_2]$ *for some* $\delta > 0$ *and* $0 \le l_1 < l_2 \le L$*, and* $v_1^{\pm}, \dots, v_{\overline{n}}^{\pm} \in C^2(S^+)$ *(see Remark A.3).*

Then the only solution to problem (2.1) *with* $f = 0$ *is* $v = 0$.

Since ε_{p} is periodic, (A.1) is equivalent to the fact that ε_{p} is invariant under the operator defined by

$$(\mathcal{S}v)(x_1, x_2) := v(-x_1, x_2), \qquad (x_1, x_2) \in S.$$

As a preparation we need the following lemma:

LEMMA A.2. *If assumptions 1 and 3 of Theorem A.1 hold true, it can be arranged by an appropriate choice of phase factors that*

$$\mathcal{S}v_n^+ = \overline{v_n^+} = v_n^- \qquad \text{for } n = 1, \dots, \overline{n}.$$

Proof. Note that $\mathcal{S}$ maps $H_\gamma^1(\Omega_0)$ into itself and that $\mathcal{S}\Delta_\xi = \Delta_{-\xi}\mathcal{S}$ for $\xi \in \mathbb{R}$. Therefore,

$$\Delta_{-\xi}\mathcal{S}\tilde{w}_{m,\xi} + \tilde{\lambda}_m(\xi)\varepsilon_{\mathrm{p}}\mathcal{S}\tilde{w}_{m,\xi} = 0, \qquad \xi \in \mathbb{R}.$$

Taking the complex conjugate shows that $(\Delta_\xi + \tilde{\lambda}_m(\xi)\varepsilon_{\mathrm{p}})\mathcal{S}\overline{\tilde{w}_{m,\xi}} = 0$. Therefore, we can replace $\tilde{w}_{m,\xi}$ by $\tilde{w}_{m,\xi} + \mathcal{S}\overline{\tilde{w}_{m,\xi}}$. (If $\tilde{w}_{m,\xi^*} + \mathcal{S}\overline{\tilde{w}_{m,\xi^*}} = 0$ we use $\tilde{w}_{m,\xi} - \mathcal{S}\overline{\tilde{w}_{m,\xi}}$ instead or equivalently multiply $\tilde{w}_{m,\xi}$ by i first.) This shows that by a proper choice of the phase factors we can arrange that

$$\mathcal{S}\overline{w_{m,\xi}} = w_{m,\xi} \qquad \text{and} \qquad \mathcal{S}w_{m,\xi} = \overline{w_{m,\xi}} = w_{m',-\xi},$$

where m' is defined as in the proof of Proposition 6.4. ☐

Proof. [Proof of Theorem A.1] Assume that $v = \sum_{n=1}^{\overline{n}} \alpha_n v_n^+ + w \in H_\gamma^{1,+}(S^+)$ be a solution to (2.1) with $f = 0$. Due to (A.1), the odd extension $v(-x_1, x_2) := v(x_1, x_2)$, $x_1 > 0$ satisfies the differential equation $\Delta v + \omega^2 \varepsilon_{\mathrm{p}} v = 0$ in all of S. Therefore, also $\Delta w + \omega^2 \varepsilon_{\mathrm{p}} w = 0$ in S, and hence $B_\alpha(\mathcal{F}w)(\cdot, \alpha) = 0$ for all α. Since (B_ξ) has at most a finite number of characteristic values on the real axis, it follows that $(\mathcal{F}w)(\cdot, \alpha) = 0$

for almost all α. As $w \in L^2(S)$, an application of the inverse Floquet transform yields $w \equiv 0$, i.e.

$$v = \sum_{n=1}^{\overline{n}} \alpha_n v_n^+ .$$

The function $\tilde{v} := v + \mathcal{S}v$ satisfies $\tilde{v}(0,\cdot) = 2f = 0$ and $\partial_{x_1}\tilde{v}(0,\cdot) = \partial_{x_1}v(0,\cdot) - \partial_{x_1}v(0,\cdot) = 0$. By Holmgren's uniqueness theorem [23, Prop. 4.3] we conclude that $\tilde{v} \equiv 0$ in $[0,\delta]\times[l_1,l_2]$. Then the unique continuation principle [3, Lemma 8.5] implies that $\tilde{v} \equiv 0$ in S^+. Since $\tilde{v} = \sum_{n=1}^{\overline{n}} \alpha_n(v_n^+ + v_n^-)$ and since $v_1^+,\dots,v_{\overline{n}}^+, v_1^-,\dots,v_{\overline{n}}^-$ are linearly independent, it follows that $\alpha_1 = \cdots = \alpha_{\overline{n}} = 0$. Therefore, $v \equiv 0$. ∎
The first part of the previous proof repeats an argument in [1, Lemma 4.3].

REMARK A.3. *The assumption $v_n^\pm \in C^2(S)$ required for the unique continuation principle rules out the interesting case of piecewise constant ε_p. However, if we assume that ε_p is piecewise constant and the interfaces are piecewise analytic instead of assuming $v_n^\pm \in C^2(S)$, the assertion of Theorem A.1 still holds true and can be shown by a repeated application of Holmgren's Theorem and the unique continuation principle.*

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