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Weber Problems With High-Speed Curves

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Abstract

The Weber problem for a given finite set of demand points $D = \{a_1, \ldots, a_M\} \subset \mathbb{R}^2$ with positive weights w_m $(m = 1, \ldots, M)$ consists of finding a facility $x \in \mathbb{R}^2$ such that $\sum_{i=1}^{M} w_i d(x, a_i)$ is minimized for some distance function d.

In this paper we extend the Weber problem in the following way: We allow traveling along given linear curves (lines, line-segments and rays) with high speed. Leaving and entering such a curve is allowed at all their points, hence a network structure is continuously integrated in the plane. This extension gives the chance to model real-world situations like highway networks or other traffic infrastructure.

The extension of the Weber problem leads to a more difficult mathematical problem, since the convexity property of the Weber problem does not hold for the extended problem. This paper presents a geometrical approach to solve the extended Weber problem and gives discretization results for polyhedral gauges.

1 Introduction

Planar location problems have been discussed and solved with various distance measures. Starting from Euclidean and rectangular distance in the early days of location theory, there are many results about norms, polyhedral norms, or gauge functions for location problems. Also non-convex distance measures as they e.g. result if barriers are present have been investigated. An overview is given e.g. in [10].

On the other hand, in many applications the location of a new facility on a network is of interest. Network location has been treated for point facilities, but also for paths or other subnetworks, a survey has been given in [7] and [8].

However, there a rather few papers combining planar and network distances in location problems. A first model has been suggested by [6] where a transportation network with fixed access points is embedded into the plane. The networks in [6] are assumed to be connected. [19] generalize this model to arbitrary networks and develop a unified model to treat Weber problems with barriers and Weber problems with embedded networks. An extension of this model is given in [13]. An other model with mixed distance is proposed in [2]. The distance between points in the plane is given by the length of a shortest path between the points, where the length is measured by network distance, if a subpath coincides with parts of the network edges, otherwise the distance is measured by a metric. Unlike previous models, an approach for an optimal placement of the given network is suggested. A similar model is presented in [4]. In [5] mixed distances are used, but the given network is fixed. A game theoretic approach is suggested, where one tries to find best pricing of the network w.r.t. the possibility of foot walks. There are many papers concerning mixed distances and Voronoi diagrams, see e.g. [3], [1], [18], [12].

In our paper we start with a planar location problem but allow some *high-speed curves* along which traveling is faster than the usual traveling in the plane. We define the resulting distance measure and then deal with median location problems under this distances. We are able to identify a finite dominating set for such locations problems if the distance measure is derived from a polyhedral gauge, and if the set of high-speed curves consists of a finite number of lines, rays or line segments. Furthermore we prove a dominance criterion for given demand points. More specific results can be obtained for special cases as we show for the case of the rectangular norm with only one high-speed line. If the given distance measure is a positive definite and symmetric, then we can show that the resulting distance is a metric.

In the paper we will use the classification scheme of location problems introduced by Hamacher and Nickel [16].Their scheme consists of five different classes:

$$Pos1/Pos2/Pos3/Pos4/Pos5.$$

$Pos1$ indicates the number of new facilities (e.g. 1 in the case of a single-facility problem), $Pos2$ gives the type of location problem (e.g. $\mathbb{R}^2$ for a planar problem), $Pos3$ contains special assumptions (e.g. forbidden regions or equal weights or a $\bullet$ if no special assumptions are to be made), $Pos4$ declares the distance function in planar case (e.g. l_1 or l_∞) and $Pos5$ gives the objective function (e.g. $\sum$ for Median problems and max for Center problems). For instance, the well known Weber problem without any further assumption with Euclidean distances will be classified as $1/\mathbb{R}^2/\bullet/l_2/\sum$.

The remainder of the paper is organized as follows: In Section 2 we formally define the *high-speed distance* and introduce the notation needed. Section 3 investigates properties of shortest paths in a polyhedral gauge distance with high-speed curves. These properties will be used in Section 4 where a finite dominating set will be constructed. Section 5 deals with the case in which the high-speed distance is derived from a arbitrary distance function. In Section 6 the rectangular metric and one straight line is investigated. The paper is concluded by a summary and possible lines of further research.

2 Weber problems with polyhedral gauges and high-speed curves

Let $H = \{h_1, \ldots, h_K\}$ be a set of linear high-speed curves (lines, line-segments and rays) in the plane with

$$|\{h_i \cap h_j\}| \ \leq \ 1 \ \ \forall i \neq j. \tag{1}$$

For every $h_i \in H$ let a real number $\lambda(h_i) \in \,]0,1[$ be given, specifying the time advantage arising from traveling along curve h_i. This means traveling along curve h_i is $\lambda(h_i)^{-1}$ times faster than traveling the same distance in the plane. We will use the notation $\Lambda = \{\lambda(h_1), \ldots, \lambda(h_K)\}$ and call $\lambda(h_i)$ *speed factor* of high-speed curve h_i. The idea will be made precise by the following explanation of distances with respect to these high-speed curves:

Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be any distance function. Let $P = [z_1, z_2, \ldots, z_L]$ be a finite sequence of points in the plane. We connect the points $z_1, z_2, \ldots, z_{L-1}, z_L$ by line segments. In this way we obtain a piecewise linear path with *intermediate points* $z_1, z_2, \ldots, z_L$. We call such a path P a *finite linear path* from x to y if $z_1 = x$ and $z_L = y$ holds. The length of such a finite linear path is defined by the sum of lengths of all line segments belonging to the path. The length of a line segment $[z_i, z_{i+1}]$ is measured by d and is equal to $d(z_i, z_{i+1})$ if the line segment is not contained in a curve $h \in H$. Otherwise, the length is $\lambda(h)d(z_i, z_{i+1})$. Finally, the *high-speed distance* $d_H(x, y)$ between two points $x, y \in \mathbb{R}^2$ is defined as the length of a shortest finite linear path (with respect to the distance function d and the pair (H, Λ)) from x to y. We give an exact definition of this idea.

Definition 1. Let $\mathcal{P}_{xy}$ be the set of all finite linear paths from x to y and let d be a distance function Then d_H is given by

$$d_H(x, y) := \min_{P=[z_1,\ldots,z_L]\in\mathcal{P}_{xy}} \left\{ \sum_{i=1}^{L-1} c(z_i, z_{i+1}) \right\} \tag{2}$$

with

$$c(z_i, z_{i+1}) \ := \ \begin{cases} \lambda(h)d(z_i, z_{i+1}) & [z_i, z_{i+1}] \in h \text{ for a } h \in H \\ d(z_i, z_{i+1}) & \text{otherwise} \end{cases} \tag{3}$$

$c(z_i, z_j)$ is well defined, since $|\{h_i \cap h_j\}| \leq 1 \forall i \neq j$ holds. Note that a path between two points x and y can enter a high-speed curve h at any point $z \in h$. The first point z on such a high-speed curve will be called *access point*, the last point on the curve will be called *exit point*. Note that each point $x \in h$ is a potential access or exit point.

In the paper we will use the following notation for specifying straight lines.

- A line segment between $y, z \in \mathbb{R}^2$ is denoted as

$$[y, z] := \{x \in \mathbb{R}^2 : x = y + \alpha(z - y) \text{ for some } \alpha \in [0, 1]\}.$$

- Given a line segment h, the straight line passing through h is denoted by $l(h)$.
- A ray starting at $z \in \mathbb{R}^2$ in direction $d \in \mathbb{R}^2$ is denoted as

$$r^{z,d} := \{x \in \mathbb{R}^2 : x = z + \alpha d \text{ for some } \alpha \in \mathbb{R}_0^+\}.$$

Furthermore we use the following notation for specifying certain points of linear curves.

- Given a linear curve h, its extreme points (i.e. the end points of h) are denoted by $\mathcal{E}(h)$.
- Given a set of linear curves H, the union of all extreme points of curves in H is denoted by $\mathcal{E}(H)$, i.e.

$$\mathcal{E}(H) := \bigcup_{h \in H} \mathcal{E}(h).$$

Using the high-speed distance d_H and denoting the set of demand points by $D = \{a_1, \ldots, a_M\}$ and the set of positive weights by $W = \{w_1, \ldots, w_M\}$ (for each demand point a_m a weight w_m is associated representing its demand) the Weber problem with high-speed curves can be formulated as $1/\mathbb{R}^2/(H, \Lambda)/d_H/\sum$. This means we search a new facility $x \in \mathbb{R}^2$ such that

$$f_H(x) \ := \ \sum_{m=1}^{M} w_m d_H(a_m, x)$$

is minimized.

If the distance function is non-symmetric, then d_H is also non-symmetric. I.e. $d_H(x, y)$ may differ from $d_H(y, x)$ for some points $x, y \in \mathbb{R}^2$. Therefore it is important to fix the travel direction. In this paper we will always travel from the given demand points D towards a new facility. We indicate this

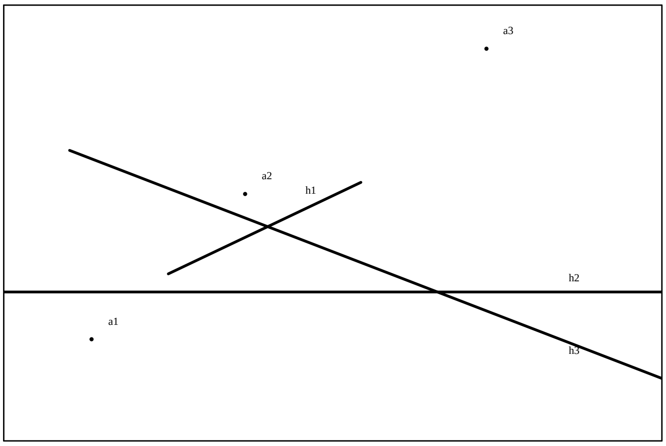


Figure 1: Problem of $1/\mathbb{R}^2/(H,\Lambda)/d_H/\sum$ with $D = \{a_1, a_2, a_3\}$ and $H = \{h_1, h_2, h_3\}$.

by using the notation $d_H(x, y)$ if the distance from x to y is required and $d_H(y, x)$ if one asks for the distance from y to x.

An example for a problem $1/\mathbb{R}^2/(H,\Lambda)/d_H/\sum$ is given in Figure 1.

Due to the high-speed curves, the distance d_H is not convex even if d is convex and therefore f_H is also not convex. Hence most of the standard methods for location problems cannot be applied, since this methods are mostly developed for convex problems. In this paper we overcome this difficulty not by non-convex optimization but use a geometric approach which enables us to develop a discretization result for the problem type $1/\mathbb{R}^2/(H,\Lambda)/\gamma_H/\sum$, where γ_H is a distance function derived from a polyhedral gauge γ.

A polyhedral gauge is given by a convex polyhedron $\mathcal{B}$ in the plane $\mathbb{R}^2$ containing the origin $0 = (0, 0)$ in its interior. With $d_1, \ldots, d_\delta$ we denote the extreme points of $\mathcal{B}$ and call them *fundamental directions* (see Figure 2). Furthermore, we define $d_{\delta+1} := d_1$ It is known [15] that for all x in the cone $\mathcal{C}(d_i, d_{i+1})$, $i = 1, \ldots, \delta$, spanned by d_i and d_{i+1} only the two fundamental directions d_i and d_{i+1} are needed to determine $\|x\|$ (see also Lemma 6). We interpret $\|x\|$ as the distance $\gamma(0, x)$ between 0 and x and extend this definition to define the *gauge distance*

$$\gamma(x, y) \quad := \quad \gamma(0, y - x)$$

between any two points $x, y \in \mathbb{R}^2$*.* From the above it follows, that the gauge distance $\gamma(x, y)$ can be represented by an arbitrary (d_i, d_{i+1})-staircase path from x to y using only the two fundamental directions d_i and d_{i+1}. In this paper we consider polyhedral gauges. To simplify our notation we use the denotation *high-speed gauge distance* for the distance function γ_H obtained

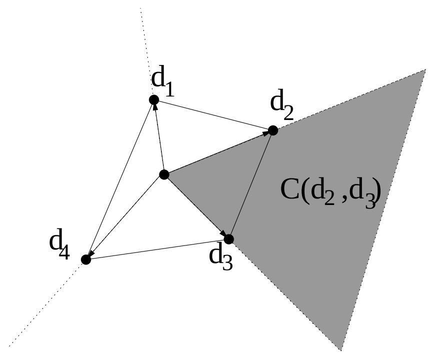


Figure 2: A polyhedral gauge with four fundamental directions.

by a set of high-speed curves H, a set of associated speed-factors Λ and a *polyhedral* gauge γ (see Definition 1).

3 Shortest paths in high-speed gauge distance

Any point of a high-speed curve $h \in H$ can be used as access or exit point to it. Hence, there may exist an uncountable number of realizations of a linear path between two points. Therefore it is not obviously, how to compute the high-speed gauge distance between two points. In this section we investigate properties of shortest linear paths in high-speed gauge distances γ_H in order to solve this problem. Without going into details, the main results of this section are the following: Given a point x, a high-speed curve h, a second point $y \in h$ and a shortest path form x to y with three intermediate points, we determine in Theorem 1 a finite set of intermediate points such that a shortest path from x to y can be constructed with this intermediate points. Furthermore we determine in Theorem 2 an analogous set for the case of two high-speed curves h_1, h_2, two points $x \in h_1$, $y \in h_2$ and a shortest path form x to y with four intermediate points. Finally we use this results to determine a finite set of intermediate points such that a shortest path between any two points can be constructed using only points form this set, see Theorem 3 and Lemma 4.

We use the notation

$$\gamma_{\mathcal{P}}(P) \quad = \quad \sum_{i=1}^{L-1} c(z_i, z_{i+1}) \tag{4}$$

with

$$c(z_i, z_{i+1}) \quad = \quad \begin{cases} \lambda(h)\gamma(z_i, z_{i+1}) & [z_i, z_{i+1}] \in h \text{ for a } h \in H \\ \gamma(z_i, z_{i+1}) & \text{otherwise} \end{cases} \quad , \tag{5}$$

$i = 1, \ldots, L-1$, for the length of a linear path $P = [z_1, \ldots, z_L]$ in the high-speed gauge distance.

In the following we need intersections between lines. It is possible that the intersection of two lines is not finite. To avoid intersections that consist of more than one point, we introduce the following definition:

Definition 2. Denote the power set of $\mathbb{R}^2$ by $P(\mathbb{R}^2)$. Then we define

$$\rho : P(\mathbb{R}^2) \to P(\mathbb{R}^2), \quad A \mapsto \begin{cases} A & if\ |A| < \infty \\ \emptyset & if\ |A| = \infty \end{cases} .$$

Now we can use ρ to eliminate infinite intersections.

Definition 3. For $h \in H$ and $z \in \mathbb{R}^2$ let $K^+_{h,z}$ be the set of intersection points between h and the rays r^{z,d_i} with origin z and fundamental directions d_i, $i = 1, \ldots, \delta$, of the gauge γ as direction vectors.

$$K^+_{h,z} : \;=\; \bigcup_{i=1}^{\delta} \rho\left(l^{z,d_i} \cap h\right).$$

Analogously, let $K^-_{h,z}$ be the set of intersection points between h and the rays $r^{z,-d_i}$ with origin z and inverse fundamental directions $-d_i$, $i = 1, \ldots, \delta$, of the gauge γ as direction vectors.

$$K^-_{h,z} : \;=\; \bigcup_{i=1}^{\delta} \rho\left(l^{z,-d_i} \cap h\right).$$

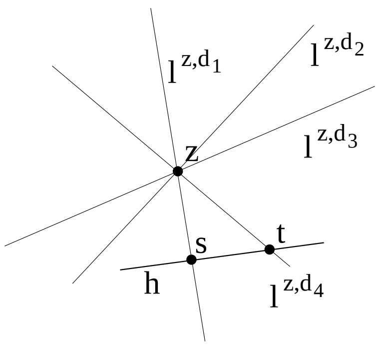
Let $K_{h,z} := K^+_{h,z} \cup K^-_{h,z}$ be the union of $K^+_{h,z}$ and $K^-_{h,z}$.

Note that $K^+_{h,z}$ and $K^-_{h,z}$ are subsets of the high-speed curve $h \in H$. If the gauge γ is symmetric, then $K^+_{h,z}$ and $K^-_{h,z}$ coincide. See Figure 3 for some examples of $K_{h,z}$.

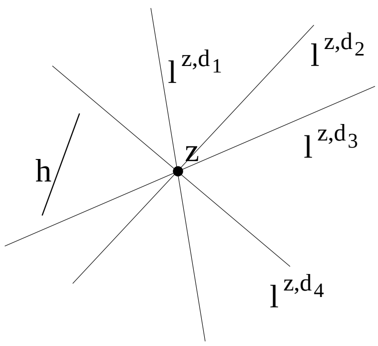
We formulate the following theorem on shortest paths with three intermediate points using exactly one high-speed curve.

Theorem 1. *Let z_1 be a point, $h \in H$ a high-speed curve, and let z_2 and z_3 be two points on h.*

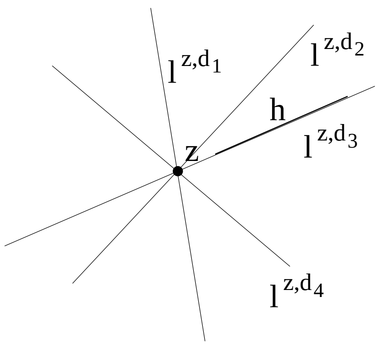
- *If $Q = [z_1, z_2, z_3]$ is a shortest path from z_1 to z_3, then there exists $s \in K^+_{h,z_1} \cup \mathcal{E}(h) \cup \{z_3\}$ such that $Q' = [z_1, s, z_3]$ is a shortest path from z_1 to z_3.*



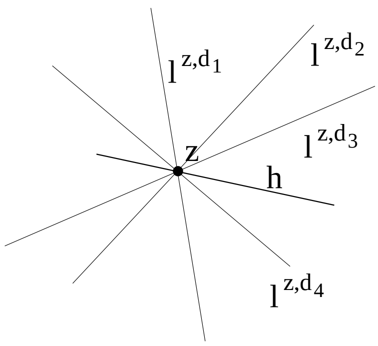
(a) $K_{h,z} = \{s, t\}$



(b) $K_{h,z} = \emptyset$



(c) $K_{h,z} = \emptyset$



(d) $K_{h,z} = \{z\}$

Figure 3: $K_{h,z}$ for different high-speed curves h and a symmetric gauge γ.

- *If* $P = [z_3, z_2, z_1]$ *is a shortest path from* z_3 *to* z_1*, then there exists* $t \in K^-_{h,z_1} \cup \mathcal{E}(h) \cup \{z_3\}$ *such that* $P^{'} = [z_3, t, z_1]$ *is a shortest path from* z_3 *to* z_1*.*

Proof. Observe that the function

$$w : h \to \mathbb{R}, \quad x \mapsto \gamma(z_1, x) + \lambda(h)\gamma(x, z_3)$$

is convex and piecewise linear with breakpoints in $K^+_{h,z_1} \cup \{z_3\}$. For any $x \in h$ the length of the path $Q(x) := [z_1, x, z_3]$ coincide with $w(x)$. Furthermore z_2 must be a minimizer of w, otherwise we obtain a contradiction to the assumption that Q is a shortest path from z_1 to z_3. Since a convex and piecewise linear function reaches its minimum in a breakpoint or in the boundary of its domain, we obtain the first part of the claim.

The function

$$v : h \to \mathbb{R}, \quad x \mapsto \lambda(h)\gamma(z_3, x) + \gamma(x, z_1)$$

is convex and piecewise linear with breakpoints in $K^-_{h,z_1} \cup \{z_3\}$. Hence, we obtain the second part of the claim analogously to the first part. □

Now we consider paths with four intermediate points and two high-speed curves.

Theorem 2. *Let* $h_1, h_2 \in H$ *be two different high-speed curves. Let* $z_1, z_2 \in h_1$ *and* $z_3, z_4 \in h_2$ *such that* $z_1 \neq z_2$ *and* $z_3 \neq z_4$*. If* $Q = [z_1, z_2, z_3, z_4]$ *is a shortest path from* z_1 *to* z_4*, then there exists a shortest path* $Q^{'} = [z_1, s, t, z_4]$ *with* $s \in h_1$ *and* $t \in h_2$ *such that at least one of the following four conditions is satisfied:*

(1) s *is an extreme point of* h_1*.*

(2) t *is an extreme point of* h_2*.*

(3) $s = t$ *is the intersection point of* h_1 *and* h_2*.*

(4) $s = z_1$ *or* $t = z_4$*, i.e. high-speed curve* h_1 *respectively* h_2 *is not used.*

Furthermore, if s *is an extreme point of* h_1*, then either* t *is also an extreme point of* h_2 *or* $t \in K^+_{h_2,s} \cup \{z_4\}$ *holds. If* t *is an extreme point of* h_2*, then either* s *is also an extreme point of* h_1 *or* $s \in K^-_{h_1,t} \cup \{z_1\}$ *holds.*

Proof. Let $l(h_k)$ be the straight line passing through h_k, $k = 1, 2$. In the case where $l(h_1)$ and $l(h_2)$ are equal, it is obvious that a path exists that fulfills conditions (1) and (2).

Let us assume that $l(h_1)$ and $l(h_2)$ are not equal. Without loss of generality we can also assume that z_2 and z_3 are not equal, otherwise they coincide with the intersection point of the high-speed curves h_1 and h_2 and we have nothing to show. Now, for any point $x \in l(h_1)$ let $m(x)$ be the intersection between the straight line

$$l^{x,z_2-z_3} := \{u \in \mathbb{R}^2 \ : \ u = x + \alpha(z_2 - z_3) \text{ for some } \alpha \in \mathbb{R}\}$$

and the straight line $l(h_2)$. Furthermore, for any $y \in l(h_2)$ let $m^{-1}(y)$ be the intersection between the straight line $l^{y,-z_2+z_3}$ and the straight line $l(h_1)$. Since $l(h_1)$ and $l(h_2)$ are not equal, m and m^{-1} are well-defined. Note that $m : l(h_1) \to l(h_2)$ is a bijective function with inverse function m^{-1}. We study the function

$$w : l(h_1) \to \mathbb{R}, \ x \mapsto \lambda(h_1)\gamma(z_1, x) + \gamma(x, m(x)) + \lambda(h_2)\gamma(m(x), z_4).$$

Due to our assumptions, the length $\gamma_{\mathcal{P}}(Q)$ of the path Q, i.e. the high-speed distance from z_1 to z_4, is equal to $w(z_2)$. Let $\mathcal{I}(h_1, h_2)$ denote the intersection between $l(h_1)$ and $l(h_2)$. If $l(h_1)$ and $l(h_2)$ have no intersection, then $\mathcal{I}(h_1, h_2)$ is empty. Observe that the function w is piecewise linear and its breakpoints are included in the set

$$B \ := \ \mathcal{E}(h_1) \cup m^{-1}(\mathcal{E}(h_2)) \cup \mathcal{I}(h_1, h_2) \cup \{z_1, m^{-1}(z_4)\} \ \subset \ l(h_1).$$

Denote the elements of B by b_i, $i = 1, \ldots, |B|$, and assume that $b_i \leq_{lex} b_{i+1}$, where $\leq_{lex}$ denotes the lexicographical order. Let $a = \max\{b_i \in B \ : \ b_i \leq_{lex} z_2\}$ and $b = \min\{b_i \in B \ : \ b_i \geq_{lex} z_2\}$. (Note that $[a, b]$ may consist of a single point.) We have $B \ \cap \ ]a, b[= \emptyset$. Therefore w is linear on $[a, b]$. Since $z_2 \in h_1$, we have $[a, b] \subseteq h_1$ and $m([a, b]) \subseteq h_2$. Hence, the length $\gamma_{\mathcal{P}}(Q(x))$ of the path $Q(x) := [z_1, x, m(x), z_4]$ coincides with $w(x)$ on $[a, b]$. A minimizer of the linear function w restricted to $[a, b]$ is included in $\{a, b\}$. Therefore we conclude that for at least one $k \in \{a, b\}$

$$\gamma_{\mathcal{H}}(z_1, z_4) \ \leq \ \gamma_{\mathcal{P}}(Q(k)) \ = \ w(k) \ \leq \ w(z_2) \ = \ \gamma_{\mathcal{H}}(z_1, z_4)$$

holds, i.e. the path $Q(k) = [z_1, k, m(k), z_4]$ is a shortest path from z_1 to z_4 that fulfills at least one of the four conditions.

In order to prove the second part of the claim, assume that $P = [z_1, u, v, z_4]$ is a shortest path from z_1 to z_4 with $u \in \mathcal{E}(h_1)$ and $u \neq v$. Now, applying Theorem 1 on the path $P' := [u, v, z_4]$ yields some $t \in K^+_{h_2,u} \cup \mathcal{E}(h) \cup \{z_4\}$. The case where $v \in \mathcal{E}(h_2)$ can be treated almost analogously. □

As a consequence of Theorem 2 it follows that is not necessary to shorten a trip on a high-speed line in order to minimize the overall distance, see Figure 4.

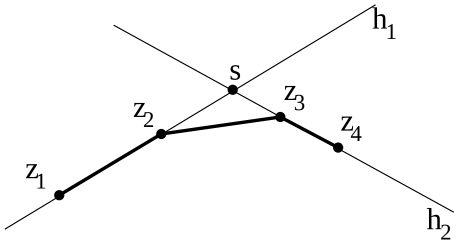


Figure 4: If $P = [z_1, z_2, z_3, z_4]$ is a shortest path from z_1 to z_4, then also the path $[z_1, s, z_4]$ is a a shortest path from z_1 to z_4.

In order to obtain a general result on shortest paths with more than four intermediate points, we need the following definitions. First we introduce a *standard representation* of a path:

Definition 4. The *standard representation* of path $P = [z_1, \ldots, z_L]$ from z_1 to z_L is a path $Q = [w_1, \ldots, w_R]$ such that

- $\gamma_{\mathcal{P}}(P) = \gamma_{\mathcal{P}}(Q)$,
- $w_1 = z_1$ and $z_L = w_R$,
- $[w_{2i}, w_{2i+1}] \subset h$ for some $h \in H$ for all $i = 1, \ldots, \frac{R}{2} - 2$,
- $|[w_{2i-1}, w_{2i}] \cap h| \leq 1$ for all $h \in H$ and $i = 1, \ldots, \frac{R}{2}$, and
- if $[w_{2i}, w_{2i+1}] \subset h_1$ for some $h_1 \in H$ and $[w_{2i+2}, w_{2i+3}] \subset h_2$ for some $h_2 \in H$ then $h_1 \neq h_2$, $i = 1, \ldots, \frac{R}{2} - 4$.

If $P = [z_1, \ldots, z_L]$ is a path in standard representation then all intermediate points but the last with even indices are access points to a high-speed curve and all intermediate points but the first with odd indices are exit points from a high-speed curve. Furthermore we have $z_{2i} \neq z_{2i+1}$ for all $i = 2, \ldots, L/2 - 2$. In particular, the number of intermediate points L of P is even. Note that we can transform any path into standard representation by deleting unnecessary intermediate points or by adding additional points. An example of a linear path and its standard representation is given in Figure 5.

Definition 5. For two linear curves $h_i, h_j \in H$, $i \neq j$ we denote their unique (see (1)) intersection point by $\mathcal{I}(h_i, h_j)$ (if it exists) and merge all these points in the set $\mathcal{I}(H)$, i.e. $\mathcal{I}(H) := \bigcup_{h_i, h_j \in H \,:\, i \neq j} \mathcal{I}(h_i, h_j)$. Then, we define

$$K \;:=\; \mathcal{I}(H) \,\cup\, \bigcup_{i,j} \bigcup_{t \in \mathcal{E}(h_i)} K_{h_j,t}.$$

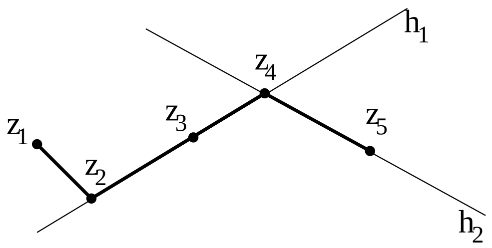


Figure 5: The standard representation of the path $P := [z_1, \ldots, z_5]$ is given by the path $Q := [w_1, \ldots, w_6]$ with $w_1 := z_1$, $w_2 := z_2$, $w_3 = z_4$, $w_4 = z_4$, $w_5 = z_5$, and $w_6 = z_5$, i.e. we have $Q = [z_1, z_2, z_4, z_4, z_5, z_5]$.

Furthermore, for any $z \in \mathbb{R}^2$ we define

$$K_z^+ := \bigcup_{h \in H} K_{h,x}^+, \text{ and } K_z^- := \bigcup_{h \in H} K_{h,x}^-.$$

Since $K_{h,x} \subseteq h$ for every $x \in \mathbb{R}^2$, we have $K \subseteq H$ and $K_z \subseteq H$ for any $z \in \mathbb{R}^2$. Furthermore, K and $K_z := K_z^+ \cup K_z^-$ are finite. An example of K and its construction is depicted in Figure 6.

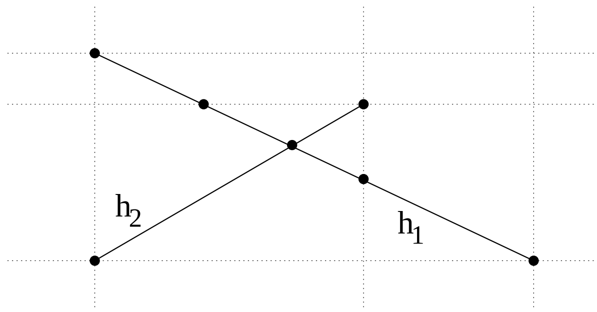


Figure 6: In the case where the set of high-speed curves is given by two line segments h_1 , h_2 and using the rectangular metric, the set K is the union of (1) the intersection between the high-speed curves h_1 and h_2, and (2) intersections between high-speed curves and lines in fundamental directions of the rectangular metric through the extreme points of the high-speed curves h_1 and h_2.

To see why we need K let $x, y \in \mathbb{R}^2$ be any points. In Theorem 3 below we will prove that there always exists a shortest path P from x to y such that all but the two first and the two last intermediate points of the path are contained in the set K. Obviously, the first and the last intermediate point of a path from x to y are x and y. In Lemma 4 we will show, that the two first and two last intermediate points of the path P are included in the set K_x^+ respectively K_y^- . Hence, the union $K \cup K_x^+ \cup K_y^- \cup \{x, y\}$ is a set similar to a finite dominating set in the sense that all intermediate points of the shortest path P from x to y are included in this set. In Corollary 5 we will show how to use this result in order to compute the length of P.

Theorem 3. *Let $z_1, z_L \in \mathbb{R}^2$ be points in the plane. Then either there exists a shortest path $P = [z_1, \ldots, z_L]$ from z_1 to z_L such that all but the two first and two last intermediate points of P are contained in the set K, i.e. $z_i \in K \; \forall\, i = 3, \ldots, L-2$, or the path $[z_1, z_L]$ is a shortest path from z_1 to z_L.*

Proof. Let $Q = [w_1, \ldots, w_R]$ be a standard representation of P. Obviously, any subpath $[w_i, \ldots, w_{i+k}]$ of Q is a shortest path from w_i to w_{i+k}. Now, let $i \in \{2, \ldots, R-4\}$ be even. Then the subpath $[w_i, w_i, w_{i+1}, w_{i+2}, w_{i+3}, w_{i+3}]$ is a shortest path from w_i to w_{i+3} in standard representation. Therefore we have $w_i \neq w_{i+1}$ and $w_{i+2} \neq w_{i+3}$. Furthermore, the line segments $[w_i, w_{i+1}]$ and $[w_{i+2}, w_{i+3}]$ are contained in different high-speed curves. Hence, we can apply Theorem 2 on $w_i, \ldots, w_{i+3}$ and the suitable high-speed curves. We obtain that there exists a shortest path $[w_i, s, t, w_{i+3}]$ with either $s, t \in K$, or with $s = w_i$, or $t = w_{i+3}$, respectively. In the latter case we remove the intermediate points w_i and w_{i+1} (if $s = w_i$) or w_{i+2} and w_{i+3} (if $t = w_{i+3}$) from Q. If $s = w_i$ and $t = w_{i+3}$ we remove the intermediate points w_i, w_{i+1}, w_{i+2} and w_{i+3}. We denote the new path with $R-2$ (or $R-4$) intermediate points again by Q. In the former case we replace the intermediate points w_{i+1}, w_{i+2} by s and t. Again, we denote this new path with R intermediate points and $w_{i+1}, w_{i+2} \in K$ by Q. Since in both cases the new path Q is still a shortest path in standard representation, we can iterate this procedure. After at most $R/2$ iterations Q is a path with the required properties, or we have deleted all but the first and the last intermediate point from Q, i.e. $[z_1, z_L]$ is a shortest path from z_1 to z_L. □

We are now able to construct a finite set such that a shortest path between two points can be constructed using only intermediate points of this set:

Lemma 4. *Let z_1, z_L be any points in the plane and let V_{z_1,z_L} be the set*

$$V_{z_1,z_L} \;:=\; K \cup K^+_{z_1} \cup K^-_{z_L} \cup \{z_1, z_L\}.$$

Then there exists a a shortest path $P = [z_1, \ldots, z_L]$ from z_1 to z_L such that all intermediate points of P are contained in the set V_{z_1,z_L}.

Proof. Due to Theorem 2, we can assume that $Q = [w_1, \ldots, w_R]$ is a shortest path in standard representation from z_1 to z_L with $w_i \in K \; \forall i = 3, \ldots, R-3$. Furthermore, we have $w_1 = z_1$ and $w_R = z_L$. Since Q is in standard representation, the intermediate points w_2 and w_3 lie on the same high-speed curve. Therfore we can apply Theorem 1 on the path $[w_1, w_2, w_3]$ and obtain that there exists a shortest path $[w_1, t, w_3]$ with $t \in V_{z_1,z_L}$. Analogously, we can apply Theorem 1 on the path $[w_{R-2}, w_{R-1}, w_R]$. $\square$

An example of $V_{x,y}$ is depicted in Figure 7.

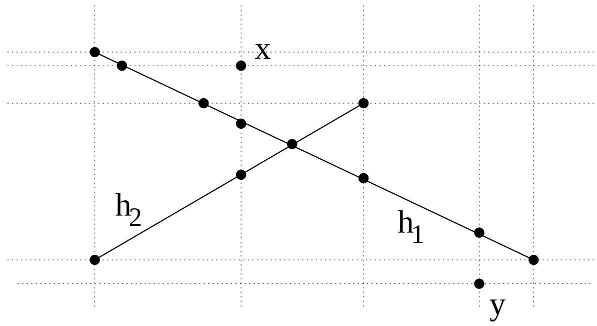


Figure 7: In the case where the set of high-speed curves is given by two line segments h_1 and h_2 and using the rectangular metric, the set $V_{x,y}$ is the union of the set $\{x, y\}$, the set K, see Figure 6, and the set of intersections between h_1, h_2 and lines in fundamental directions of the rectangular metric through the points of x and y.

In order to compute the length of a shortest path in high-speed gauge distance of any pair of points $x, y \in \mathbb{R}^2$ we take the points contained in the set $V_{x,y}$ as vertices of a directed graph $G_{x,y}$. We construct the edges of $G_{x,y}$ in the following way: We add an edge (u, v) with weight $\lambda(h)\gamma(u, v)$ for any $h \in H$, $u \in K^+_{h,x}$, and $v \in K^-_{h,y}$. We add an edge (x, v) with weight $\gamma(x, v)$ or (u, y) with weight $\gamma(u, y)$, respectively, for any $v \in K^+_x \cup \mathcal{E}(H)$ and any $u \in K^-_y \cup \mathcal{E}(H)$. Furthermore, for any pair $h_1, h_2 \in H$ with non-empty intersection $z := h_1 \cap h_2$ we add an edge (u, z) with weight $\lambda(h_1)\gamma(u, z)$ for any $u \in K^+_{h_1,x} \cup \mathcal{E}(h_1)$, and we add an edge (z, v) with weight $\lambda(h_1)\gamma(z, v)$ for any $v \in K^-_{h_1,y} \cup \mathcal{E}(h_1)$. At last, for any pair $h_1, h_2 \in H$ we add an edge (u, v) with weight $\gamma(u, v)$ for any $u \in \mathcal{E}(h_1)$ and any $v \in \mathcal{E}(h_2) \cup K^+_{h_1,u}$. An example of the graph $G_{x,y}$ is given in Figure 8.

Corollary 5. *Let x, y be points in the plane. Then the high-speed gauge distance from x to y is given by the length of a shortest path from x to y in the graph $G_{x,y}$.*

Proof. Due to Lemma 4, there exists a shortest path from x to y using only intermediate points included in $V_{x,y}$. Let $P = [z_1, \ldots, z_L]$ be a path with $z_i \in V_{x,y}$ $\forall i = 1, \ldots, L$. Since the length of P is given by $\sum_i c(z_i, z_{i+1})$, see Definition 1, the length of the path P in G coincides with the length of P in the plane. $\square$

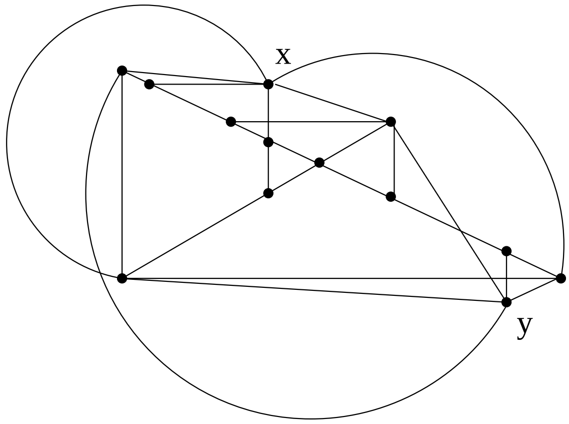


Figure 8: The graph $G_{x,y}$ for Figure 7.

4 Discretization of Weber problems with high-speed curves

Discretization of planar location problem to discrete location problems is not a new concept. Probably, the most famous discretization of a planar location problem is the paper of Durier and Michelot [11]. They showed, that in the case of the unrestricted Weber problem with polyhedral gauges $1/\mathbb{R}^2/\bullet/\gamma/\Sigma$ the fundamental directions rooted at the demand points D define a grid tessellation of the plane such that the grid points (intersection of fundamental directions) contain at least one optimal location. This result is based on the fact that the objective function is linear in each cell.

In this section we will show that a grid tessellation exists such that our objective function is *concave* on each cell. This yields a grid point which is an optimal location for $1/\mathbb{R}^2/(H,\Lambda)/\gamma_H/\sum$. We will show that the number of grid points depends polynomially on D and H and that a solution of $1/\mathbb{R}^2/(H,\Lambda)/\gamma_H/\sum$ can be computed in polynomial time.

We first define a grid in $\mathbb{R}^2$ consisting of the high-speed curves H and of all rays starting from points in

$$\mathcal{K} \ := \ K \cup D \cup \bigcup_{z \in D} K_z^+$$

along the fundamental directions of the gauge γ, i.e.,

$$\mathcal{G} := \ \bigcup_{x \in \mathcal{K}} \bigcup_{i=1}^{\delta} r^{x,d_i} \cup H.$$

We use the notation $\mathcal{P}(\mathcal{G})$ for the set of intersections points between curves in $\mathcal{G}$ and the notation $\mathcal{C}(\mathcal{G})$ for cells defined by $\mathcal{G}$. Note that $\mathcal{K} \subseteq \mathcal{P}(\mathcal{G})$.

A cell $C \in \mathcal{C}(\mathcal{G})$ is a convex polyhedron with extreme points in $\mathcal{P}(\mathcal{G})$ such that C does not contain any other convex polyhedron with edges in $\mathcal{G}$; see Figure 10 for a grid and Figure 11 for the corresponding grid points $\mathcal{P}(\mathcal{G})$.

Recall the following well-known basic lemma.

Lemma 6 ([9]). *The gauge distance $\gamma(x, a)$ is linear in $x = (x_1, x_2)$ on each cone spanned by two neighbored fundamental directions $d_i = (d_{i,1}, d_{i,2})$ and $d_{i+1} = (d_{i+1,2}, d_{i+1,2})$ with origin a. Furthermore, the value of $\gamma(x, a)$ can be evaluated by*

$$\gamma(x, a) = \alpha_i + \alpha_{i+1},$$

where α_i and α_{i+1} can be calculated as

$$\alpha_i = \frac{d_{i+1,1}x_2 - d_{i+1,2}x_1}{d_{i,2}d_{i+1,1} - d_{i,1}d_{i+1,2}} \quad \text{and} \quad \alpha_{i+1} = \frac{d_{i,2}x_1 - d_{i,1}x_2}{d_{i,2}d_{i+1,1} - d_{i,1}d_{i+1,2}}.$$

With this observation we can prove the main result of the paper:

Theorem 7. *At least one of the grid points $\mathcal{P}(\mathcal{G})$ is optimal for $1/\mathbb{R}^2/(H, \Lambda)/\gamma_H/\sum$.*

Proof. We show that $\gamma_H(a, x)$ is concave on each cell $C \in \mathcal{C}(\mathcal{G})$ for every demand point $a \in D$. Then the objective function f_H of $1/\mathbb{R}^2/(H, \Lambda)/\gamma_H/\sum$ is concave on C as positive linear combination of concave functions and therefore an extreme point of a cell is a minimizer i.e. $\mathcal{P}(\mathcal{G})$ contains an optimal solution.

Let $a \in D$ be a demand point and let $x \in \mathbb{R}^2$. From Theorem 3 and Lemma 4 we know, that either a shortest path $Q = [z_1, \ldots, z_L]$ from a to x exists such that all intermediate points z_i ,$i \leq L$, are contained in $V_{a,x}$, or that $[a, x]$ is a shortest path from a to x. Furthermore, in the former case we have $z_i \in \mathcal{K} \forall i \leq L - 2$ and $z_{L-1} \in K_x^- \cup \mathcal{E}(H)$. Hence, for any point $x \in \mathbb{R}^2$ one of the following sets contains a shortest path from a to x:

(1) $\mathcal{P}_{ax}^1 := \{P = [a, x]\}$.

(2) $\mathcal{P}_{ax}^2 := \{P = [z_1, z_2, \ldots, z_L] \; : \; z_1 = a, \; z_L = x, \; z_{L-1} \in \mathcal{E}(H)\}$.

(3) $\mathcal{P}_{ax}^3 := \{P = [z_1, z_2, \ldots, z_L] \; : \; z_1 = a, \; z_L = x, \; z_{L-2} \in K \cap h, \; z_{L-1} \in K_{h,x}^-$ for some $h \in H\}$.

Therefore we obtain

$$\gamma_H(a,x) \quad = \quad \min_{i=1,2,3} \min_{P \in \mathcal{P}^i_{ax}} \gamma_{\mathcal{P}}(P).$$

We now show that all three functions

$$f^i(x) := \min_{P \in \mathcal{P}^i_{ax}} \gamma_{\mathcal{P}}(P), \quad i = 1,2,3$$

are concave functions on every cell $C \in \mathcal{C}(\mathcal{G})$. To this end, we fix a cell $C \in \mathcal{C}(\mathcal{G})$.

Concavity of f^1. For $x \in C$ we obtain

$$f^1(x) = \begin{cases} \lambda(h)\gamma(a,x) & \text{if } a, x \in h \text{ for some } h \in H \\ \gamma(a,x) & \text{else} \end{cases}.$$

First, note that C lies in a cone spanned by two neighbored fundamental directions with origin a. If $f^1(x) = \gamma(a,x) \, \forall \, x \in C$ we can apply Lemma 6 and obtain that f^1 is linear on C. If there exists $x \in C$ such that $f^1(x) = \lambda(h)\gamma(a,x)$ for some $h \in H$, then h defines a facet of C. Furthermore there are at most two high-speed curves such that $f^1(x) = \lambda(h)\gamma(a,x)$ for some $x \in C$. In the case that only one facet of C coincides with a high-speed curve h through a, we modify a fundamental direction of the gauge γ as follows: If h is not a multiple of a fundamental direction we add h as a new fundamental direction to γ, otherwise we adjust the length of the fundamental direction by the speed factor λ of h. By applying Lemma 6 to the modified gauge we obtain that f^1 is linear on C.

If two facets of C are given by high-speed curves h_1, h_2 through a, we modify γ w.r.t. h_1 and a copy $\gamma^{'}$ of γ w.r.t. h_2. Now the value of f^1 is given by the minimum of $\gamma(a,x)$ and $\gamma^{'}(a,x)$ and therefore f^1 is concave on C.

Concavity of f^2. For f^2 we obtain

$$f^2(x) = \min_{z \in \mathcal{E}(H)} \{\gamma_H(a,z) + \gamma(z,x).\}$$

For each $z \in \mathcal{E}(H)$ the function $\gamma_H(a,z) + \gamma(z,x)$ is linear in x, again due to Lemma 6 and the definition of $\mathcal{G}$. Hence, f^2 is concave as the minimum over a finite set of linear functions.

Concavity of f^3. Let $x \in C$ and $P = [z_1, z_2, \ldots, z_{L-2}, z_{L-1}, z_L]$ be a shortest path between a and x in $\mathcal{P}^3_{ax}$. Then there exists a curve h with $z_{L-2} \in h$ and a fundamental direction d_i such that

$$x = z_{L-2} + \alpha_0 d_h + \alpha_1 d_i$$

where d_h is the direction vector of h. Moreover, we know that

- $z_{L-1} = z_{L-2} + \alpha_0 d_h \in h$ and that
- d_i and d_h are linearly independent or $\alpha_1 = 0$ (if d_i and d_h are linearly dependent and $\alpha_1 \neq 0$ then $z_{L-1} \in \mathcal{E}(H)$ and this has been already treated in our second case).

If $\alpha_1 = 0$ we can choose d_i arbitrarily and therfore we can assume that d_i and d_h are linearly independent. Hence, the point z_{L-1} and the distance from z_{L-2} to x is determined by the curve h (with its direction vector d_h) and the fundamental direction d_i used in the last segment of the path. Since $x \in C$ was chosen arbitrarily, there exists for every $x \in C$ some $h \in H$, $z \in K \cap h$ and a direction vector d_i such that the value of $f^3(x)$ is given by $\gamma_H(a, z) + \lambda(h)\gamma(0, \alpha_0 d_h) + \gamma(0, \alpha_1 d_i)$. We will now use this fact to show that there exists a representation of f^3 on C as minimum of linear functions, i.e. we will show that f^3 is concave on C. To this end we need the term *feasible constellation* of two vectors which we define as follows.

Two vectors e, f are called *feasible constellation* w.r.t. h and x if

- e is a fundamental direction of the gauge considered,
- f is a direction vector of h,
- e and f are linearly independent, and
- $l(h) \cap r^{x,-e} \in h$, i.e. the intersection between the straight line passing through h and the ray $r^{x,-e}$ lies in h.

We claim that e, f are a feasible constellation w.r.t. h and some $x \in \text{int}(C)$ if and only if e, f are a feasible constellation w.r.t h and *all* $x \in C$.

To justify this statement, let by $z_{h,i}(x)$ denote the (unique) intersection point between the ray $r^{x,-d_i}$ and the line $l(h)$ passing through h. Now assume $x, x' \in C$ and $z_{h,i}(x) \in h$ but $z_{h,i}(x') \notin h$. Then h is a line segment or a ray which ends between the two points $z_{h,i}(x), z_{h,i}(x') \in l(h)$. Hence there is an extreme point $z \in h$ and the ray in direction d_i starting at z belongs to $\mathcal{G}$. Since this ray separates $z_{h,i}(x)$ from $z_{h,i}(x')$ it also separates x from x' and hence lies in the interior of the cell C, a contradiction. See Figure 9 for an example of feasible and infeasible constellations.

Now let d_i be a fundamental direction of the gauge considered and d_h a direction vector of a curve $h \in H$ such that d_i, d_h are a feasible constellation. Furthermore $z \in K \cap h$. We are going to show that

$$g_{z,h,i} : C \to \mathbb{R},\ y \mapsto \gamma_H(a, z) + \lambda(h)\gamma(z, z_{h,i}(y)) + \gamma(z_{h,i}(y), y)$$

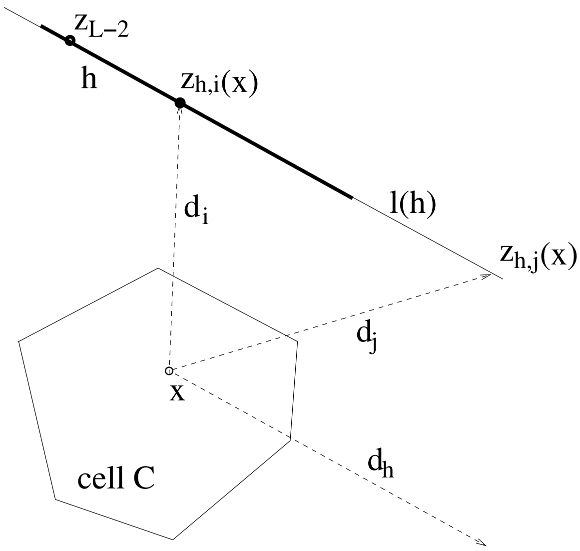


Figure 9: The notation in the proof of Theorem 7. d_i is a feasible direction w.r.t h , but d_j is not.

is linear. The first term of $g_{z,h,i}(x)$ is constant. The others describe the distance from z to x using the curve h and the fundamental direction d_i. Since d_h and d_i are linearly independent, there exist unique scalars $\alpha_0(x), \alpha_1(x)$ such that

$$x = z_{L-2} + \alpha_0(x) d_h + \alpha_1(x) d_i. \tag{6}$$

We rewrite (6) as

$$(d_h, d_i) \begin{pmatrix} \alpha_0(x) \\ \alpha_1(x) \end{pmatrix} = (x - z_{L-2})$$

with regular 2×2 matrix $M := (d_h, d_i)$. Hence M^{-1} exists and

$$\begin{pmatrix} \alpha_0(x) \\ \alpha_1(x) \end{pmatrix} := M^{-1}(x - z_{L-2})$$

are linear functions. We add a new fundamental direction $d_{\delta+1} := \frac{\lambda(h)}{\|d_h\|} d_h$ to γ and by applying Lemma 6 on the modified gauge γ we obtain

$$g_{z,h,i}(x) = \gamma_H(a, z) + \alpha_0(x) + \alpha_1(x) \, \forall \, x \in C.$$

(Note that due to the definition of high-speed curves the modified gauge γ is always a convex gauge).

Summarizing, for every $x \in C$ there exists a feasible constellation of vectors d_h, d_i and $z \in K \cap h$ such that the length of a shortest path $P \in \mathcal{P}^3_{a,x}$ is given by the linear function $g_{z,h,i}(x)$. Hence, using the conditions for feasible constellations, we can rewrite the distance from a to x for paths $P \in \mathcal{P}^3_{a,x}$ as

$$f^3(x) = \min_{z \in K} \min_{\substack{h \in \mathcal{H} \\ \text{s.t.} z \in h}} \min_{\substack{i=1,\ldots,\delta \text{ s.t. } d_i, d_h \text{ are feasible} \\ \text{constellation w.r.t. } h \text{ and } x}} g_{z,h,i}$$

□

Lemma 8. *The size of $\mathcal{P}(\mathcal{G})$ is of order $O(\delta^4 L^4 M^2)$ where δ denotes the number of fundamental directions of the used gauge γ, M is the number of given demand points and L the number of high-speed lines.*

Proof. The size of $\mathcal{P}(\mathcal{G})$ is given by the number of intersection between linear curves in $\mathcal{G}$. Hence $|\mathcal{P}(\mathcal{G})| \leq \binom{|\mathcal{G}|}{2}$. $\mathcal{G}$ consists of

- δ rays through M demand points,
- δ rays through $|\mathcal{K}|$ points, and
- all L high-speed curves.

$\mathcal{K}$ consists of at most

- $\binom{L}{2}$ intersections between high-speed curves and
- $\delta ML + 2\delta L(L-1)$ points lying on high-speed curves.

So $|\mathcal{K}|$ is of order $O(\delta L^2 M))$, $|\mathcal{G}|$ is of order $O(\delta^2 L^2 M)$ and finally $|\mathcal{P}(\mathcal{G})|$ is of order $O(\delta^4 L^4 M^2)$. □

Note that for fixed δ and L the size of $\mathcal{P}(\mathcal{G})$ is of order $O(M^2)$. So the size of the FDS $\mathcal{P}(\mathcal{G})$ is not too bad. Nevertheless, it is possible to identify a smaller FDS for special cases. For the case of only one high-speed line and of the rectangular metric this will be done in Section 6.

5 The problem with more general distances γ

In this section we investigate high-speed metrics that are derived from more general distance functions d. We first investigate the properties of the resulting high-speed distance.

Theorem 9. *Let $d : \mathbb{R}^2 \to \mathbb{R}$ be a function and let H be a set of linear curves. Then the following properties for the distance function d_H apply:*

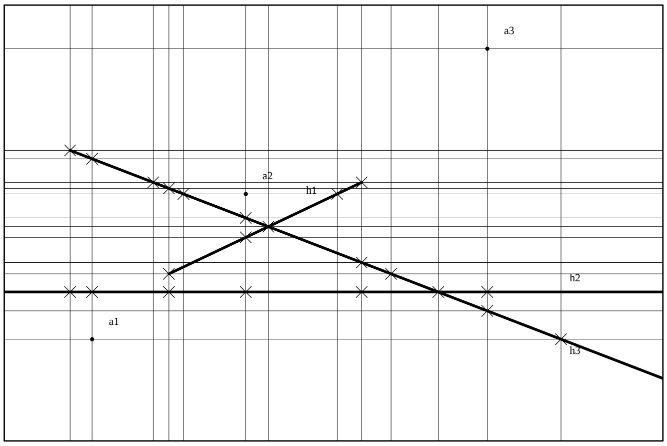


Figure 10: Grid $\mathcal{G}$ for the example introduced in Figure 1 with rectangular metric. 2.

(1) *If d is positive definite then also d_H is.*

(2) *If d is symmetric then also d_H is.*

(3) *d_H satisfies the triangle inequality.*

Proof. **Positive definiteness**. Let $x, y \in \mathbb{R}^2$ and $d_H(x, y) = 0$. This means that a path $P = [z_1, z_2, \ldots, z_L] \in \mathcal{P}_{x,y}$ exists such that

$$\sum_{i=1}^{L-1} c(z_i, z_{i+1}) \quad = \quad 0,$$

see (3) for the definition of c. Since $c(z_i, z_{i+1}) \geq 0 \forall i = 1, \ldots, L-1$ and $c(z_i, z_{i+1}) = 0$ if and only if $z_i = z_{i+1}$ we obtain $z_1 = z_2 = \cdots = z_L$ and therefore $x = y$.

On the other hand, if $x = y$, we obtain $d_H(x, y) = 0$ using the path $P := [x, y] \in \mathcal{P}_{xy}$.

Symmetry. Note that each path $P \in \mathcal{P}_{x,y}$ can be traversed in opposite direction. Since d is a metric, it satisfies $c(x, y) = c(y, x)$ for all $x, y \in \mathbb{R}^2$. Hence, given a path $P \in \mathcal{P}_{x,y}$ there exists a path $\bar{P} \in \mathcal{P}_{y,x}$ with

same length. This shows symmetry of d_H.

Triangle inequality. Let $x, y, z \in \mathbb{R}^2$, $P_1 := [v_1, v_2, \ldots, v_M] \in \mathcal{P}_{x,y}$ and $P_2 := [w_1, w_2, \ldots, w_N] \in \mathcal{P}_{y,z}$ such that $d_H(x, y) = \sum_{i=1}^{M-1} c(v_i, v_{i+1})$ and

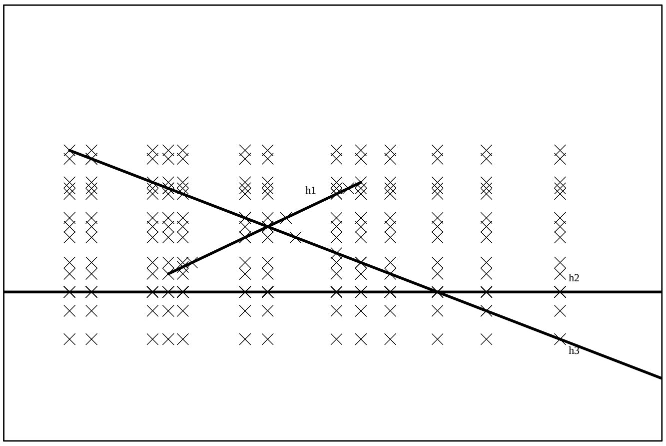


Figure 11: Set $\mathcal{P}(\mathcal{G})$ = {crossed dots} defined by the grid of Figure 10. $|\mathcal{P}(\mathcal{G})| = 175$.

$d_H(y,z) = \sum_{i=1}^{N-1} c(w_i, w_{i+1})$. Let P_3 be the composition of P_1 and P_2, i.e.

$$P_3 \quad := [v_1, \dots, v_M, w_1, \dots, w_N].$$

Then P_3 is a finite linear path from x to z, i.e. $P_3 \in \mathcal{P}_{xz}$. Hence

$$\begin{aligned} d_H(x,z) &= \min_{P=[z_1,\dots,z_L]\in\mathcal{P}_{xz}} \left\{ \sum_{i=1}^{L-1} c(z_i, z_{i+1}) \right\} \leq \gamma_{\mathcal{P}}(P_3) \\ &= \sum_{i=1}^{M-1} c(v_i, v_{i+1}) + \sum_{i=1}^{N-1} c(w_i, w_{i+1}) \\ &= d_H(x,y) + d_H(y,z), \end{aligned}$$

proving that the triangle inequality holds. □

Note, if a function d is positive definite and symmetric, then the derived highs-speed distance d_H is a metric. I.e. d_H fulfills the triangle inequality, also if the triangle inequality does not hold for d.

We next provide a simple dominance criterion for the optimality of a demand point in the case of a high-speed distance d_H which is derived from any function d.

Theorem 10. *If*

$$w_k \geq \frac{1}{2} \sum_{i=1}^{M} w_i \tag{7}$$

holds for some $k \in \{1, \ldots, M\}$ then demand point a_k is a minimizer of $1/\mathbb{R}^2/(H, \Lambda)/d_H/\sum$. If inequality (7) is strict, then demand point a_k is a unique minimizer of $1/\mathbb{R}^2/(H, \Lambda)/d_H/\sum$.

Proof. Let x^* be a optimal solution of $1/\mathbb{R}^2/(H, \Lambda)/d_H/\sum$. Without lost of generality we can assume that demand point a_1 satisfy inequality (7). Then we obtain

$$\begin{array}{lr} f_H(x^*) - d_H(x^*, a_1)\left(\omega_1 - \sum_{m=2}^{M} \omega_m\right) & = \\ \sum_{m=1}^{M} \omega_m d_H(x^*, a_m) - d_H(x^*, a_1)\left(\omega_1 - \sum_{m=2}^{M} \omega_m\right) & = \\ \sum_{m=2}^{M} \omega_m d_H(x^*, a_m) + \sum_{m=2}^{M} \omega_m d_H(x^*, a_1) & = \\ \sum_{m=2}^{M} \omega_m (d_H(x^*, a_m) + d_H(x^*, a_1)). & \end{array}$$

Since d_H satisfy the triangle inequality (see Theorem 9) we obtain $d_H(x^*, a_m) + d_H(x^*, a_1) \geq d_H(a_m, a_1)$ for all $m = 1, \ldots, M$ and hence

$$f_H(x^*) - d_H(x^*, a_1)\left(\omega_1 - \sum_{m=2}^{M} \omega_m\right) \geq \sum_{m=1}^{M} \omega_m d_H(a_m, a_1) = f_H(a_1).$$

Note that $d_H(a_1, a_1) = 0$. Since $\omega_1 \geq \sum_{m=2}^{M} \omega_m$ it follows $\omega_1 - \sum_{m=2}^{M} \omega_m \geq 0$. Therefore $f_H(x^*) \geq f_H(a_1)$ holds. Hence we obtain that a_1 is a minimizer of $1/\mathbb{R}^2/(H, \Lambda)/d_H/\sum$.

Now assume that inequality (7) is strict, i.e. $\omega_1 > \frac{1}{2}\sum_{i=1}^{M} \omega_i$. From the first part of the proof we know that a_1 is a minimizer of $1/\mathbb{R}^2/(H, \Lambda)/d_H/\sum$. To the contrary let us assume that there exists another minimizer $x^* \neq a_1$. Then $d_H(x^*, a_1) > 0$ and we obtain analogously to the first part of the proof that $f_H(x^*) > f_H(a_1)$. This is a contradiction to the assumption that a_1 is a minimizer.

□

6 The case of a single high-speed line in rectangular metric

We now treat the special case of $1/\mathbb{R}^2/(H, \Lambda)/d_H/\sum$ where the used distance function is induced by one straight line and the rectangular metric. Furthermore the high-speed curve, denoted by h, is assumed to be parallel to one fundamental line of the rectangular metric, i.e. the line is either parallel to the x_1-axis or to the x_2-axis. Therefore the x_1-axis can assumed to be the high-speed curve. As in Definition 1 we define the distance between two points in the plane by the length of a shortest path between the points. In contrast to the previous sections, we now use the denotation l_H for this distance function to underline that it is derived from the rectangular metric l_1. From Theorem 9 it follows that l_H is a metric. Since we have one high-speed curve, we use the following representation of l_H that does not use shortest paths.

Lemma 11. *For every* $x = (x_1, x_2),\ y = (y_1, z_2) \in \mathbb{R}^2$ *we have*

$$l_H(x, y) = \min \{l_1(x, y),\ \lambda(h) \mid x_1 - y_1 \mid + \mid x_2 \mid + \mid y_2 \mid\}.$$

Proof. Let us assume that $l_H(x, y) < l_1(x, y)$ holds, otherwise we have nothing to show. This assumption implies $x_1 \neq y_1$. Let us denote the x_1-axis by $h = \{(z_1, z_2) \in \mathbb{R}^2 \ :\ z_2 = 0\}$. Since there exists only one high-speed line, we have to prove that $\hat{v} := (x_1, 0)$ and $\hat{w} := (y_1, 0)$ are minimizers of

$$\begin{array}{rl} \text{minimize} & l_1(x, v) + \lambda(h) l_1(v, w) + l_1(w, y) \\ \text{s.t.} & v, w \in h. \end{array} \tag{8}$$

A minimizer of (8) can be found by solving

$$\begin{array}{rl} \text{minimize} & |x_1 - v| + \lambda(h)|v - w| + |y_1 - w| \\ \text{s.t.} & v, w \in \mathbb{R}. \end{array} \tag{9}$$

$|x_1 - v|$ and $|y_1 - w|$ are piecewise linear and convex with breakpoints in x_1 or y_1, respectively. Furthermore $\lambda(h)|v - w|$ is linear in each of the four cones defined by the lines $l_1 :\ y = x$ and $l_2 :\ y = -x$. Summarizing, the vertical line through $(x_1, 0)$, the horizontal line through $(0, y_1)$, and the lines l_1, l_2 induce a partition of the plane into polygonal cells such that (9) is linear in each cell. Let Z be the set that contains all extreme points of this cells. From the Fundamental Theorem of Linear Programming we know, that Z must contain a minimizer of (9). Furthermore, except the point (x_1, y_1) all points in Z are of the form (z, z) or $(z, -z)$, respectively. A point of the form (z, z) can not be a minimizer, since in this case we obtain a contradiction to our assumption $l_H(x, y) < l_1(x, y)$. Also a point of the form $(z, -z) \neq (x_1, y_1)$ can not be a minimizer, since in this case we can always improve the objective value by moving in x_1-direction or in x_2-direction, respectively. Hence, only (x_1, y_1) is a valid minimizer of (9). □

Using Lemma 11 it is possible to solve $1/\mathbb{R}^2/(H, \lambda)/l_H/\sum$ by linear programming. But in the following we will investigate a geometrical approach. Since h is parallel to a fundamental direction of the rectangular metric, the size of the FDS computed in Section 4 decreases. Now the size of the FDS (i.e. the set $\mathcal{C}(\mathcal{G})$) is in order of $O(M^2)$ where M is the number of given demand points. But in this special case we are able to calculate a smaller FDS by identifying a certain subset of $\mathcal{C}(\mathcal{G})$. To this we will use some well known properties of $1/\mathbb{R}/\bullet/|\cdot|/\sum$.

Theorem 12. *([14]) Let $B = \{b_1, \ldots, b_n\}$ be a set of real numbers such that $b_1 \leq b_2 \leq \ldots \leq b_n$ and let w_i be a non-negative weight associated to b_i for all $i = 1, \ldots, M$. Let $m^* := \min\left\{m : \sum_{i=1}^{m} w_i \geq \frac{1}{2}\sum_{i=1}^{M} w_i\right\}$. If $\sum_{i=1}^{m^*} w_i > \frac{1}{2}\sum_{i=1}^{M} w_i$, then b_{m^*} is the unique minimizer of $g(x) := \sum_{i=1}^{n} w_i|b_i - x|$. Or else $[b_{m^*}, b_{m^*+1}]$ is the set of minimizer of g.*

The objective function of $1/\mathbb{R}^2/(H,\lambda)/l_H/\sum$ is given by

$$f_H(x) = \sum_{i=1}^{M} w_i l_H(a_i, x).$$

We use the standard notation, i.e. every given demand point is represented by a point a_i with associated weight w_i. To investigate properties of f_H, we define for every $\epsilon \in \{0,1\}^M$ a function $z_\epsilon : \mathbb{R}^2 \to \mathbb{R}$,

$$x \mapsto \sum_{i=1}^{M} w_i \left(\epsilon_i l_1(a_i, x) + (1-\epsilon_i)\left(\lambda(h) \mid x_1 - a_{i1} \mid + \mid x_2 \mid + \mid a_{i2} \mid\right)\right).$$

Note that for $\epsilon = (1, \ldots, 1)$ z_ϵ is equal to the objective function of $1/\mathbb{R}^2/\cdot/l_1/\sum$. Furthermore, for every $x \in \mathbb{R}^2$ there exists an $\epsilon \in \{0,1\}^M$ such that $z_\epsilon(x) = f_H(x)$ holds. Since $z_\epsilon(x) \geq f_H(x)$ for all $x \in \mathbb{R}^2$ and for all $\epsilon \in \{0,1\}^M$, we obtain that a minimizer of $z(x) := \min_{\epsilon \in \{0,1\}^M} z_\epsilon(x)$ is also a minimizer of f_H and vice versa. Each point in the set

$$\mathrm{OPT}(z) := \{x \in \operatorname{argmin} z(x)\}$$

is a minimizer of f_H. So $\mathrm{OPT}(z)$ is a FDS of $1/\mathbb{R}^2/(H,\lambda)/l_H/\sum$. In the following we will not compute $\mathrm{OPT}(z)$ exactly but use the observation that

$$\mathrm{OPT}(z) \subset \mathrm{OPT}(z_\epsilon) := \left\{x \in \operatorname{argmin} z_\epsilon(x) \, : \, \epsilon \in \{0,1\}^M\right\}$$

holds. We calculate four elements of $\mathrm{OPT}(z_\epsilon)$, namely the element with the smallest x_1 value, the element with the largest x_1 value and the both elements with the smallest and largest x_2 values. With the help of this four elements we obtain a rectangle that contains $\mathrm{OPT}(z_\epsilon)$ and therefore also $\mathrm{OPT}(z)$.

First we compute the smallest and largest x_1 values of elements in $\mathrm{OPT}(z_\epsilon)$. To this, we assume that the given demand points $D = \{a_1, \ldots, a_M\}$ satisfy $a_{11} < a_{21} < \ldots < a_{M1}$. If there are demand points with the same x_1 value we can combine them if we accumulate the associated weights.

Theorem 13. *Let* $x = (x_1, x_2)$ *be in* $\mathrm{OPT}(z_\epsilon)$,

$$s^* := \min\left\{ s \in \{1, \ldots, M\} \ : \ \sum_{i=1}^{s} w_i \geq \lambda(h) \sum_{i=s+1}^{M} w_i \right\}$$

and

$$t^* := \max\left\{ t \in \{1, \ldots, M\} \ : \ \sum_{i=t}^{M} w_i > \lambda(h) \sum_{i=1}^{t-1} w_i \right\}.$$

Then $x_1 \in [a_{s^*1}, a_{(t^*+1)1}]$ *holds.*

Proof. Let $x = (x_1, x_2)$ be in $\mathrm{OPT}(z_\epsilon)$. Since there exists $\epsilon \in \{0,1\}^M$ such that x is a minimizer of z_ϵ, x_1 is a minimizer of

$$z_\epsilon^1 : \mathbb{R} \to \mathbb{R},\ x \mapsto \sum_{i\,:\,\epsilon_i=0} w_i \lambda(h) \mid x_1 - a_{i1} \mid + \sum_{i\,:\,\epsilon_i=1} w_i \mid x_1 - a_{i1} \mid .$$

Now we can apply Theorem 12 on $\{a_{11}, \ldots, a_{M1}\}$, $\{w_1, \ldots, w_M\}$ and z_ϵ^1 and obtain that there exists $m^* \in \{1, \ldots, M\}$ such that $a_{m^*1} = x_1$ or $x_1 \in [a_{m^*1}, a_{(m^*+1)1}]$. Furthermore m^* satisfies the inequality

$$\sum_{i\,:\,i \leq m^*,\, \epsilon_i=0} w_i \lambda(h) + \sum_{i\,:\,i \leq m^*,\, \epsilon_i=1} w_i \geq \frac{1}{2} \sum_{i\,:\,\epsilon_i=0} w_i \lambda(h) + \frac{1}{2} \sum_{i\,:\,\epsilon_i=1} w_i. \quad (10)$$

Inequality (10) is equivalent to

$$\sum_{\substack{i\ :\ i \leq m^*,\\ \epsilon_i=0}} w_i \lambda(h) + \sum_{\substack{i\ :\ i \leq m^*,\\ \epsilon_i=1}} w_i \ \ \geq \ \ \sum_{\substack{i\ :\ i > m^*,\\ \epsilon_i=0}} w_i \lambda(h) + \sum_{\substack{i\ :\ i > m^*,\\ \epsilon_i=1}} w_i. \quad (11)$$

From inequality (11) we obtain

$$\sum_{i=1}^{M} w_i \geq \lambda(h) \sum_{i=m^*+1}^{M} w_i,$$

i.e. $m^* \geq s^*$.

From Theorem 12 we also conclude

$$\sum_{i\,:\,i < m^*,\, \epsilon_i=0} w_i \lambda(h) + \sum_{i\,:\,i < m^*,\, \epsilon_i=1} w_i < \frac{1}{2} \sum_{i\,:\,\epsilon_i=0} w_i \lambda(h) + \frac{1}{2} \sum_{i\,:\,\epsilon_i=1} w_i. \quad (12)$$

Inequality (12) is equivalent to

$$\sum_{i\,:\,i < m^*,\, \epsilon_i=0} w_i \lambda(h) + \sum_{i\,:\,i < m^*,\, \epsilon_i=1} w_i < \sum_{i\,:\,i \geq m^*\,:\,\epsilon_i=0} w_i \lambda(h) + \sum_{i\,:\,i \geq m^*\,:\,\epsilon_i=1} w_i.$$

From inequality (6) we obtain

$$\lambda(h) \sum_{i=1}^{m^*-1} w_i < \sum_{i=m^*}^{M} w_i,$$

i.e. $m^* \leq t^*$.
Altogether we obtain $a_{s^*1} \leq a_{m^*1} \leq x_1 \leq a_{(m^*+1)1} \leq a_{(t^*+1)1}$, i.e. $x_1 \in [a_{s^*1}, a_{(t^*+1)1}]$. □

Now we compute the smallest and largest x_2 value of elements in $\mathrm{OPT}(z_\epsilon)$. To this end let $\mathrm{OPT}(z_1^2)$ be the set of minimizers of $z_1^2(x) := \sum_{i=1}^{M} w_i \mid x - a_{i2} \mid$. We denote the smallest and largest elements of $\mathrm{OPT}(z_1^2)$ by l and u, respectively.

Theorem 14. *Let* $x = (x_1, x_2)$ *be in* $\mathrm{OPT}(z_\epsilon)$. *Then*

$$x_2 \in [\min\{l, 0\}, \max\{0, u\}].$$

Proof. Let $x = (x_1, x_2)$ be in $\mathrm{OPT}(z_\epsilon)$. Since there exists $\epsilon \in \{0,1\}^M$ such that x is a minimizer of z_ϵ, we conclude that x_2 is a minimizer of

$$z_\epsilon^2 : \mathbb{R} \to \mathbb{R},\ x \mapsto \sum_{i\,:\,\epsilon_i=0} w_i \mid x_1 \mid + \sum_{i\,:\,\epsilon_i=1} w_i \mid x_1 - a_{i2} \mid .$$

Obviously, if $\epsilon = (1, \ldots, 1)$ then $x_2 \in [l, u]$. If $\epsilon_i = 0$ the location of demand point a_{i2} changes to zero (note that the demand points are points in $\mathbb{R}$). We can merge this demand points to a new demand point a_0 and sum up their weights. Hence, with each $i \in \{1, \ldots, M\}$ such that $\epsilon_i = 0$ the importance of the demand point a_0 increases. Therefore if $\epsilon \neq (1, \ldots, 1)$ a minimizer of z_ϵ^2 is closer to zero. □

With Theorem 13 and Theorem 14 we obtain a rectangle, namely

$$[a_{s^*1}, a_{t^*1}] \times [\min\{l, 0\}, \max\{0, u\}]$$

that contains $\mathrm{OPT}(z)$. We are hence able to define a smaller finite dominating set FDS for $1/\mathbb{R}^2/(H,\lambda)/l_H/\sum$ through

$$\mathcal{C} := [a_{s^*1}, a_{t^*1}] \times [a_{u^*2}, a_{v^*2}] \cap \mathcal{P}(\mathcal{G}).$$

Figure 6 shows an example of a problem of type $1/\mathbb{R}^2/(H,\lambda)/l_H/\sum$. The new FDS $\mathcal{C}$ is 85% smaller than $\mathcal{P}(\mathcal{G})$.

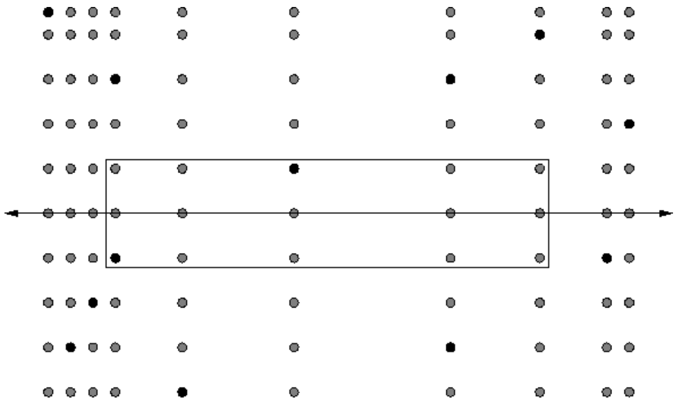


Figure 12: Example of $1/\mathbb{R}^2/(H,\lambda)/l_H/\sum$ with $D = \{\text{black points}\}$, $\lambda = 0.5$, and weight 1 for each demand point in D. The grey and black points belongs to $\mathcal{P}(\mathcal{G})$. $\mathcal{C}$ consists of the 15 points included in the rectangular.

7 Conclusion

In this paper we studied an extended Weber problem with non-convex distance function. We proved a discretization result to solve this new type of problem and were able to identify a finite dominating set. Our results can be used to solve small location problems. However, for a large number of given high-speed curves or of fundamental directions of the used polyhedral gauge, the FDS becomes too large. Approaches to reduce the size of the FDS are possible for special cases as we have shown for the case of rectangular distance and one straight line. We also looked at block norms with only two fundamental directions (see [17]).

A first step to obtain a smaller FDS in the general case could be to study the convex hull of the given demand points and enlarge it with respect to the high-speed curves to obtain a bounded region which contains all optimal solutions. The idea extends the fact, that for many location problems a minimizer is contained in the convex set of the given demand points. Furthermore, the theoretical results of this paper can be used to reduce the given continuous location problem to a discrete problem with a network structure. It has to be evaluated if this network structure leads to faster algorithms than testing all points of the FDS.

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