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A second order Newton method for sound soft inverse obstacle scattering

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Abstract

A new second order Newton method for reconstructing the shape of a sound soft scatterer from the measured far-field pattern for scattering of time harmonic plane waves is presented. This method extends a hybrid between regularized Newton iterations and decomposition methods that has been suggested and analyzed in a number of papers by Kress and Serranho [7, 8, 9, 11, 12] and has some features in common with the second degree method for ill-posed nonlinear problems as considered by Hettlich and Rundell [6]. The main idea of our iterative method is to use Huygen's principle, i.e., represent the scattered field as a single-layer potential. Given an approximation for the boundary of the scatterer, this leads to an ill-posed integral equation of the first kind that is solved via Tikhonov regularization. Then, in a second order Taylor expansion, the sound soft boundary condition is employed to update the boundary approximation. In an iterative procedure, these two steps are alternated until some stopping criterium is satisfied. We describe the method in detail and illustrate its feasibility through examples with exact and noisy data.

1 Introduction

Inverse scattering problems for time harmonic waves are of fundamental importance in applications such as radar and sonar, nondestructive evaluation, geophysical exploration, medical imaging and others. In principle, in these applications the wave scattered by an unknown object is measured at a number of discrete locations and

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information such as shape parameters, location parameters and electromagnetic parameters of the scatterer are extracted from these data. As opposed to classical techniques of imaging such as computerized tomography that is based on the fact that x-rays travel along straight lines, inverse scattering problems take into account that the propagation of acoustic, electromagnetic and elastic waves have to be modeled by a wave equation. This means that inverse scattering requires a nonlinear model, whereas inverse tomography is a linear approximation of inverse scattering problems.

In this study, we are interested in shape reconstruction of sound soft obstacles from measurements of the far field pattern by using a new variant of a second order Newton method. As a major advantage, this iterative algorithm does not require a forward solver at each iteration step. Our approach extends a corresponding first order Newton method as suggested and analyzed by Kress and Serranho for shape reconstruction for sound soft [7, 12] and sound hard obstacles [9] and for the reconstruction of both the shape and the boundary impedance [11]. Although, in the present paper the obstacle is assumed to be smooth, since the approach of Kress and Serranho has also been extended to cracks [8] we expect that our method also can be carried over to this case.

Given an open bounded obstacle $D \subset \mathbb{R}^2$ with an unbounded and connected complement and a smooth boundary ∂D and an incident field u^i, the direct scattering problem consists of finding the total field $u = u^i + u^s$ as the sum of the known incident field u^i and the scattered field u^s, such that the Helmholtz equation

$$\triangle u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \tag{1.1}$$

and the sound soft or Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\bar{D} \tag{1.2}$$

are fulfilled and the scattered wave u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|\,, \tag{1.3}$$

uniformly with respect to all directions. The latter ensures an asymptotic behavior of the scattered wave of the form

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \tag{1.4}$$

uniformly in all directions with the far field pattern u_∞ defined on the unit circle Ω.

The *inverse scattering problem* we are interested in is to determine the location and the shape of the scatterer D from a knowledge of the far field pattern u_∞ for one or several incident plane waves $u^i = e^{ik\,x\cdot d}$ with incident direction $d \in \Omega$. We note that the inverse scattering problem we have just formulated is ill-posed in the sense that the determination of D does not depend continuously on the measured far field data in any reasonable norm. This issue of ill-posedness will be handled using Tikhonov regularization in our reconstruction algorithm.

This algorithm starts from Green's representation formula for the scattered wave

$$u^s(x) = -\frac{i}{4} \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, H_0^{(1)}(k\,|x-y|)\; ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{1.5}$$

and its far field pattern

$$u_\infty(\hat{x}) = -\frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, e^{-ik\,\hat{x}\cdot y}\; ds(y), \quad \hat{x} \in \Omega, \tag{1.6}$$

that is, Huygen's principle (see Theorem 3.12 in [2]). Here, $H_0^{(1)}$ denotes the Hankel function of order zero and of the first kind and ν is the outward unit normal to ∂D. We view (1.6) as the data equation in terms of the measured far field pattern u_∞. Given an approximation for the boundary ∂D, we solve (1.6) for the unknown flux $\varphi := -\partial u/\partial \nu$. Then we insert the scattered wave (1.5) into the boundary condition (1.2) and consider this as the field equation which we solve for updating the approximation for ∂D. This is achieved via a Taylor expansion up to order two for u along the normal direction and solving the quadratic equation for the update function. In an obvious way, these two steps are iterated until an appropriate stopping criterium is satisfied.

To some extend our approach is related to a method that has been suggested and investigated more recently in Çayören et al [1]. The main difference is that in [1] the curve on which the Taylor expansion is employed remains fixed throughout the algorithm and that, in order to compensate for this, higher order Taylor expansions are used.

The idea of our algorithm differs from the traditional regularized Newton type iterations for the inverse obstacle scattering problem. The latter approach is based on the observation that the solution to the direct scattering problem with a fixed incident plane wave u^i defines an operator

$$A : \partial D \mapsto u_\infty$$

that maps the boundary ∂D of the scatterer D onto the far field pattern u_∞ of the scattered wave. In terms of this operator, given a far field pattern u_∞, the inverse problem just consists in solving the nonlinear and ill-posed operator equation

$$A(\partial D) = u_\infty$$

for the unknown surface ∂D which, for example, can be done via regularized Newton iterations as has been suggested for the first time by Roger [10]. For details on this approach we refer to [2]. Numerical examples in three dimensions have been more recently reported by Farhat et al [3] and by Harbrecht and Hohage [5]. A related second order Newton scheme has been considered by Hettlich and Rundell [6].

2 The reconstruction algorithm

Since we assume the boundary ∂D to be smooth, i.e., analytic, the scattered wave can be extended as solution to the Helmholtz equation across the boundary into a neighborhood of ∂D. We now assume that an initial estimate Γ_0 for the boundary of the scatterer D is at our disposal such that this extension of u^s is defined in the closed exterior of Γ_0. Further we assume that k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the interior of Γ_0. Then the scattered field can be expressed as a single-layer potential

$$u^s(x) = \frac{i}{4} \int_{\Gamma_0} H_0^{(1)} \left(k\,|x-y|\right) \varphi(y)\, ds(y) \tag{2.1}$$

for all x in the exterior of Γ_0 and a uniquely determined density $\varphi \in L^2(\Gamma_0)$ (see [2]). Passing to the far field in (2.1) we obtain

$$\frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\Gamma_0} e^{-ik\,\hat{x}\cdot y} \varphi(y)\, ds(y) = u_\infty(\hat{x}), \quad \hat{x} \in \Omega, \tag{2.2}$$

as an integral equation of the first kind for the unknown density φ. Due to its analytic kernel, this integral equation is severely ill-posed. However, the operator $S_\infty : L^2(\Gamma_0) \to L^2(\Omega)$ defined by

$$(S_\infty \varphi)(\hat{x}) := \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\Gamma_0} e^{-ik\,\hat{x}\cdot y} \varphi(y)\, ds(y), \quad \hat{x} \in \Omega,$$

is known to be injective and have dense range (see Theorem 5.17 in [2]). Therefore, Tikhonov regularization can be applied for a stable approximate solution of (2.2), that is, the ill-posed equation (2.2) is replaced by

$$\alpha \varphi_\alpha + S_\infty^* S_\infty \varphi_\alpha = S_\infty^* u_\infty \tag{2.3}$$

with some positive regularization parameter α and the adjoint S_∞^* of S_∞.

For the further description of the reconstruction scheme we represent the curve Γ_0 by a regular parameterization

$$\Gamma_0 = \{z_0(t) : t \in [0, 2\pi)\} \tag{2.4}$$

with a 2π-periodic function $z_0 : \mathbb{R} \to \mathbb{R}^2$. Searching the location where the boundary condition (1.2) is satisfied we approximate the total field u by the Taylor formula of order two with respect to the normal direction at Γ_0. For this we try to update in the form

$$\Gamma_1 = \{z_1(t) = z_0(t) + h(t)\nu_0(t) : t \in [0, 2\pi)\} \tag{2.5}$$

where ν_0 denotes the outward normal vector to Γ_0 and $h : \mathbb{R} \to \mathbb{R}$ is a sufficiently small 2π periodic function. The normal vector can be expressed through the parameterization (2.4) via

$$\nu_0(t) = \frac{[z_0'(t)]^\perp}{|z_0'(t)|}, \quad t \in [0, 2\pi),$$

where for any vector $a = (a_1, a_2)$ we set $a^\perp = (a_2, -a_1)$. Then the second order Taylor formula requires the update function h to satisfy

$$u \circ z_0 + \frac{\partial u}{\partial \nu_0} \circ z_0\, h + \frac{1}{2} \frac{\partial^2 u}{\partial \nu_0^2} \circ z_0\, h^2 = 0. \tag{2.6}$$

Once the single layer density φ is known from (2.3), the values u and $\partial u/\partial \nu_0$ of the total field on Γ_0 can be obtained through the jump relations for the single-layer potential [2], that is, by

$$u(x) = u^i(x) + \frac{i}{4} \int_{\Gamma_0} H_0^{(1)}(k\,|x - y|)\, \varphi(y)\, ds(y), \quad x \in \Gamma_0, \tag{2.7}$$

and

$$\frac{\partial u}{\partial \nu_0}(x) = \frac{\partial u^i}{\partial \nu_0}(x) + \frac{i}{4} \int_{\Gamma_0} \frac{\partial H_0^{(1)}(k\,|x - y|)}{\partial \nu_0(x)}\, \varphi(y)\, ds(y) - \frac{1}{2}\, \varphi(x), \quad x \in \Gamma_0. \tag{2.8}$$

The second order derivative $\partial^2 u/\partial \nu_0^2$ can be obtained by using the fact that the total field satisfies the Helmholtz equation outside the object, that is, it is given by

$$\frac{\partial^2 u}{\partial \nu_0^2} \circ z_0 = -k^2 u \circ z_0 + \frac{z_0' \cdot z_0''}{|z_0'|^4} \frac{\partial (u \circ z_0)}{\partial t} - \frac{1}{|z_0'|^2} \frac{\partial^2 (u \circ z_0)}{\partial t^2} - \frac{z_0' \cdot \nu_0'}{|z_0'|^2} \frac{\partial u}{\partial \nu_0} \circ z_0 \tag{2.9}$$

in terms of the parameterization (2.4) (see, for example, [9]). The integrals in (2.7) and (2.8) can be accurately evaluated by the quadrature rules as described in [2] and the first and second order derivatives of $u \circ z_0$ with respect to the parameter t occurring in (2.9) can be obtained via trigonometric differentiation.

As in the work of Hettlich and Rundell [6] and following Halley [4] the nonlinear equation (2.6) is solved in two steps, a predictor and a corrector step. In the predictor step, one has to solve

$$u \circ z_0 + \frac{\partial u}{\partial \nu} \circ z_0\, h_0 = 0 \tag{2.10}$$

for h_0. Since the solution of (2.10) is sensitive to errors in the normal derivative of u in the vicinity of zeros, equation (2.10) is solved in a stable way by a least squares method. For this we express h_0 in terms of basis functions $\omega_1, \omega_2, \dots, \omega_J$ by

$$h_0 = \sum_{j=1}^{J} a_j \omega_j \tag{2.11}$$

with possible choices of basis functions given by splines or trigonometric polynomials. Then, we satisfy (2.10) in a penalized least squares sense, that is, the coefficients $a_1, a_2, \dots, a_J$ in (2.11) are chosen such that for a set of collocation points $t_1, t_2, \dots, t_N$ in $[0, 2\pi)$ the penalized least squares sum

$$\sum_{n=1}^{N} \left| u(z_0(t_n)) + \frac{\partial u}{\partial \nu}\,(z_0(t_n)) \sum_{j=1}^{J_1} a_j \omega_j(t_n) \right|^2 + \beta_1 \sum_{j=1}^{J} a_j^2$$

with some regularization parameter $\beta_1 > 0$ is minimized. Once h_0 has been obtained, in the corrector step the equations

$$u \circ z_0 + \frac{\partial u}{\partial \nu_0} \circ z_0\; h_m + \frac{1}{2}\,\frac{\partial^2 u}{\partial \nu_0^2} \circ z_0\; h_{m-1}\, h_m = 0 \tag{2.12}$$

are solved recursively for h_m, $m = 1, \dots, M$, again in a penalized least squares sense, that is,

$$\sum_{n=1}^{N} \left| u(z_0(t_n)) + \left[\frac{\partial u}{\partial \nu}\,(z_0(t_n)) + \frac{1}{2}\,\frac{\partial^2 u}{\partial \nu^2}\,(z_0(t_n)) h_0(t_n) \right] \sum_{j=1}^{J} b_{jm} \omega_j(t_n) \right|^2 + \beta_2 \sum_{j=1}^{J} b_{jm}^2$$

with some regularization parameter $\beta_2 > 0$ is minimized to obtain

$$h_m = \sum_{j=1}^{J} b_{jm} \omega_j.$$

Then, finally $h = h_M$ is inserted in (2.5) to obtain the updated boundary Γ_1. In our numerical examples, we used $M = 4$ for the number of inner iterations. This procedure of alternating solving (2.3) and (2.6) now is iterated in an obvious fashion until some stopping criterium is satisfied.

3 Numerical examples

In our examples we employed trigonometric polynomials of degree less than or equal to J for the approximation of the boundary in the predictor and corrector step.

In all examples, we used $N = 50$ collocation points. In order to avoid an inverse crime, the synthetic data were obtained by solving the combined single- and double-layer boundary integral equation for the direct scattering problem by the Nyström method as described in [2] with 100 quadrature points. The wave number is chosen as $k = 1$ and the penalty factors for the least squares approach as $\beta_1 = \beta_2 = 0.001$. The number of iteration steps is denoted by T.

In the first example, we consider the identification of a peanut-shaped object with the parameterization

$$\partial D = \left\{ \sqrt{\cos^2 t + 0.25 \sin^2 t}\, (\cos t, \sin t) : t \in [0, 2\pi) \right\}. \tag{3.1}$$

The iterations were started with a circle of radius 0.5 centered at the origin and we worked with the parameters $J = 5$, $T = 6$, and $\alpha = (0.5)^n \times 10^{-8}$, $\alpha = (0.9)^n \times 10^{-5}$ depending on the iteration number n for noiseless and noisy data, respectively.

In the second example, the reconstruction of a kite-shaped object with parameterization

$$\partial D = \{(\cos t + 0.65 \cos 2t - 0.65, 1.5 \sin t) : t \in [0, 2\pi)\} \tag{3.2}$$

is considered. Here we started the iterations with a circle of radius 1.5 centered at the origin. As parameters we choose $J = 10$, $T = 6$, and $\alpha = (0.9)^n \times 10^{-8}$, $\alpha = (0.9)^n \times 10^{-5}$ for noiseless and noisy data, respectively.

As seen from the figures, our second order Newton method gives slightly better reconstructions than the first order Newton iterations and is more stable against noisy data. Here, by first order Newton iteration we understand the variant where only the predictor step is carried out at each iteration as in the method of Kress and Serranho. We observed that if the noise level exceeds 4% then the reconstructions start to deteriorate. As to be expected we have better reconstructions in the illuminated region than in the shadow region.

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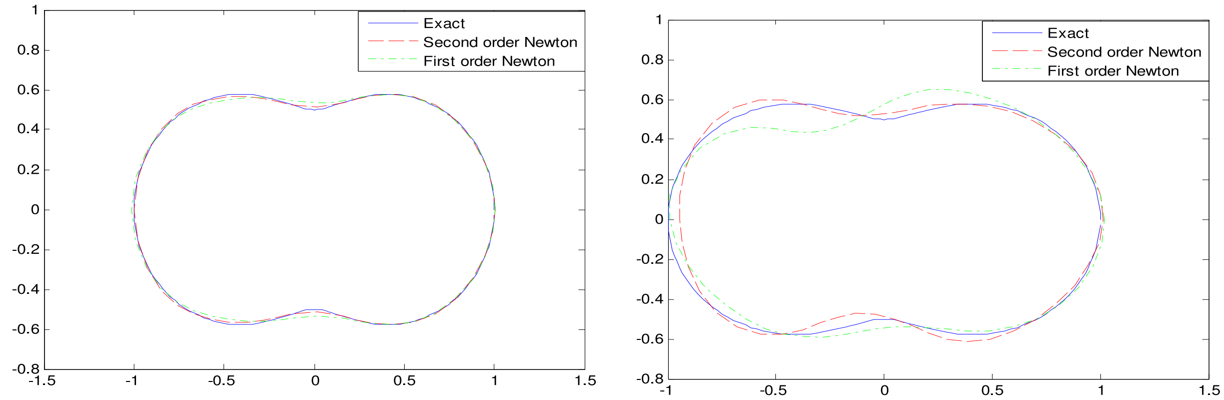

Figure 1: Reconstruction of the peanut for $d = (-1, 0)$ without noise (left) and with 3% noise (right)

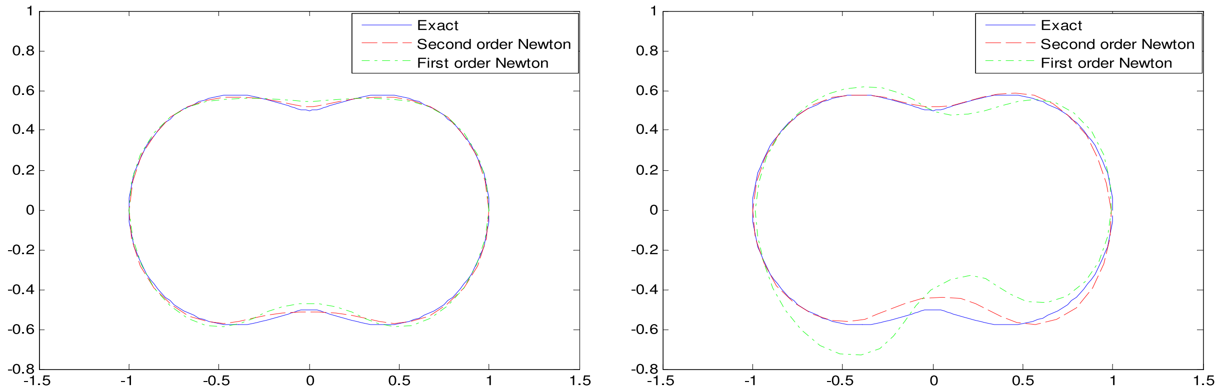

Figure 2: Reconstruction of the peanut for $d = (0, -1)$ without noise (left) and with 3% noise (right)

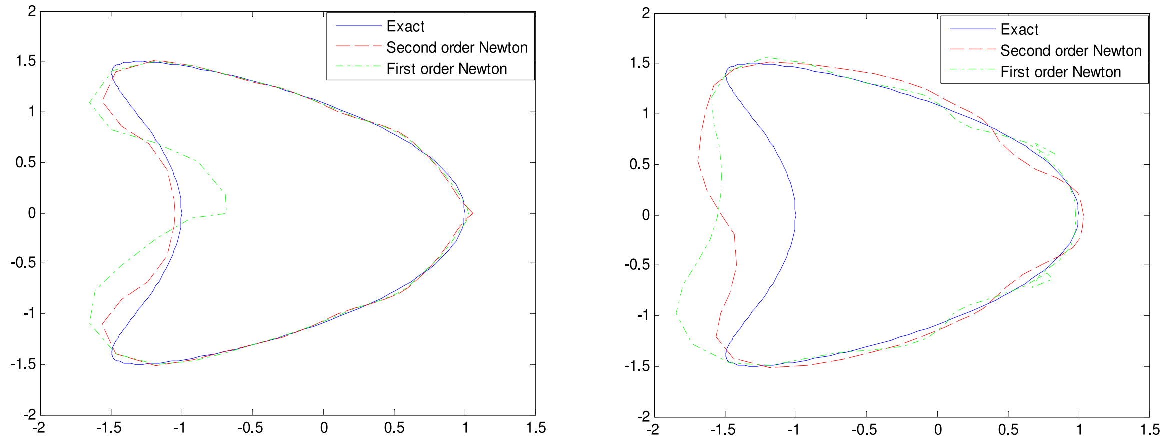

Figure 3: Reconstruction of the kite for $d = (-1, 0)$ without noise (left) and with 3% noise (right)

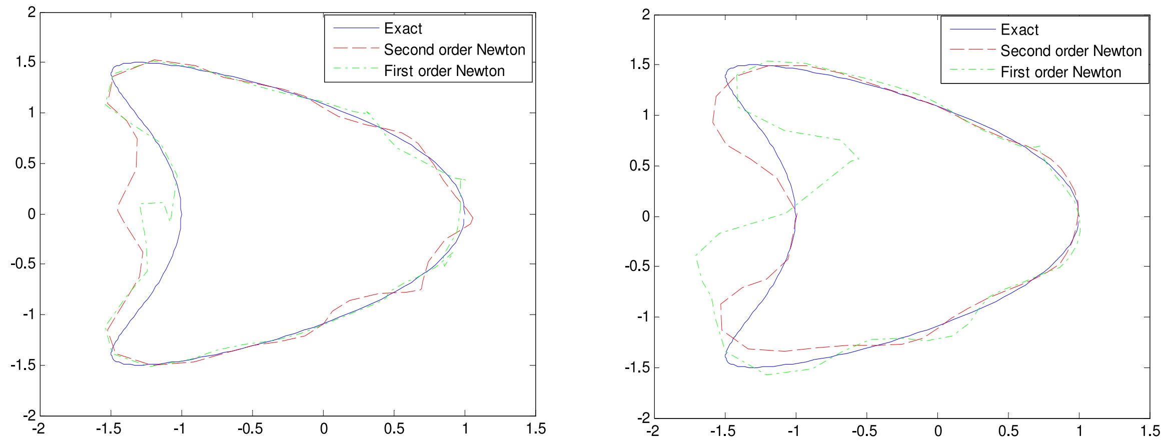

Figure 4: Reconstruction of the kite for $d = \left(-\sqrt{1/2}, -\sqrt{1/2}\right)$ without noise (left) and with 3% noise (right)

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