

Georg-August-Universität Göttingen



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Preprint Nr. 2005-37

Preprint-Serie des
Instituts für Numerische und Angewandte Mathematik
Lotzestr. 16-18
D - 37083 Göttingen

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Abstract

We provide a brief description of recent results in inverse scattering theory having as a common mathematical framework the exploitation of the behavior of the fundamental solution to the Helmholtz equation, in particular the fact that for the source point on the boundary ∂D of the scattering object such a solution is not in the Sobolev space $H^{1/2}(\partial D)$. Included in our discussion are uniqueness theorems, decomposition methods (including the point-source method), the method of singular sources, the linear sampling method and the factorization method.

1 Introduction

In the past twenty years considerable progress has been made on the development of the mathematical theory of inverse scattering problems for time-harmonic acoustic and electromagnetic waves. A survey of these results up until 1998 was given in the second edition of our book [17]. However, since that time a number of important new developments have occurred as well as an understanding of a common mathematical structure underlying many of the results obtained to date. Hence the purpose of our paper is two-fold: to describe some of the basic recent results in the mathematical theory of inverse scattering problems as well as to indicate the common mathematical framework that forms the basis of these new developments. In addition we will provide some historical remarks and personal anecdotes arising out of our participation in the growth of the field of inverse scattering theory over the past twenty years. Before beginning, we warn the reader that our account is

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personal and strongly influenced by the close connection for many years between the Department of Mathematical Sciences at the University of Delaware and the Institut für Numerische und Angewandte Mathematik at the University of Göttingen together with research groups growing out of this collaboration. In particular, we do not claim to cover all of the new developments in inverse scattering theory!

We now present a brief overview of what the reader can expect to find in this paper. We will restrict our attention to the case of the scattering of time-harmonic acoustic waves by either a sound-soft obstacle or a penetrable inhomogeneous medium (although we will not hesitate to consider more general situations from time to time where appropriate). In particular, factoring out $e^{-i\omega t}$, we will always assume that the incident field is given by the time-harmonic plane wave

$$u^i(x) = e^{ik \cdot x \cdot d}$$

where $k = \omega/c$ is the (fixed) wave number, ω the frequency, c the speed of sound, d the direction of propagation and $x \in \mathbb{R}^3$. In this case, the scattering of the incident field u^i by a penetrable medium with support $D \subset \mathbb{R}^3$ is a special case of the scattering problem

$$\nabla \cdot A \nabla w + k^2 n w = 0 \quad \text{in } D \tag{1.1}$$

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{1.2}$$

$$u(x) = e^{ik \cdot x \cdot d} + u^s(x) \tag{1.3}$$

$$w = u, \quad \nu \cdot A \nabla w = \nu \cdot \nabla u \quad \text{on } \partial D \tag{1.4}$$

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial u^s}{\partial r} - i k u^s \right) = 0 \tag{1.5}$$

where $A \in C^1(\bar{D})$ is a matrix, $n \in C(\bar{D})$ is a scalar, D is a bounded domain with smooth boundary ∂D having unit outward normal ν , $\mathbb{R}^3 \setminus D$ is connected and the Sommerfeld radiation condition (1.5) is assumed to hold uniformly with respect to $\hat{x} = x/|x|$ as $r = |x|$ tends to infinity. We always assume that $\Re n(x) > 0$ and $\Im n(x) \geq 0$ for $x \in D$ and that the matrix A is symmetric such that

$$\xi \cdot \overline{\Im A(x)} \bar{\xi} \geq \gamma_1 |\xi|^2 \quad \text{and} \quad \xi \cdot \Re A(x) \bar{\xi} \geq \gamma_2 |\xi|^2 \tag{1.6}$$

for all $x \in D$ and $\xi \in \mathbb{C}^3$ where γ_1 and γ_2 are positive constants. The physical problem of scattering by acoustic waves corresponds to the case when A is of the

form aI where a is a scalar and I is the identity matrix. The special case of scattering by a sound-soft obstacle corresponds to the case when $w = 0$ in D and the boundary condition (1.4) is replaced by $u = 0$ on ∂D , i.e.,

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \quad (1.7)$$

$$u(x) = e^{ik \cdot x \cdot d} + u^s(x) \quad (1.8)$$

$$u = 0 \quad \text{on } \partial D \quad (1.9)$$

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0. \quad (1.10)$$

The existence of a unique solution to the scattering problem (1.1)–(1.5) such that $w \in H^1(D)$, $u \in H_{loc}^1(\mathbb{R}^3 \setminus \bar{D})$, and to the scattering problem (1.7)–(1.10) such that $u \in H_{loc}^1(\mathbb{R}^3 \setminus \bar{D})$ is well known [17, 32].

Our main interest in the paper is not with the direct scattering problems (1.1)–(1.5) and (1.7)–(1.10) but rather with the corresponding inverse problem. In particular, it is easily seen [17] that the scattered fields u^s corresponding to both (1.1)–(1.5) and (1.7)–(1.10) have the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{r} \left\{ u_\infty(\hat{x}, d) + O\left(\frac{1}{r}\right) \right\} \quad (1.11)$$

as $r = |x| \rightarrow \infty$. The function u_∞ is known as the *far field pattern* of the scattered wave and is an analytic function of \hat{x} and d on the unit sphere $\Omega := \{x \in \mathbb{R}^3 : |x| = 1\}$. The *inverse scattering problem* that the two of us have been concerned with for the past twenty years is to determine D from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x} \in \Omega_0 \subset \Omega$ and $d \in \Omega_1 \subset \Omega$ where Ω_0 and Ω_1 are subsets of Ω . We note that in general this is the most we can expect to determine from u_∞ . In particular, the matrix A is in general not uniquely determined from u_∞ nor is the scalar n except in certain circumstances [28]. We further note that the inverse scattering problem we have just formulated is *ill-posed* in the sense that the determination of D does not depend continuously on the measured far field data in any reasonable norm. This issue of ill-posedness will be handled using standard regularization techniques, e.g. Tikhonov regularization. Hence our main concern is with issues of uniqueness and (stabilized) reconstruction algorithms.

The methods that we will discuss in this paper for establishing uniqueness and reconstruction techniques are characterized by two salient features. The first is that the methods that we and our collaborators have developed are in general independent of the boundary conditions, i.e., the material properties of the scatterer. In

particular, for uniqueness and many of the reconstruction algorithms we will present it is not necessary to know a priori whether the far field pattern is associated with (1.1)–(1.5) or (1.7)–(1.10) (or indeed other boundary conditions instead of (1.9) such as the Neumann or impedance boundary conditions). This is important since in many (if not most) situations one cannot assume that the material properties of the scatterer are known a priori. A second feature of our approach to the inverse scattering problem is that both our uniqueness theorems and reconstruction algorithms are based on the exploration of the behavior of the fundamental solution to the Helmholtz equation defined by

$$\Phi(x, z) := \frac{e^{ik|x-z|}}{4\pi|x-z|}, \quad x \neq z, \quad (1.12)$$

in particular the trivial fact that for the source point z on the boundary $\Phi(\cdot, z)$ is not in $H^{1/2}(\partial D)$. In our opinion, the commonality of the mathematical techniques used in uniqueness and reconstruction provides a particularly satisfying approach to the inverse scattering problem that we are considering in this paper.

The plan of our paper is as follows. In the next section we will present uniqueness theorems for the inverse scattering problems associated with (1.1)–(1.5) and (1.7)–(1.10). These uniqueness theorems are motivated by the fundamental paper of Isakov [36] but differ from his approach in a number of significant aspects. We then apply the techniques used to prove uniqueness to establish a number of reconstruction algorithms, in particular the point-source method and the method of singular sources developed by Potthast [56]. A different class of reconstruction algorithms than the point-source and singular source methods is the class of methods based on determining the range of the far field operator $F : L^2(\Omega_1) \rightarrow L^2(\Omega_0)$ defined by

$$(Fg)(\hat{x}) := \int_{\Omega_1} u_\infty(\hat{x}, d)g(d) ds(d), \quad \hat{x} \in \Omega_0, \quad (1.13)$$

where $g \in L^2(\Omega_1)$ (or the range of other operators derived from F). In this regard, we will discuss both the linear sampling method and the factorization method [7, 27]. In particular we will outline a new derivation of the factorization method based on the operator $(F^*F)^{1/4}$ that has recently been developed by Grinberg and Kirsch [27].

2 Uniqueness

The first question to ask in inverse scattering problems is that of uniqueness, i.e., whether a scatterer can be identified from a knowledge of its far field pattern. As an important ingredient of all uniqueness results, we note that by Rellich's lemma

(see [17]) the far field pattern u_∞ uniquely determines the scattered wave u^s . The following classical uniqueness result for sound-soft scatterers is due to Schiffer.

Theorem 2.1 *Assume that D_1 and D_2 are two sound-soft scatterers such that their far field patterns coincide for an infinite number of incident plane waves with distinct directions and one fixed wave number. Then $D_1 = D_2$.*

Proof. Assume that $D_1 \neq D_2$. By Rellich's lemma for each incident plane wave u^i the scattered waves u_1^s and u_2^s for the obstacles D_1 and D_2 coincide in the unbounded component G of the complement of $D_1 \cup D_2$. Without loss of generality, we can assume that $D^* := (\mathbb{R}^3 \setminus G) \setminus \bar{D}_2$ is nonempty. Then u_2^s is defined in D^* , and the total field $u = u^i + u_2^s$ satisfies the Helmholtz equation in D^* and the homogeneous boundary condition $u = 0$ on ∂D^* . Hence, u is a Dirichlet eigenfunction of $-\Delta$ in the domain D^* with eigenvalue k^2 . The proof can now be completed by showing that the total fields for distinct incoming plane waves are linearly independent, since this contradicts the fact that for a fixed eigenvalue there exist only finitely many linearly independent Dirichlet eigenfunctions of $-\Delta$ in $H_0^1(D^*)$. \square

Schiffer's uniqueness result was obtained around 1960 and was never published by Schiffer himself. It appeared as a private communication in the monograph by Lax and Philipps [51]. This is notable since nowadays in a time of permanent evaluation and competition for grants nobody would want to give away such a valuable result as a private communication! Noting that the proof presented in [51] contains a slight technical fault since the fact that the complement of $D_1 \cup D_2$ might be disconnected was overlooked, it is comforting to observe that even eminent authors can also have errors in their books.

By analyticity the far field pattern is completely determined on the whole unit sphere by only knowing it on some surface patch. Therefore, Schiffer's result and, simultaneously, all other results of this section carry over to the case of limited aperture problems where the far field is only known on some open subset Ω_0 of Ω .

Using the strong monotonicity property of the eigenvalues of $-\Delta$, extending Schiffer's ideas, Colton and Sleeman [21] showed that a sound-soft scatterer is uniquely determined by the far field pattern for one incident plane wave under the a priori assumption that it is contained in a ball of radius R such that $kR < \pi$. More recently, exploiting the fact that the wave functions are complex-valued, this bound was improved to $kR < 4.49$ by Gintides [26].

Schiffer's proof cannot be generalized to other boundary conditions. This is due to the fact that the finiteness of the dimension of the eigenspaces for eigenvalues of $-\Delta$ for the Neumann or impedance boundary condition requires the boundary of the intersection D^* from the proof of the Theorem 2.1 to be sufficiently smooth. Therefore, for a long time uniqueness for other inverse scattering problems both for

impenetrable and penetrable obstacles remained open. In 1990, Isakov [36] wrote: *As pointed out by Rainer Kress [45] in his lecture at the Conference on Inverse Problems in Partial Differential Equations in Arcata, there are no uniqueness results for inverse transmission problems, so convergence of numerical algorithms is not justified. In this paper we obtain uniqueness theorems for the inverse transmission problem.* Assuming two different scatterers producing the same far field patterns for all incident directions, Isakov obtained a contradiction by considering a sequence of solutions with a singularity moving towards a boundary point of one scatterer that is not contained in the other scatterer. He used weak solutions and the proofs are technically involved. During a hike in the Dolomites, on a long downhill walk from Rifugio Treviso to Passo Cereda, Andreas Kirsch and Rainer Kress [44] realized that these proofs can be simplified by using classical solutions rather than weak solutions and by obtaining the contradiction by considering pointwise limits of the singular solutions rather than limits of L^2 integrals. Only after this new uniqueness proof was published, it was also observed by the authors that for scattering from impenetrable objects it is not required to know the boundary condition of the scattered wave on the boundary of the scatterer. Furthermore, as stated in the following theorem, one can conclude that in addition to the shape ∂D of the scatterer the boundary condition is also uniquely determined by the far field pattern for infinitely many incident plane waves (see also Alves and Ha-Duong [2], Cakoni and Colton [6] and Kress and Rundell [48]).

We consider boundary conditions of the form $Bu = 0$ on ∂D , where $Bu = u$ for a sound-soft scatterer and $Bu = \partial u / \partial \nu + ik\lambda u$ for the impedance boundary condition. In the latter case, the real-valued function λ is assumed to be continuous and non-negative to ensure well-posedness of the direct scattering problem. In the proof of the uniqueness theorem, in addition to scattering of plane waves, we also need to consider scattering of point sources $\Phi(\cdot, z)$ with source location $z \in \mathbb{R}^3 \setminus \bar{D}$. We denote the corresponding scattered wave by $w^s(\cdot, z)$ and its far field pattern by $w_\infty(\cdot, z)$. Scattering by plane waves and by point sources is related through the *mixed reciprocity relation* (see [46, 56])

$$u^s(z, d) = 4\pi w_\infty(-d, z), \quad z \in \mathbb{R}^3 \setminus \bar{D}, \quad d \in \Omega, \quad (2.1)$$

which is valid both for the sound-soft and impedance boundary condition and also for transmission conditions.

Theorem 2.2 *Assume that D_1 and D_2 are two scatterers with boundary conditions B_1 and B_2 such that the far field patterns coincide for an infinite number of incident plane waves with distinct directions and one fixed wave number. Then $D_1 = D_2$ and $B_1 = B_2$*

Proof. Following Potthast [56] we simplify the approach of Kirsch and Kress through the use of the mixed reciprocity relation (2.1). Let $u_{\infty,1}$ and $u_{\infty,2}$ be the far field patterns for plane wave incidence and let w_1^s and w_2^s be the scattered waves for point source incidence corresponding to D_1 and D_2 , respectively. With (2.1) and two applications of Rellich's lemma, first for scattering of plane waves and then for scattering of point sources, from the assumption $u_{\infty,1}(\hat{x}, d) = u_{\infty,2}(\hat{x}, d)$ for all $\hat{x}, d \in \Omega$ it can be concluded that $w_1^s(x, z) = w_2^s(x, z)$ for all $x, z \in G$. Here, as in the previous proof, G denotes the unbounded component of the complement of $D_1 \cup D_2$.

Now assume that $D_1 \neq D_2$. Then, without loss of generality, there exists $x^* \in \partial G$ such that $x^* \in \partial D_1$ and $x^* \notin \bar{D}_2$. In particular we have

$$z_n := x^* + \frac{1}{n} \nu(x^*) \in G, \quad n = 1, 2, \dots,$$

for sufficiently large n . Then, on one hand we obtain that

$$\lim_{n \rightarrow \infty} B_1 w_2^s(x^*, z_n) = B_1 w_2^s(x^*, x^*),$$

since $w_2^s(x^*, \cdot)$ is continuously differentiable in a neighborhood of $x^* \notin \bar{D}_2$ due to reciprocity and the well-posedness of the direct scattering problem with boundary condition B_2 on ∂D_2 . On the other hand we find that

$$\lim_{n \rightarrow \infty} B_1 w_1^s(x^*, z_n) = \infty,$$

because of the boundary condition $B_1 w_1^s(x^*, z_n) = -B_1 \Phi(x^*, z_n)$ on ∂D_1 . This contradicts $w_1^s(x^*, z_n) = w_2^s(x^*, z_n)$ for all sufficiently large n , and therefore $D_1 = D_2$.

Finally, to establish that $\lambda_1 = \lambda_2$ for the case of two impedance boundary conditions B_1 and B_2 we set $D = D_1 = D_2$ and assume that $\lambda_1 \neq \lambda_2$. Then from Rellich's lemma and the boundary conditions, considering one incident field, we have that

$$\frac{\partial u}{\partial \nu} + ik\lambda_1 u = \frac{\partial u}{\partial \nu} + ik\lambda_2 u = 0 \quad \text{on } \partial D,$$

for the total wave $u = u_1 = u_2$. Hence, $(\lambda_1 - \lambda_2)u = 0$ on ∂D . From this, in view of the fact that $\lambda_1 \neq \lambda_2$, by Holmgren's theorem and the boundary condition we obtain that $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. This leads to the contradiction that the incident field must satisfy the radiation condition. Hence, $\lambda_1 = \lambda_2$. The case when one of the boundary conditions is the sound-soft boundary condition is dealt with analogously. \square

Clearly, the above method exploits the fact that the scattered wave for point source incidence becomes singular at the boundary as the source point approaches a boundary point. It also has been employed by Kirsch and Kress [44] for the

transmission problem and by Hettlich [33] and by Gerlach and Kress [25] for the conductive boundary condition. Here the analysis becomes more involved due to the fact that from the transmission or conductive boundary conditions it is not immediately obvious that the scattered wave becomes singular since the singularity of the incident wave, in principle, could be compensated by a singularity of the transmitted wave. In the proofs this possibility is excluded through a somewhat tedious analysis of boundary integral operators that, in particular, require the fundamental solution both inside and outside the homogeneous scatterer. In electromagnetics corresponding uniqueness results with the above approach were obtained for scattering from perfect conductors by Colton and Kress [17, Theorem 7.1], for scattering from homogeneous dielectrics by Hähner [31], for scattering from homogeneous chiral media by Gerlach [24], and for scattering from homogeneous orthotropic media by Colton, Kress, and Monk [19].

Hähner [32] considered uniqueness for the inverse scattering problem to determine the shape of the scatterer D for the inhomogeneous transmission problem (1.1)–(1.5) that, in particular, includes scattering from an inhomogeneous orthotropic medium. Hähner’s approach restructures Isakov’s [36] original idea in such a way that, in general, it can be applied provided the direct scattering problem is well-posed with a sufficiently regular solution and an associated interior transmission problem is a compact perturbation of a well-posed problem. It differs from the analysis in [44] by using weak solution techniques rather than boundary integral equations and as opposed to both [36, 44] it does not need the fundamental solution inside the scatterer. In the context of this survey it is also notable that the occurrence of the interior transmission problem within the proof connects Hähner’s uniqueness analysis with the mathematical foundations of the linear sampling method as considered in Section 4. The ideas of Hähner have been extended to Maxwell’s equations by Cakoni and Colton [4].

In closing this section we briefly mention a challenging open problem. Although there is widespread belief that the far field pattern for one single incident direction and one single wave number determines a sound-soft scatterer without any additional a priori information, establishing this is still open. Some progress has recently been obtained by Cheng and Yamamoto [11], Allesandrini and Rondi [1], and Liu and Zou [52] who established uniqueness with one incident plane wave for polyhedral scatterers. Assuming that there exist two polyhedral scatterers producing the same far field pattern for one incident plane wave, the main idea of their proofs is to use the reflexion principle to construct a zero field line extending to infinity. However, in view of the fact that the scattered wave tends to zero uniformly at infinity, this contradicts the property that the incident plane wave has modulus one everywhere.

3 Decomposition methods

The main idea of so-called decomposition methods is to break up the inverse obstacle scattering problem into two parts: the first part deals with the ill-posedness by constructing the scattered wave u^s from its far field pattern u_∞ and the second part deals with the nonlinearity by determining the unknown boundary ∂D of the scatterer as the location where the boundary condition for the total field $u^i + u^s$ is satisfied in a least-squares sense. In the decomposition method due to Kirsch and Kress [43], for the first part, enough a priori information on the unknown scatterer D is assumed so one can place a closed surface Γ inside D . Then the scattered field u^s is represented as a single-layer potential

$$u^s(x) = \int_{\Gamma} \varphi(y) \Phi(x, y) ds(y) \quad (3.1)$$

with an unknown density $\varphi \in L^2(\Gamma)$. In this case the far field pattern u_∞ has the representation

$$u_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\Gamma} e^{-ik\hat{x}\cdot y} \varphi(y) ds(y), \quad \hat{x} \in \Omega. \quad (3.2)$$

Given the far field pattern u_∞ , the density φ is now found by solving the integral equation of the first kind (3.2). Due to its analytic kernel, the integral equation (3.2) is severely ill-posed. This reflects the ill-posed nature of the problem to determine u^s from its far field pattern. For a stable numerical solution of (3.2) Tikhonov regularization can be applied.

Given an approximation of the scattered wave by inserting a solution of the regularized form of (3.2) into the potential (3.1), the unknown boundary ∂D is then determined by requiring the sound-soft boundary condition

$$u^i(x) + \int_{\Gamma} \varphi(y) \Phi(x, y) ds(y) = 0, \quad x \in \partial D, \quad (3.3)$$

to be satisfied in a least-squares sense by minimizing the defect over all surfaces contained in some suitable class of admissible boundaries.

Although, in principle, this decomposition method does not make special use of the fundamental solution, we mention it here for several reasons. Firstly, it was invented within the Delaware–Göttingen connection in the late 1980s as an alternative to the dual-space method (see the next section) and has the advantage over this method in that it requires as data the far field pattern for only one incident wave. Secondly, we want to mention two modifications of this approach that have been suggested more recently. One possibility is to view Γ as an approximation for the boundary ∂D and, keeping φ fixed as a regularized solution of (3.2), update Γ via linearizing the boundary condition (3.3) around Γ . Then, in an obvious fashion, the

two steps are repeated iteratively [47, 50]. Another possibility is to first change Γ into ∂D in both (3.2) and (3.3) and then view these equations as a system of two nonlinear equations for the two unknowns ∂D and φ that can be solved by regularized Newton iterations [37, 49]. Since the equations (3.2) and (3.3), with Γ changed into ∂D , can also be related to the reciprocity gap principle, this approach exhibits some connections with the reciprocity gap version of the linear sampling method of the next section. Finally, it should be noted that all of the above approaches do not require the solution of the forward problem within the iteration procedure.

A third reason for briefly considering the decomposition method, and the main reason we have included a description of it in this paper, is that it is closely related to the *point-source method* of Potthast that we are going to now describe. This method was invented by Potthast [53, 54, 56] in 1996 and since then its presentation and motivation has undergone some metamorphosis. In the sequel we adopt the more recent interpretation by Potthast [23, 57, 58] to explain the basic ideas of the method. It starts from Green's representation for the scattered wave for scattering from a sound-soft obstacle

$$u^s(x) = - \int_{\partial D} \frac{\partial u}{\partial \nu}(y) \Phi(x, y) ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \quad (3.4)$$

and its far field pattern

$$u_\infty(\hat{x}) = - \frac{1}{4\pi} \int_{\partial D} \frac{\partial u}{\partial \nu}(y) e^{-ik\hat{x}\cdot y} ds(y), \quad \hat{x} \in \Omega, \quad (3.5)$$

that is, Huygen's principle (see Theorem 3.12 in [17]). For $z \in \mathbb{R}^3 \setminus \bar{D}$ we choose a domain B_z such that $z \notin B_z$ and $\bar{D} \subset B_z$ and approximate the point source $\Phi(\cdot, z)$ by a Herglotz wave function

$$\Phi(y, z) \approx \int_{\Omega} e^{ik y \cdot d} g_z(d) ds(d), \quad y \in B_z \quad (3.6)$$

with kernel g_z . Provided the wave number k^2 is not an eigenvalue for $-\Delta$ in B_z the Herglotz wave functions are dense on $H^{1/2}(\partial B_z)$ [18, 22] and consequently the approximation (3.6) can be achieved uniformly with respect to y on compact subsets of B_z . Then we can insert (3.6) into (3.4) and use (3.5) to obtain

$$u^s(z) \approx 4\pi \int_{\Omega} g_z(\hat{x}) u_\infty(-\hat{x}) ds(\hat{x}) \quad (3.7)$$

as an approximation for the scattered wave u^s . Note that the reconstruction formula (3.7) may be considered as a backprojection of the far field pattern via the kernel g_z . Knowing the scattered wave, the boundary ∂D of the sound-soft scatterer can be

obtained either by finding the location where the boundary condition $u^i + u^s = 0$ is satisfied in a least-squares sense or by visualizing the modulus $|u|$ of the total field.

The approximation (3.6), for example, can be obtained by solving the ill-posed linear integral equation

$$\int_{\Omega} e^{ik y \cdot d} g_z(d) ds(d) = \Phi(y, z), \quad y \in \partial B_z, \quad (3.8)$$

via Tikhonov regularization. Note that, in principle, the integral operator in (3.8) may be interpreted as the L^2 adjoint of the integral operator in (3.2) with Γ replaced by ∂B_z . (This connection has been exploited more recently by Potthast and Schulz [59] to suggest a modification of the classical decomposition method due to Kirsch and Kress that does not require any a priori information and, in addition, overcomes the difficulty that, in general, the integral equation (3.2) is not solvable.) The computational effort of solving (3.8) for a full grid of points z outside the scatterer D can be largely reduced by fixing a reference domain B not containing the origin, for example a ball, and then choose

$$B_z = MB + z$$

that is, first apply an orthogonal matrix M to the reference domain B and then translate it. Straightforward calculations show that if the Herglotz wave function with kernel g approximates the point source $\Phi(\cdot, 0)$ located at the origin with error less than ε with respect to $L^2(\partial B)$ then the Herglotz wave function with kernel

$$g_z(d) = e^{-ik z \cdot d} g(M^* d)$$

approximates the point source $\Phi(\cdot, z)$ located at z with error less than ε with respect to $L^2(\partial B_z)$. Hence, it suffices to solve (3.8) via Tikhonov regularization only once for $z = 0$ and ∂B .

In the practical implementation for a grid of points z_ℓ , $\ell = 1, \dots, L$, the above procedure is carried out for a finite number of matrices M_j , $j = 1, \dots, J$, representing various directions that are used to move the approximating domain around. As an indicator to decide whether the crucial condition $\bar{D} \subset B_z$ is satisfied, one can use two different error levels in the solution of the integral equation (3.8), that is, two different regularization parameters in the Tikhonov regularization and keep the approximation obtained via (3.7) only if the two results are close.

We finish this section by connecting the above ideas to those of sampling methods that will be the general topic of the last two sections of this survey. Roughly speaking, sampling methods rely on choosing an indicator function f on \mathbb{R}^3 such that its value $f(z)$ decides whether z lies inside or outside the scatterer D . For Potthast's [55, 56, 58] *singular source method* this indicator function is given by

$f(z) := w^s(z, z)$ through the value of the scattered wave $w^s(\cdot, z)$ for the singular source $\Phi(\cdot, z)$ as incident field evaluated at the source point z . The values $w^s(z, z)$ will be small for points $z \in \mathbb{R}^3 \setminus \bar{D}$ that are away from the boundary and will blow up when z approaches the boundary due to the singularity of the incident field. Clearly, the singular source method can be viewed as a straightforward numerical implementation of the uniqueness proof for Theorem 2.2.

Assuming the far field pattern for plane wave incidence to be known for all incident and observation directions, the indicator function $w^s(z, z)$ can be obtained by two applications of the backprojection (3.7) and the mixed reciprocity principle (2.1). Combining (2.1) and (3.7) we obtain the approximation

$$w_\infty(-d, z) = \frac{1}{4\pi} u^s(z, d) \approx \int_{\Omega} g_z(\hat{x}) u_\infty(-\hat{x}, d) ds(\hat{x}).$$

Inserting this into the backprojection (3.7) as applied to w^s yields the approximation

$$w_s(z, z) \approx \int_{\Omega} \int_{\Omega} g_z(d) g_z(\hat{x}) u_\infty(-\hat{x}, d) ds(\hat{x}), ds(d). \quad (3.9)$$

We note that, as opposed to the point source method described above, for the singular source method the boundary condition need not be known. Furthermore, using higher order multipoles as the incident field instead of the fundamental solution, it can also be applied to the inverse medium problem. We also note that if we use the reflexion operator $R : L^2(\Omega) \rightarrow L^2(\Omega)$ defined by $(Rg)(\hat{x}) = g(-\hat{x})$ the approximation (3.9) can be expressed in terms of the far field operator F as introduced in (1.13) (in the special case $\Omega_0 = \Omega_1 = \Omega$) by the $L^2(\Omega)$ scalar product

$$w_s(z, z) \approx (Fg_z, R\bar{g}_z).$$

The *probe method* as suggested by Ikehata [34, 35] uses as indicator function an energy integral for $w^s(\cdot, z)$ instead of the point evaluation $w_s(z, z)$. In this sense, it follows the uniqueness proof of Isakov whereas the singular source method mimics the uniqueness proof of Kirsch and Kress.

4 The linear sampling method

The linear sampling method is a technique for determining the support D of a scattering object by solving the *far field equation*

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z), \quad \hat{x} \in \Omega_0, \quad (4.1)$$

where the far field operator F is defined by (1.13) and

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x} \cdot z} \quad (4.2)$$

is the far field pattern of the radiating fundamental solution to the Helmholtz equation with source point $z \in \mathbb{R}^3$ given by (1.12). However, the origins of the linear sampling method go back to the *dual space method* for determining D as developed by Colton and Monk during the years 1985–1990 and described in our book [17]. Focusing on the scattering problem (1.7)–(1.10), the dual space method (in the simplest version) seeks to determine D by assuming that D contains the origin and then solving

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, 0), \quad \hat{x} \in \Omega_0.$$

Having determined g through the use of Tikhonov regularization, ∂D is then determined by first constructing the *Herglotz wave function* with kernel g defined by

$$(Hg)(x) := \int_{\Omega_1} e^{ikx \cdot d} g(d) ds(d), \quad x \in \mathbb{R}^3, \quad (4.3)$$

and then using nonlinear optimization techniques to determine ∂D as the locus of points satisfying

$$(Hg)(x) + \Phi(x, 0) = 0.$$

The linear sampling method was born in Kennedy Airport in New York while Andreas Kirsch had several hours to wait for his flight back to Germany. On his laptop he had the dual space method programmed (in \mathbb{R}^2) and for amusement he shifted the origin to arbitrary points z . In doing this, he noted that $\|g(\cdot, z)\|_{L^2(\Omega_1)}$ became unbounded as z approached ∂D and, by plotting the level curves of $\|g(\cdot, z)\|_{L^2(\Omega_1)}$, the shape ∂D miraculously appeared! Soon after his return to Germany, David Colton met him at the University of Erlangen and the paper [14] was written. In particular, the linear sampling method uses Tikhonov regularization to solve (4.1) for z on a grid of points containing D and then looks for the locus of points where $\|g(\cdot, z)\|_{L^2(\Omega_1)}$ becomes unbounded.

A problem with the linear sampling method as described above is that in general there does not exist a solution of (4.1) for noise free data and hence it is not clear what “solution” is obtained by using Tikhonov regularization (However, see [3]). Hence the mathematical justification of the linear sampling method is based on approximation arguments, in particular on the fact that, with respect to the $H^1(D)$ norm, the set of Herglotz wave functions is dense in the space of solutions to the Helmholtz equation [18, 22]. (For the case of limited aperture far field data (i.e. $\Omega_1 \neq \Omega$) more is needed, in particular that the denseness property remains true if the kernels of the set of Herglotz wave functions have their support on a fixed compact subset of the unit sphere Ω [5]). Given this approximation property, and the fact that if k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D then the far field operator corresponding to (1.7)–(1.10) is injective with dense range [17], one can prove the following theorem:

Theorem 4.1 Assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D and let F be the far field operator corresponding to (1.7)–(1.10). Then

1. For $z \in D$ and a given $\epsilon > 0$ there exists a $g_\epsilon^z \in L^2(\Omega_1)$ such that

$$\|Fg_\epsilon^z - \Phi_\infty(\cdot, z)\|_{L^2(\Omega_0)} < \epsilon$$

and the corresponding Herglotz wave function Hg_ϵ^z converges to a solution of

$$\Delta u + k^2 u = 0 \quad \text{in } D \quad (4.4)$$

$$u = -\Phi(\cdot, z) \quad \text{on } \partial D \quad (4.5)$$

in $H^1(D)$ as $\epsilon \rightarrow 0$.

2. For $z \in D$ and a fixed $\epsilon > 0$ we have that

$$\lim_{z \rightarrow \partial D} \|Hg_\epsilon^z\|_{H^1(D)} = \infty \quad \text{and} \quad \lim_{z \rightarrow \partial D} \|g_\epsilon^z\|_{L^2(\Omega_1)} = \infty.$$

3. For $z \in \mathbb{R}^3 \setminus \bar{D}$ and a given $\epsilon > 0$, every $g_z^\epsilon \in L^2(\Omega_1)$ that satisfies

$$\|Fg_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2(\Omega_0)} < \epsilon$$

is such that

$$\lim_{\epsilon \rightarrow 0} \|Hg_z^\epsilon\|_{H^1(D)} = \infty \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \|g_z^\epsilon\|_{L^2(\Omega_1)} = \infty.$$

This theorem is the basis of the linear sampling method for the scattering problem (1.7)–(1.10). An analogous theorem for the linear sampling method for the scattering problem (1.1)–(1.5) is more difficult to establish. The following theorem indicates why this is true (For simplicity we will assume that we have full aperture data, i.e. $\Omega_0 = \Omega_1 = \Omega$).

Theorem 4.2 Let $F : L^2(\Omega) \rightarrow L^2(\Omega)$ be the far field operator corresponding to the scattering problem (1.1)–(1.5). Then F is injective with dense range if and only if there does not exist a solution $v, w \in H^1(D)$ of the interior transmission problem

$$\nabla \cdot A \nabla w + k^2 n w = 0 \quad \text{in } D \quad (4.6)$$

$$\Delta v + k^2 v = 0 \quad \text{in } D \quad (4.7)$$

$$w = v, \quad \nu \cdot A \nabla w = \nu \cdot \nabla v \quad \text{on } \partial D \quad (4.8)$$

such that v is a Herglotz wave function with kernel $g \neq 0$.

Proof. By the reciprocity relation $u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x})$ (see Theorem 8.8 of [17]), it is easily verified that the adjoint operator F^* satisfies

$$(F^*h)(d) = \overline{(Fg)(-d)}$$

where $g(\hat{x}) = \overline{h(-\hat{x})}$. Since $N(F^*)^\perp = \overline{F(L^2(\Omega))}$ where $N(F^*)$ denotes the null space of F^* , we see that it suffices to prove injectivity of F .

Assume now that $(Fg)(\hat{x}) = 0$ for $\hat{x} \in \Omega$. Then by Rellich's lemma [17] we have that

$$U^s(x) := \int_{\Omega} u^s(x, d)g(d) ds(d) \quad (4.9)$$

is equal to zero for $x \in \mathbb{R}^3 \setminus \bar{D}$. Hence if $v = Hg$ then from (1.1)–(1.5) we see that $U = U^s + v$ satisfies

$$\nabla \cdot A \nabla W + k^2 n W = 0 \quad \text{in } D$$

$$\Delta v + k^2 v = 0 \quad \text{in } D$$

$$w = v, \quad \nu \cdot A \nabla W = \nu \cdot \nabla v \quad \text{on } \partial D$$

for

$$W(x) := \int_{\Omega} w(x, d)g(d) ds(d).$$

Hence, by the hypothesis of the theorem, $g = 0$, that is, F is injective.

Conversely, it is easily verified that if there exists a nontrivial solution to (4.6)–(4.8) such that v is a Herglotz wave function with kernel $g \neq 0$ then $(Fg)(\hat{x}) = 0$ for $\hat{x} \in \Omega$ (multiply (1.2)–(1.4) by g , integrate over Ω and note that if U^s is defined by (4.9) then $U^s(x) = 0$ for $x \in \mathbb{R}^3 \setminus \bar{D}$ by the uniqueness of the solution to (1.1)–(1.5)). The theorem is now proved. \square

Corollary 4.3 *Assume that $\xi \cdot \Im \overline{A(x_0)} \xi < 0$ in a point $x_0 \in D$ for all $\xi \in \mathbb{C}^3$ or $\Im n(x_0) > 0$ for a point $x_0 \in D$. Then the far field operator corresponding to the scattering problem (1.1)–(1.5) is injective with dense range.*

Proof. Let $v, w \in H^1(D)$ be a solution to the interior transmission problem (4.6)–(4.8). Then by the divergence theorem and (4.6)–(4.8) we have that

$$\int_D \nabla \bar{w} \cdot A \nabla w dx = k^2 \int_D n |w|^2 dx - k^2 \int_D |v|^2 dx + \int_D |\nabla v|^2 dx.$$

Hence by the assumptions on A and n given in the introduction we have that

$$\Im \left(\int_D \nabla \bar{w} \cdot A \nabla w dx \right) = 0 \quad \text{and} \quad \Im \int_D n |w|^2 dx = 0.$$

By the hypothesis of the corollary and the continuity of A and n we can now conclude by the unique continuation principle that $w(x) = 0$ for $x \in D$ and hence $v(x) = 0$ also for $x \in D$. The corollary now follows from Theorem 4.2. \square

If $k > 0$ is such that the interior transmission problem has a nontrivial solution then k is called a *transmission eigenvalue*. From the above corollary it is seen that k can be a transmission eigenvalue only if $\Im A(x) = 0$ and $\Im n(x) = 0$ for $x \in D$. Except in the special case when $A = I$ and n is spherically symmetric (c.f. Theorem 8.13 of [17]), it is an open problem whether or not transmission eigenvalues exist. However, from the point of view of the linear sampling method, it is sufficient to know that if transmission eigenvalues exist they form a discrete set. For the case $A = I$ and $n(x) > 1$ for $x \in D$ (or $n(x) < 1$ for $x \in D$) this has been established by Colton, Kirsch and Päiväranta [15] and Rynne and Sleemann [60]. Analogous results for the case when $A \neq I$ have been given by Cakoni, Colton and Haddar [8]. In addition to the discreteness of the set of transmission eigenvalues, in order to establish the analogue of Theorem 4.1 for the scattering problem (1.1)–(1.5) it is necessary to establish the well-posedness of an inhomogeneous interior transmission problem analogous to the interior Dirichlet problem (4.4)–(4.5) and this has been done by Colton, Piana and Potthast [20] for the case when $A = I$ and by Cakoni, Colton and Haddar [8] for the case when $A \neq I$. Putting all of this together, one now arrives at a result analogous to Theorem 4.1 for the case of the scattering problem (1.1)–(1.5) where instead of assuming that k^2 is not a Dirichlet eigenvalue one must now assume that k is not a transmission eigenvalue and instead of the inhomogeneous interior Dirichlet problem (4.4)–(4.5) one has an inhomogeneous interior transmission problem (c.f. [7]).

The linear sampling method has also been extended to the case of Maxwells's equations by Colton, Haddar and Monk [13], Haddar and Monk [30], Cakoni and Colton [5], Cakoni, Colton and Monk [9] and Haddar [29]. For a brief description of these more recent developments and further references, see the last chapter of [7].

We note that in the first two papers on the linear sampling method [14, 20], this method for solving the inverse scattering problem was called the “simple method” since it was a simple linear method to solve a rather complicated nonlinear inverse problem. At the time, Michele Piana from Genova was visiting David Colton at the University of Delaware and was searching for a permanent position in Italy. Over a beer at the local pub, Michele said it would perhaps not reflect well on his application if the method he helped develop was called a “simple” method. So it was suggested that he come up with a better name and so he did: the *linear sampling method* since it is a method by which you examine a function $g(\cdot, z)$ at certain “sampling” points $z = z_1, z_2, \dots, z_L$. Of course not everyone liked this choice since sampling methods typically refer to something quite different, but nevertheless the name has stuck.

We close this section by briefly describing a version of the linear sampling method based on the reciprocity gap functional which is applicable to objects situated in a piecewise homogeneous background medium. In particular, assume that the unknown scattering object is embedded in a portion B of a piecewise inhomogeneous medium where the index of refraction is constant with wave number k . Let $B_0 \subset B$ be a domain in B having a smooth boundary ∂B_0 such that $D \subset B_0$, let ν be the unit outward normal to ∂B_0 and define the *reciprocity gap functional* by

$$R(u, v) := \int_{\partial B_0} \left(u \frac{\partial u}{\partial \nu} - v \frac{\partial v}{\partial \nu} \right) ds \quad (4.10)$$

where u and v are solutions of the Helmholtz equation in $B_0 \setminus \overline{D}$. In particular, we want u to be the total field due to a point source situated at $x_0 \in B \setminus \overline{B_0}$ and $v = v_g$ to be a Herglotz wave function with kernel g . We then consider the integral equation

$$R(u, v_g) = R(u, \Phi_z) \quad (4.11)$$

where $\Phi_z = \Phi(\cdot, z)$ is the radiating fundamental solution (1.12) and $u = u(\cdot, x_0)$ where x_0 is now assumed to be on a smooth surface C in $B \setminus \overline{B_0}$ that is homotopic to ∂B_0 . If D is a sound-soft obstacle, we assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D and if D is an inhomogeneous medium we assume that the medium is isotropic and absorbing. Then the following theorem is valid [12]:

Theorem 4.4 *Assume that the above assumptions on D are satisfied. Then*

1. *If $z \in D$ then there exists a sequence $\{g_n\}$, $g_n \in L^2(\Omega)$, such that*

$$\lim_{n \rightarrow \infty} R(u, v_{g_n}) = R(u, \Phi_z), \quad x_0 \in C,$$

and v_{g_n} converges in $L^2(D)$.

2. *If $z \in B_0 \setminus D$ then for every sequence $\{g_n\}$, $g_n \in L^2(\Omega)$, such that*

$$\lim_{n \rightarrow \infty} R(u, v_{g_n}) = R(u, \Phi_z), \quad x_0 \in C,$$

we have that

$$\lim_{n \rightarrow \infty} \|v_{g_n}\|_{L^2(D)} = \infty.$$

In particular, Theorem 4.4 provides a method for determining D from a knowledge of the Cauchy data of u on ∂B_0 in a manner analogous to that of the linear sampling method.

Numerical examples using Theorem 4.4 to determine D from Cauchy data on ∂B_0 , as well as to objects buried in an absorbing half-space, can be found in [12]. The extension of Theorem 4.4 to Maxwell's equations, together with numerical examples, can be found in [10].

5 The factorization method

As mentioned in the previous section, a problem with the linear sampling method from the mathematical point of view is that in general there does not exist a solution of the far field equation for noise free data. The *factorization method* as developed by Kirsch [38]–[42] overcomes this problem under certain assumptions, e.g. full aperture scattering data or, at best, the case when $\hat{x}, d \in \Omega_0 \subset \Omega$. Here, based on the presentation in the forthcoming book by Grinberg and Kirsch [27], we present a brief outline of the simplest version of the factorization method, i.e., that corresponding to the case of the operator $(F^*F)^{1/4}$ and the scattering problem (1.7)–(1.10) corresponding to a sound-soft obstacle. As in the reconstruction methods presented in Sections 3 and 4, the idea is again based on the fact that the fundamental solution to the Helmholtz equation is singular at the source point.

We begin with a slight generalization of the scattering problem (1.7)–(1.10), i.e., given the boundary data $f \in H^{1/2}(\partial D)$, find a solution $v \in H_{loc}^1(\mathbb{R}^3 \setminus \bar{D})$ satisfying

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \quad (5.1)$$

$$v = f \quad \text{on } \partial D \quad (5.2)$$

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial v}{\partial r} - ikv \right) = 0. \quad (5.3)$$

The existence of a unique solution to (5.1)–(5.3) is well known (c.f. [17]). We now define the operator $G : H^{1/2}(\partial D) \rightarrow L^2(\Omega)$ by $Gf = v_\infty$ where v_∞ is the far field pattern of the solution of the boundary value problem (5.1)–(5.3) and note that $\Phi_\infty(\cdot, z) \in G(H^{1/2}(\partial D))$ if and only if $z \in D$. It is furthermore easy to verify that G is compact, injective and has dense range. The factorization method gets its name from the following theorem. In the theorem $S : H^{-1/2}(\partial D) \rightarrow H^{1/2}(\partial D)$ denotes the single layer potential defined by

$$(S\varphi)(x) := \int_{\partial D} \Phi(x, y) \varphi(y) ds(y), \quad x \in \partial D. \quad (5.4)$$

Theorem 5.1 *Let F be the far field operator corresponding to (1.7)–(1.10) and let G and S be defined as above. Then*

$$F = -4\pi GS^*G^*$$

where $G^* : L^2(\Omega) \rightarrow H^{-1/2}(\partial D)$ and $S^* : H^{-1/2}(\partial D) \rightarrow H^{1/2}(\partial D)$ are the adjoints of G and S respectively with respect to $L^2(\Omega)$ and the duality pairing $\langle H^{-1/2}(\partial D), H^{1/2}(\partial D) \rangle$.

Proof. Define $H : L^2(\Omega) \rightarrow H^{1/2}(\partial D)$ by

$$(Hg)(x) := \int_{\Omega} e^{ikx \cdot d} g(d) ds(d), \quad x \in \partial D, \quad (5.5)$$

i.e. Hg is the trace on ∂D of the Herglotz wave function with kernel g . The adjoint $H^* : H^{-1/2}(\partial D) \rightarrow L^2(\Omega)$ is given by

$$(H^*\varphi)(\hat{x}) = \int_{\partial D} e^{-ik\hat{x} \cdot y} \varphi(y) ds(y), \quad \hat{x} \in \Omega, \quad (5.6)$$

and note that this, up to a factor 4π , is the far field pattern of the single layer potential (5.4). Hence, for continuous densities φ , we have that $H^*\varphi = 4\pi GS\varphi$ and hence, by a denseness argument, $H^* = 4\pi GS$ and therefore

$$H = 4\pi S^*G^*. \quad (5.7)$$

But Fg is the far field pattern of (5.1)–(5.3) corresponding to the boundary values $-Hg$ on ∂D , and hence

$$Fg = -GHg. \quad (5.8)$$

(5.7) and (5.8) now imply the theorem. \square

Theorem 5.1 suggests that the following theorem from functional analysis [27] will be relevant for our purposes.

Theorem 5.2 *Let X be a Hilbert space with duality pairing $\langle \cdot, \cdot \rangle$. Furthermore, let H be a Hilbert space with inner product (\cdot, \cdot) and let $F : H \rightarrow H$ and $B : X \rightarrow H$ be bounded linear operators such that*

1. $F = BAB^*$ for some bounded linear operator $A : X^* \rightarrow X$.
2. There exists $c > 0$ such that

$$|\langle \varphi, A\varphi \rangle| \geq c\|\varphi\|^2 \quad \text{for all } \varphi \in B^*(H).$$

Then for any $\varphi \in H$, $\varphi \neq 0$, we have that

$$\varphi \in B(X) \iff \inf\{ |(\psi, F\psi)| : \psi \in H, (\psi, \varphi) = 1 \} > 0.$$

We will now apply Theorem 5.2 to the factorization of the far field operator given in Theorem 5.1. In particular, we choose $H = L^2(\Omega)$, $X = H^{1/2}(\partial D)$, $B = G$ and $A = -S^*$. To this end, we note that if k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D

then there exists a positive constant c such that with respect to the duality pairing $\langle H^{-1/2}(\partial D), H^{1/2}(\partial D) \rangle$ we have that [27]

$$|\langle \varphi, S\varphi \rangle| \geq c \|\varphi\|_{H^{-1/2}(\partial D)}^2 \quad (5.9)$$

for all $\varphi \in H^{-1/2}(\partial D)$. From this fact we now see that the following corollary immediately follows from Theorem 5.2 (noting that $\langle \varphi, S^*\varphi \rangle = \overline{\langle \varphi, S\varphi \rangle}$):

Corollary 5.3 *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D . Then if F is the far field operator corresponding to (1.7)–(1.10) then for any $\varphi \in H$, $\varphi \neq 0$, we have that*

$$\varphi \in G(L^2(\partial D)) \iff \inf\{|\langle \psi, F\psi \rangle| : \psi \in L^2(\Omega), (\psi, \varphi) = 1\} > 0.$$

Since $\Phi_\infty(\cdot, z)$ is in the range of G if and only if $z \in D$, the above corollary provides a variational method for determining D from a knowledge of the far field pattern u_∞ [40]. However such an approach is very time consuming since it involves solving a minimization problem for every sampling point z . A more efficient approach, and one more closely related to the linear sampling method, is based on the following theorem which can be proved using Theorem 5.2 [27].

Theorem 5.4 *Let H and X be Hilbert spaces and let the compact operator $F : H \rightarrow H$ have a factorization of the form*

$$F = BAB^*$$

with bounded linear operators $B : X \rightarrow H$ and $A : X^ \rightarrow X$ such that $\Im \langle \varphi, A\varphi \rangle \neq 0$ for $\varphi \in \overline{B^*(H)}$ and $\varphi \neq 0$. Assume that*

1. *A is of the form $A = A_0 + C$ where C is compact and A_0 is self-adjoint and coercive on $B^*(H)$.*
2. *F is injective and, for some $\gamma > 0$, $I + i\gamma F$ is unitary.*

*Then the ranges of B and $(F^*F)^{1/4}$ coincide.*

In order to apply Theorem 5.4 to the inverse scattering problem associated with (1.7)–(1.10) we set $H = L^2(\Omega)$, $X = H^{1/2}(\partial D)$, $B = G$ and $A = -S^*$ and need to verify the conditions of the above theorem. The first condition follows from the coercivity of the single layer potential for the Laplace equation [27]. For the second condition, we have already noted in the previous section that F is injective, provided k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D . Furthermore, from [16] we have the basic identity

$$F - F^* = \frac{ik}{2\pi} F^* F \quad (5.10)$$

where F^* denotes the L^2 adjoint of F and from this one easily sees by direct computation that the *scattering operator*

$$S := I + \frac{ik}{2\pi} F \quad (5.11)$$

is unitary. Hence, we have the following corollary to Theorem 5.4 [38]:

Corollary 5.5 *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D . Then the ranges of G and $(F^*F)^{1/4}$ coincide.*

In particular, we have that if k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D we have that

$$z \in D \iff \Phi_\infty(\cdot, z) \in (F^*F)^{1/4}(L^2(\Omega)). \quad (5.12)$$

This means that, in contrast to the linear sampling method, if Tikhonov regularization (with the regularization parameter chosen by the Morozov discrepancy principle) is used to solve the operator equation

$$(F^*F)^{1/4}g_z = \Phi_\infty(\cdot, z) \quad (5.13)$$

with noisy data u_∞ , then $\|g_z\|$ converges as the noise level tends to zero if and only if $z \in D$.

The factorization method for an inhomogeneous medium has been considered in [39], where the analysis is again based on the interior transmission problem. The case of Maxwell's equation for an inhomogeneous medium was investigated in [42], although the factorization method for a perfect conductor remains an open problem as is the problem of the discreteness of the set of transmission eigenvalues. Finally, it is possible to remove the assumption that the scattering operator is unitary (i.e. F is normal) and thus allow the scattering object to be absorbing [41, 42].

Acknowledgment. This research was supported in part by grants from the Deutsche Forschungsgemeinschaft and the US Air Force Office of Scientific Research and was written while DC was visiting the University of Göttingen in November, 2005.

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