Georg-August-Universität Göttingen

Optimal control of singularly perturbed advection-diffusion-reaction problems

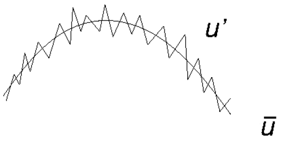
Lube, G., Tews, B.

Nr. 2008-14

Preprint-Serie des
Instituts für Numerische und Angewandte Mathematik
Lotzestr. 16-18
D - 37083 Göttingen

Optimal control of singularly perturbed advection-diffusion-reaction problems

G. Lube and B. Tews



Institute for Numerical and Applied Mathematics
Georg-August-University of Göttingen
D-37083 Göttingen, Germany

International Conference on Boundary and Interior Layers 2008
University of Limerick (Irland)

Outline

1. A singularly perturbed mixed boundary value problem
2. Continuous linear-quadratic optimization problem
3. Stabilized discrete optimality system
4. A-priori error analysis for optimal control problem
5. Numerical experiments
6. Regularized Dirichlet control

Outline

Problem statement

- $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$ – bounded polyhedral (!) domain
- $\partial\Omega = \overline{\Gamma_R \cup \Gamma_D}, \Gamma_D \cap \Gamma_R = \emptyset$ – Lipschitz boundary

Mixed elliptic BVP:

$$\begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + \sigma u &= \tilde{f} \quad \text{in } \Omega, \\ \varepsilon\nabla u\cdot\mathbf{n} + \beta u &= \tilde{g} \quad \text{on } \Gamma_R \\ u &= 0 \quad \text{on } \Gamma_D. \end{aligned} \tag{1}$$

Variational form:

Find $u \in V := \{v \in H^1(\Omega) : \; v|_{\Gamma_D} = 0\}$, s.t. $\quad a(u,v) = f(v) \quad \forall v \in V.$

$$\begin{aligned} a(u,v) &:= \varepsilon(\nabla u, \nabla v)_\Omega + (\mathbf{b}\cdot\nabla u + \sigma u, v)_\Omega + (\beta u, v)_{\Gamma_R} \\ f(v) &:= (\tilde{f}, v)_\Omega + (\tilde{g}, v)_{\Gamma_R}. \end{aligned}$$

Solvability of the mixed BVP

Lemma 1:

$\exists!$ solution $u \in H^1(\Omega)$ of mixed BVP under the assumptions:

i) $b_i \in L^\infty(\Omega),\ i \in \{1, \cdots, d\}, \quad \tilde{f} \in L^2(\Omega), \quad \tilde{g} \in L^2(\Gamma_R),\ \beta \in L^\infty(\Gamma_R),$

ii) $\epsilon > 0,\ \sigma \geq 0$ and $\nabla \cdot \mathbf{b} = 0$ a.e. in Ω,

iii) $\beta \geq 0$ and $\tilde{\beta} := \beta + \frac{1}{2}(\mathbf{b} \cdot \mathbf{n}) \geq \beta_0 \geq 0$ on Γ_R,

iv) Let at least one of the following conditions be valid:

$$(iv)_1 \quad \mu_{n-1}(\Gamma_D) > 0, \qquad (iv)_2 \quad \sigma > 0 \text{ and } \beta_0 > 0$$

Problem:

Solution of mixed BVP in a polyhedral domain is in general not in $W^{2,p}(\Omega)$.

Regularity of the solution I

- $\mathcal{S}$ – set of points (for $d = 2$) or edges (for $d = 3$) which subdivide polyhedral boundary $\partial\Omega$ into smooth disjoint connected components.
- Weighted Sobolev spaces $V_\beta^{k,p}(\Omega)$ – closure of $C^\infty(\Omega)$ w.r.t.

$$\|v\|_{V_\beta^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \int_\Omega r^{p(\beta - k + |\alpha|)} |D^\alpha u|^p \, dx \right)^{\frac{1}{p}}$$

  where $r = r(x) = \mathrm{dist}(x, \mathcal{S})$, $\beta \in \mathbb{R}$, $k \in \mathbb{N}$ and $p > 1$.
- Parameter β defined via eigenvalues of certain eigenvalue problems (in local coordinate systems at parts of $\mathcal{S}$)

Sufficient conditions for the solution of mixed BVP to belong to $V_\beta^{k,p}(\Omega)$, see Grisvard [1992, 1996], Kufner [1987].

Regularity of the solution II

- **Here:** Sobolev-Slobodeckij spaces on $G \subseteq \Omega$

$$W^{k+\lambda,p}(G) := \{v \in W^{k,p}(G) : \underbrace{\Big(\sum_{|\alpha|=k} \int_G \int_G \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{\|x-y\|^{d+p\lambda}} dx\, dy \Big)^{\frac{1}{p}}}_{=:\|u\|_{k+\lambda,p,G}}$$

with $k \in \mathbb{N}_0$, $\lambda \in [0,1)$, $p \in (1,\infty)$ and obvious variant for $p = \infty$.

- $W^{k+\lambda,p}(\Gamma_R)$ similarly defined.

Remark

- *Continuous embeddings* $V^{2,2}_\beta(\Omega) \subset W^{\frac{d}{2}+\kappa,2}(\Omega) \subset C(\overline{\Omega})$ *valid for* $\beta < 2 - \frac{d}{2} + \kappa,\ \kappa > 0.$
- *Dirichlet case* $\partial\Omega = \Gamma_D$*:* $\beta \le \frac{1}{2} + \kappa, \kappa > 0.$
- *Motivates regularity assumption* $u \in W^{1+\lambda,2}(\Omega),\ 1 + \lambda > d/2$

Outline

Linear-quadratic optimization problem

Minimization problem: $\min J(u, q_\Omega, q_\Gamma), \ \ (u, q_\Omega, q_\Gamma) \in V \times Q_\Omega \times Q_\Gamma$

- $J(u, q_\Omega, q_\Gamma) = \frac{\lambda_\Omega}{2}\|u - u_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda_\Gamma}{2}\|u - u_\Gamma\|^2_{L^2(\Gamma_R)} + \frac{\alpha_\Omega}{2}\|q_\Omega\|^2_{L^2(\Omega)} + \frac{\alpha_\Gamma}{2}\|q_\Gamma\|^2_{L^2(\Gamma_R)}$
- $\lambda_\Omega, \lambda_\Gamma \geq 0$ with $\lambda_\Omega^2 + \lambda_\Gamma^2 > 0$ and $\alpha_\Omega, \alpha_\Gamma \geq 0$ with $\alpha_\Omega^2 + \alpha_\Gamma^2 > 0$
- $V = \{v \in H^1(\Omega) : \ u|_{\Gamma_D} = 0\}, \quad Q_\Omega = Q_\Omega = L^2(\Omega), \quad Q_\Gamma = Q_\Gamma = L^2(\Gamma_R)$

subject to elliptic BVP with distributed control and Robin boundary control

$$
\begin{aligned}
-\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + \sigma u &= f + q_\Omega && \text{in } \Omega,\\
\varepsilon\nabla u\cdot\mathbf{n} + \beta u &= g + q_\Gamma && \text{on } \Gamma_R\\
u &= 0 && \text{on } \Gamma_D.
\end{aligned}
$$

Theorem 1:

Assumptions of Lemma 1 $\rightsquigarrow$ $\exists\,!$ *solution* $(\bar{u}, \bar{q}_\Omega, \bar{q}_\Gamma) \in V \times Q_\Omega \times Q_\Gamma$

Reduced optimization problem

- Linear continuous solution operator

$$S : L^2(\Omega) \times L^2(\Gamma_R) \to V, \qquad u = S(q_\Omega + f, q_\Gamma + g).$$

well-defined as mixed BVP admits unique solution.

- Affine linear mapping $(q_\Omega, q_\Gamma) \mapsto u$

Reduced cost functional: $u = S(q_\Omega + f, q_\Gamma + g), \;\; u|_{\Gamma_R} = \gamma \circ S(q_\Omega + f, q_\Gamma + g)$

$$\begin{aligned} j(q_\Omega, q_\Gamma) \;&:=\; J(q_\Omega, q_\Gamma, S(q_\Omega, q_\Gamma)) = \frac{\lambda_\Omega}{2} \|S(q_\Omega + f, q_\Gamma + g) - u_\Omega\|^2_{0;\Omega} \\ &+ \;\frac{\lambda_\Gamma}{2} \|\gamma \circ S(q_\Omega + f, q_\Gamma + g) - u_\Gamma\|^2_{0;\Gamma_R} + \frac{\alpha_\Omega}{2} \|q_\Omega\|^2_{0;\Omega} + \frac{\alpha_\Gamma}{2} \|q_\Gamma\|^2_{0;\Gamma_R} \end{aligned}$$

Reduced optimization problem:

$$\text{Minimize} \quad j(q_\Omega, q_\Gamma), \quad (q_\Omega, q_\Gamma) \in Q_\Omega \times Q_\Gamma.$$

Optimality system (KKT-system) I

Notation: Optimal control $(\overline{q}_\Omega, \overline{q}_\Gamma)$ and optimal state $\overline{u} = S(\overline{q}_\Omega + f, \overline{q}_\Gamma + g)$

Lemma 2: Necessary and sufficient optimality conditions

State problem for optimal state $\overline{u}$:

$$\text{Find } \overline{u} \in V : \quad a(\overline{u}, v) = (f + \overline{q}_\Omega, v)_\Omega + (g + \overline{q}_\Gamma, v)_{\Gamma_R} \quad \forall v \in V,$$

$$a(u,v) := \varepsilon(\nabla u, \nabla v)_\Omega + (\mathbf{b} \cdot \nabla u, v)_\Omega + \sigma(u,v)_\Omega + (\beta u, v)_{\Gamma_R}.$$

Adjoint state problem for optimal adjoint state $\overline{p} \in V$:

$$\text{Find } \overline{p} \in V : \quad a_{adj}(\overline{p}, v) = \lambda_\Omega(\overline{u} - u_\Omega)_\Omega + \lambda_\Gamma(\overline{u} - u_\Gamma)_{\Gamma_R} \quad \forall v \in V,$$

$$a_{adj}(p,v) := \varepsilon(\nabla p, \nabla v)_\Omega - (\mathbf{b} \cdot \nabla p, v)_\Omega + \sigma(p,v)_\Omega + ((\beta + \mathbf{b} \cdot \mathbf{n})p, v)_{\Gamma_R}.$$

Complementary conditions:

$$\alpha_\Omega \overline{q}_\Omega + \overline{p} = 0 \text{ in } \Omega, \qquad \alpha_\Gamma \overline{q}_\Gamma + \overline{p} = 0 \text{ on } \Gamma_R.$$

Optimality system (KKT-system) II

Second order derivatives of $j(q_\Omega, q_\Gamma)$ independent of (q_Ω, q_Γ) and satisfy

$$\begin{aligned} D_{q_\Omega q_\Omega} j(q_\Omega, q_\Gamma) \cdot (k_\Omega, k_\Omega) &\geq \alpha_\Omega \|k_\Omega\|^2_{0,\Omega}, \quad \forall k_\Omega \in Q_\Omega \\ D_{q_\Gamma q_\Gamma} j(q_\Omega, q_\Gamma) \cdot (k_\Gamma, k_\Gamma) &\geq \alpha_\Gamma \|k_\Gamma\|^2_{0,\Gamma_R}, \quad \forall k_\Gamma \in Q_\Gamma. \end{aligned}$$

Assumption:

(A.0) Solution u of mixed BVP (1) belongs to $W^{1+\lambda,2}(\Omega)$.

Lemma 2: Regularity of optimal control

Assume that $\alpha_\Omega, \alpha_\Gamma > 0$ and sufficiently smooth data $f, g, \beta, u_\Omega, u_\Gamma$. Then assumption A.0 implies

$$(\overline{u}, \overline{p}, \overline{q}_\Omega, \overline{q}_\Gamma) \in [W^{1+\lambda,2}(\Omega)]^3 \times W^{\frac{1}{2}+\lambda,2}(\Gamma_R) \text{ with } 1+\lambda > \frac{d}{2}$$

Outline

Finite element spaces

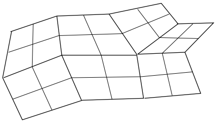
- Shape-regular, admissible decomposition $\mathcal{T}_h$ of Ω into d-dimensional simplices, quadrilaterals ($d = 2$) or hexahedra ($d = 3$)
- h_T – diameter of cell $T \in \mathcal{T}_h$ and $h = \max_{T \in \mathcal{T}_h} h_T$
- Assume that, for each $T \in \mathcal{T}_h$, there $\exists$ affine mapping $F_T : \hat{T} \to T$
- $\mathcal{E}_h$ – FE mesh induced by $\mathcal{T}_h$ on $\partial\Omega$.
- Assume exact triangulation of Robin part Γ_R by elements of $\mathcal{E}_h$.

Variants of FE spaces:

$$\mathbb{V}_h = \{v_h \in V \subset H^1(\Omega) : v_h \circ F_T \in \mathbb{P}_1(\hat{T}), T \in \mathcal{T}_h\}$$
$$\mathbb{V}_h = \{v_h \in V \subset H^1(\Omega) : v_h \circ F_T \in \mathbb{Q}_1(\hat{T}), T \in \mathcal{T}_h\}$$

Local projection stabilization I

Two-level setting with FE spaces: $\mathcal{T}_h \subseteq \mathcal{M}_h$



i) $\mathbb{V}_h \subset H_0^1(\Omega)$ acting on $\mathcal{T}_h$

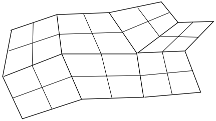
ii) $\mathbb{D}_h \subset L^2(\Omega)$ acting on $\mathcal{M}_h$

Local projection stabilization I

Two-level setting with FE spaces: $\mathcal{T}_h \subseteq \mathcal{M}_h$

i) $\mathbb{V}_h \subset H_0^1(\Omega)$ acting on $\mathcal{T}_h$

ii) $\mathbb{D}_h \subset L^2(\Omega)$ acting on $\mathcal{M}_h$



Local L^2-projection on $\mathcal{M}_h$:

- Local projection: $\pi_M : L^2(M) \to \mathbb{D}_h(M) := \{q_h|_M : \; q_h \in \mathbb{D}_h\}$
- defines global projection: $\pi_h : L^2(\Omega) \to \mathbb{D}_h, \quad (\pi_h w)|_M := \pi_M(w|_M)$
- Fluctuation operator: $\kappa_h : L^2(\Omega) \to L^2(\Omega), \qquad \kappa_h := id - \pi_h$

Local projection stabilization (LPS) scheme:

Find $u_h \in \mathbb{V}_h : \quad (a + s_h)(u_h, v_h) = (\tilde{f}, v_h)_\Omega + (\tilde{g}, v)_{\Gamma_R} \quad \forall v_h \in \mathbb{V}_h.$

$$s_h(u_h, v_h) := \sum_{M \in \mathcal{M}_h} \tau_M \big(\kappa_h(\mathbf{b} \cdot \nabla u_h), \kappa_h(\mathbf{b} \cdot \nabla v_h)\big)_M, \qquad \tau_M \geq 0$$

Some technical ingredients of LPS-analysis

Lagrangian interpolation on $\mathcal{T}_h$:

(A.1) $\exists i_h : H^1_0(\Omega) \cap W^{\frac{d}{2}+\sigma}(\Omega) \to \mathbb{V}_h$ s.t. $\forall w \in H^k(T), \ \forall T \in \mathcal{T}_h$

$$\|w - i_h w\|_{m,T} \leq C h_T^{l-m} \|w\|_{k,T}, \qquad 0 \leq m \leq l = \min(2,k)$$

Approximation property of L^2-projector $\kappa_h := id - \pi_h$:

(A.2) $\|\kappa_h q\|_{0,M} \leq C h_M^l |q|_{l,M} \quad \forall q \in H^l(M), \ \forall M \in \mathcal{M}_h, \ 0 \leq l \leq 1$

Compatibility condition between coarse and fine mesh:

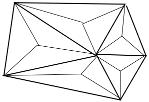
(A.3) $\forall h > 0, M \in \mathcal{M}_h \ \exists \beta > 0 :$

$$\inf_{q_h \in \mathbb{D}_h(M)} \sup_{v_h \in \mathbb{V}_h(M)} \frac{(w_h, q_h)_M}{\|v_h\|_{0,M} \|q_h\|_{0,M}} \geq \beta$$

Two variants for given simplicial mesh $\mathcal{T}_h$:

Consider here: simplicial case

(similarly for hexahedral case)



Variant 1: Local projection onto coarser mesh $\mathcal{M}_h = \mathcal{T}_{2h}$

- $\mathbb{V}_h = \{v \in H^1(\Omega) : \ v|_T \circ F_T \in \mathbb{P}_1(\hat{T}) \quad \forall T \in \mathcal{T}_h\}$
- $\mathbb{D}_h = \{v \in L^2(\Omega) : \ v|_M \circ F_M \in \mathbb{P}_0(\hat{M}) \ \ \forall M \in \mathcal{T}_{2h}\}$

Variant 2: Enrichment of approximation spaces $\mathcal{M}_h = \mathcal{T}_h$

- $\mathbb{V}_h = \{v \in H^1(\Omega) : \ v|_T \circ F_T \in \mathbb{P}_1^{bub} \ \forall T \in \mathcal{T}_h\}$

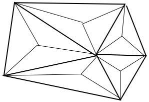
  $$\mathbb{P}_1^{bub}(\hat{T}) := \mathbb{P}_1(\hat{T}) + \hat{b} \cdot \mathbb{P}_0(\hat{T}), \qquad \hat{b}(\hat{x}) := (d+1)^{d+1} \prod_{i=1}^{d+1} \hat{\lambda}_i(\hat{x})$$

- $\mathbb{D}_h = \{v \in L^2(\Omega) : \ v|_T \circ F_T \in \mathbb{P}_0(\hat{T}) \ \forall T \in \mathcal{T}_h\}$

Two variants for given simplicial mesh $\mathcal{T}_h$:

Consider here: simplicial case

(similarly for hexahedral case)



Variant 1: Local projection onto coarser mesh $\mathcal{M}_h = \mathcal{T}_{2h}$

- $\mathbb{V}_h = \{v \in H^1(\Omega) : \ v|_T \circ F_T \in \mathbb{P}_1(\hat{T}) \quad \forall T \in \mathcal{T}_h\}$
- $\mathbb{D}_h = \{v \in L^2(\Omega) : \ v|_M \circ F_M \in \mathbb{P}_0(\hat{M}) \ \forall M \in \mathcal{T}_{2h}\}$

Variant 2: Enrichment of approximation spaces $\mathcal{M}_h = \mathcal{T}_h$

- $\mathbb{V}_h = \{v \in H^1(\Omega) : \ v|_T \circ F_T \in \mathbb{P}_1^{bub} \ \forall T \in \mathcal{T}_h\}$

  $$\mathbb{P}_1^{bub}(\hat{T}) := \mathbb{P}_1(\hat{T}) + \hat{b} \cdot \mathbb{P}_0(\hat{T}), \qquad \hat{b}(\hat{x}) := (d+1)^{d+1} \prod_{i=1}^{d+1} \hat{\lambda}_i(\hat{x})$$
- $\mathbb{D}_h = \{v \in L^2(\Omega) : \ v|_T \circ F_T \in \mathbb{P}_0(\hat{T}) \ \forall T \in \mathcal{T}_h\}$

Lemma 3.

(A.1), (A.2) are valid for $\mathbb{V}_h$ and $\mathbb{D}_h$ with L^2-projector π_h.
(A.3) is valid on a shape-regular simplicial mesh with $\beta \neq \beta(h)$.

Discrete optimal control problem

- FE space for control variables: $\mathbb{Q}_{h,\Omega} \subset H^1(\Omega)$ and $\mathbb{Q}_{h,\Gamma} = \mathbb{Q}_{h,\Omega}|_{\Gamma_R}$

Discretized control problem:

$$\min J(u_h, q_{h,\Omega}, q_{h,\Gamma}), \quad (u_h, q_{h,\Omega}, q_{h,\Gamma}) \in \mathbb{V}_h \times \mathbb{Q}_{h,\Omega} \times \mathbb{Q}_{h,\Gamma}$$

subject to

$$(a + s_h)(u_h, v_h) = (f + q_{h,\Omega}, v_h) + (g + q_{h,\Gamma}, v_h)_{\Gamma_R}, \quad \forall v_h \in \mathbb{V}_h.$$

with unique solution $(\overline{u}_h, \overline{q}_{h,\Omega}, \overline{q}_{h,\Gamma})$

Discrete solution operator $S_h : Q_{h,\Omega} \times Q_{h,\Gamma} \to \mathbb{V}_h$ by

$$(a + s_h)(S_h(q_{h,\Omega}, q_{h,\Gamma}), v_h) = (f + q_{h,\Omega}, v_h)_\Omega + (g + q_{h,\Gamma}, v_h)_{\Gamma_R} \quad \forall v_h \in \mathbb{V}_h.$$

induces discrete reduced cost functional:

$$j_h(q_{h,\Omega}, q_{h,\Gamma}) = J(S_h(q_{h,\Omega}, q_{h,\Gamma}), (q_{h,\Omega}, q_{h,\Gamma})).$$

Discrete optimality (KKT) system

Lemma 4: Necessary and sufficient optimality conditions

State problem for optimal state $\overline{u}_h \in \mathbb{V}_h$:

$$\text{Find } \overline{u}_h \in \mathbb{V}_h : \quad a(\overline{u}_h, v) = (f + \overline{q}_\Omega, v)_\Omega + (g + \overline{q}_\Gamma, v)_{\Gamma_R} \quad \forall v \in \mathbb{V}_h,$$

Adjoint state problem for optimal adjoint state $\overline{p}_h \in \mathbb{V}_h$:

$$\text{Find } \overline{p}_h \in \mathbb{V}_h : \quad a_{adj}(\overline{p}_h, v) = \lambda_\Omega(\overline{u}_h - u_\Omega)_\Omega + \lambda_\Gamma(\overline{u}_h - u_\Gamma)_{\Gamma_R} \quad \forall v \in \mathbb{V}_h,$$

Complementary conditions: $\alpha_\Omega \overline{q}_{h,\Omega} + \overline{p}_h = 0, \qquad \alpha_\Gamma \overline{q}_{h,\Gamma} + \overline{p}_h = 0.$

Lemma 5:

LPS schemes for discrete state and adjoint state admit unique solutions.

Remark

Symmetric LPS term $\rightsquigarrow$ *Operations "optimize" and "discretize" commute.*

Outline

Some analysis tools

Lemma 6: Special interpolation operator

(A.1), (A.3) $\rightsquigarrow \ \exists \ j_h : V \longrightarrow \mathbb{V}_h$ s.t.

$$(v - j_h v, q_h)_\Omega = 0, \ \ \forall q_h \in \mathbb{D}_h, \forall v \in V,$$

and for all $M \in \mathcal{M}_h$ and for $v \in V \cap W^{1+\lambda,2}(\Omega)$ with $1 + \lambda > \frac{d}{2}$

$$\|v - j_h v\|_{0,M} + h_M |v - j_h v|_{1,M} + h_M^{\frac{1}{2}} \|v - j_h v\|_{0,E} \lesssim h_M^{1+\lambda} \|v\|_{1+\lambda,2,M}.$$

Analysis with mesh-dependent norm:

$$|||v||| := \left(\varepsilon |v|_{1,\Omega}^2 + \sigma \|v\|_{0,\Omega}^2 + \|\tilde{\beta}^{\frac{1}{2}} v\|_{0,\Gamma_R}^2 + s_h(v, v) \right)^{\frac{1}{2}}, \qquad \forall v \in V.$$

Analysis of the state problems I

Guideline: Fix $(p_\Omega, p_\Gamma) \in Q_\Omega \times Q_\Gamma$ (later on: $p_\Omega := i_h \overline{q}_\Omega,\ p_\Gamma = \gamma \circ i_h \overline{q}_\Gamma$)

Lemma 7: Auxiliary estimate for state problem

- $u = S(q_\Omega, q_\Gamma) \in V$ – state for $(q_\Omega, q_\Gamma) \in Q_\Omega \times Q_\Gamma$
- $w_h = S_h(p_\Omega, p_\Gamma) \in \mathbb{V}_h$ for $(p_\Omega, p_\Gamma) \in Q_\Omega \times Q_\Gamma$ – solution of

$$(a + s_h)(w_h, v_h) = (f + p_\Omega, v_h)_\Omega + (g + p_\Gamma, v_h)_{\Gamma_R} \qquad \forall v_h \in \mathbb{V}_h.$$

Set $\tau_M \sim \frac{h_M}{\|\boldsymbol{b}\|_{[L^\infty(M)]^d}}$. Then, **(A.0)** - **(A.3)** imply:

$$\begin{aligned} |||u - w_h||| \quad &\leq \quad C_\Omega \|q_\Omega - p_\Omega\|_{0,\Omega} + C_\Gamma \|q_\Gamma - p_\Gamma\|_{0,\Gamma_R} \\ &\quad + C \left(\sum_{M \in \mathcal{M}_h} h_M^{2\lambda+1} \left\{ \frac{|\boldsymbol{b} \cdot \nabla u|^2_{\lambda,2,M}}{\|\boldsymbol{b}\|_{[L^\infty(M)]^d}} + C_M \|u\|^2_{1+\lambda,2,M} \right\} \right)^{\frac{1}{2}} \end{aligned}$$

$$C_M \quad := \quad \frac{\varepsilon}{h_M} + \sigma h_M + \|\boldsymbol{b}\|_{[L^\infty(M)]^d} + \|\beta\|_{L^\infty(\partial M \cap \Gamma_R)} + \|\mathbf{b} \cdot \mathbf{n}\|_{L^\infty(\partial M \cap \Gamma_R)},$$

$$C_\Omega \quad := \quad \min\{\frac{1}{\sqrt{\sigma}}; \frac{C_P}{\sqrt{\varepsilon}}\}; \qquad C_\Gamma := \min\{\frac{1}{\sqrt{\beta_0}}; \frac{C_P}{\sqrt{\varepsilon}}\}.$$

Analysis of the state problems II

Remark

Obtain optimal convergence rate $\mathcal{O}(h_M^{\frac{3}{2}})$ in limit case $\lambda = 1$, i.e. for $u \in H^2(\Omega)$.

Lemma 8: Auxiliary estimate for adjoint state problem

- $p \in V$ – adjoint state associated to $(q_\Omega, q_\Gamma) \in Q_\Omega \times Q_\Gamma$
- $y_h \in V_h$ – discrete adjoint state associated to some $(p_\Omega, p_\Gamma) \in Q_\Omega \times Q_\Gamma$.

Then:

$$\begin{aligned} |||p - y_h||| &\leq (C_\Omega^2 \lambda_\Omega + C_\Gamma^2 \lambda_\Gamma)|||u - w_h||| \\ &+ C\Big(\sum_{M \in \mathcal{M}_h} h_M^{2\lambda+1} \Big\{ \frac{|\boldsymbol{b} \cdot \nabla p|^2_{\lambda,2,M}}{\|\boldsymbol{b}\|_{[L^\infty(M)]^d}} + C_M \|p\|^2_{1+\lambda,2,M} \Big\} \Big)^{\frac{1}{2}} \end{aligned}$$

with C_M, C_Ω and C_Γ as in previous Lemma.

Guideline: Estimate $|||u - w_h|||$ via Lemma 7.

Main result

Theorem 2: A-priori estimate for controls

- Let **(A.0)** - **(A.3)** be valid and let $\alpha_\Omega, \alpha_\Gamma > 0$.
- $(\overline{u}, \overline{q}_\Omega, \overline{q}_\Gamma)$ – solution of optimal control problem
- $(\overline{u}_h, \overline{q}_{h,\Omega}, \overline{q}_{h,\Gamma})$ – solution of the discretized problem

$\rightsquigarrow$ $\exists$ constant $C > 0$ depending on $\lambda_\Omega, \lambda_\Gamma, \alpha_\Omega, \alpha_\Gamma, C_\Omega, C_\Gamma$ s.t.:

$$
\begin{aligned}
&\|\overline{q}_\Omega - \overline{q}_{h,\Omega}\|_{0;\Omega} + \|\overline{q}_\Gamma - \overline{q}_{h,\Gamma}\|_{0;\Gamma_R} \\
\leq\ & C \Bigg\{ \Big(\sum_{M \in \mathcal{M}_h} h_E^{1+2\lambda} |\overline{q}_\Omega|^2_{1+\lambda;2,M} \Big)^{\frac{1}{2}} + \Big(\sum_{E \in \mathcal{E}_h \cap \Gamma_R} h_M^{1+2\lambda} |\overline{q}_\Gamma|^2_{1+\lambda;2,E} \Big)^{\frac{1}{2}} \\
& + \Big(\sum_M h_M^{1+2\lambda} \Big(\frac{|\mathbf{b} \cdot \nabla \overline{u}|^2_{\lambda;2,M}}{\|\mathbf{b}\|_{[L^\infty(M)]^d}} + C_M \|\overline{u}\|^2_{1+\lambda,2,M} \Big) \Big)^{\frac{1}{2}} \\
& + \Big(\sum_M \Big(h_M^{1+2\lambda} \frac{|\mathbf{b} \cdot \nabla \overline{p}|^2_{\lambda;2,M}}{\|\mathbf{b}\|_{[L^\infty(M)]^d}} + C_M \|\overline{p}\|^2_{1+\lambda,2,M} \Big) \Big)^{\frac{1}{2}} \Bigg\}
\end{aligned}
$$

with C_M, C_Ω and C_Γ as in Lemma 7.

Outline

1. A singularly perturbed mixed boundary value problem
2. Continuous linear-quadratic optimization problem
3. Stabilized discrete optimality system
4. A-priori error analysis for optimal control problem
5. Numerical experiments
6. Regularized Dirichlet control

Example 1: Effect of LPS stabilization

$$\min J(q_\Omega, q_\Gamma, u) := \frac{1}{2}\|u - u_\Omega\|^2_{L^2(\Omega)} + \frac{\alpha_\Omega}{2}\|q_\Omega\|^2_{L^2(\Omega)}$$

$$\text{s.t.} \quad \begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + \sigma u &= q_\Omega && \text{in } \Omega = (0,1)^2 \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

with $\varepsilon = 10^{-3}$, $\mathbf{b} = (-1,-2)^t$, $\sigma = 1$

- Prescribe control as $q_{\Omega,ref}(x) = (\sin(\pi x_1))^{0.3}(\sin(\pi x_2))^{0.3}$, then compute u with given q_Ω and prescribe it as desired state u_Ω.
- Study convergence of control in the sense of $q_\Omega \to q_{\Omega,ref}$ for $\alpha_\Omega \to 0$

Convergence of control for $\alpha_\Omega \to 0$

	Control			State		
α_Ω	L^∞	L^2	H^1	L^∞	L^2	H^1
1e+0	9.47E-01	6.97E-01	5.98E+00	4.01E-01	1.54E-01	3.45E+00
1e-1	6.92E-01	5.16E-01	9.54E+00	2.54E-01	1.02E-01	2.73E+00
1e-2	7.23E-01	2.63E-01	1.68E+01	1.51E-01	3.48E-02	4.32E+00
1e-3	2.43E+00	3.41E-01	4.48E+01	1.24E-01	2.07E-02	4.43E+00
1e-4	1.04E+01	1.11E+00	1.97E+02	7.67E-02	1.11E-02	2.35E+00
1e-5	2.23E+01	2.07E+00	3.87E+02	2.38E-02	2.84E-03	5.68E-01
1e-6	2.64E+01	2.43E+00	4.55E+02	3.19E-03	3.66E-04	7.18E-02

Unstabilized scheme with fixed $h = 2^{-5}$

	Control			State		
α_Ω	L^∞	L^2	H^1	L^∞	L^2	H^1
1e+0	9.46E-01	6.97E-01	5.89E+00	4.09E-01	1.54E-01	3.55E+00
1e-1	6.87E-01	5.12E-01	5.31E+00	2.79E-01	1.03E-01	2.60E+00
1e-2	5.57E-01	2.23E-01	6.74E+00	8.54E-02	2.77E-02	9.67E-01
1e-3	2.96E-01	8.04E-02	5.29E+00	1.94E-02	4.37E-03	2.35E-01
1e-4	1.64E-01	2.74E-02	2.85E+00	3.57E-03	5.81E-04	4.77E-02
1e-5	4.95E-02	6.79E-03	9.53E-01	4.81E-04	7.06E-05	7.77E-03
1e-6	7.08E-03	9.81E-04	1.56E-01	5.12E-05	7.64E-06	9.45E-04

Stabilized scheme with fixed $h = 2^{-5}$

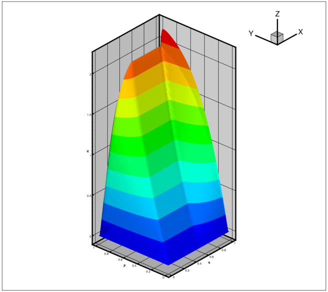
Example 2: Boundary layers

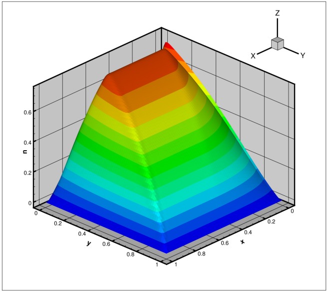
$$\min J(q_\Omega, q_\Gamma, u) := \frac{1}{2}\|u - u_\Omega\|^2_{L^2(\Omega)} + \frac{\alpha_\Omega}{2}\|q_\Omega\|^2_{L^2(\Omega)},$$

$$\text{s.t.} \qquad -\varepsilon\Delta u + (\mathbf{b}\cdot\nabla)u + \sigma u = f + q_\Omega \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

$q_\Omega \in L^2(\Omega)$, $\varepsilon = 10^{-5}$, $\beta = (-1,-2)^t$, $\sigma = 1$, $f = 1$, $u_\Omega = 1$, $\alpha_\Omega = 0.1$.





Optimal discrete control and state with $\varepsilon = 10^{-5}$ and LPS parameters $\tau = 0.1\,h$ on coarse grid

Convergence history of cost functional

$h = 2^{-l}$	$J(\bar{q}_h, \bar{u}_h)$	$J(\bar{q}_h, \bar{u}_h) - J(\bar{q}_{2h}, \bar{u}_{2h})$	num. conv. rate
2	3.0819062205E-01	-	-
3	2.7667469999E-01	3.1515922055E-02	-
4	2.6390427735E-01	1.2770422639E-02	1.30327460
5	2.6015638611E-01	3.7478912390E-03	1.76865528
6	2.5924253025E-01	9.1385586600E-04	2.03604054
7	2.5906824138E-01	1.7428887200E-04	2.39048618
8	2.5905749238E-01	1.0748996000E-05	4.01920664

Table: h-convergence of the cost functional

Remark: Spurious oscilations of two-level LPS-solution in boundary layer (reported by Becker/Vexler [2007]) are strongly reduced.

Outline

Regularized Dirichlet control I

Goal:

Approximation of Dirichlet control $u = q$ by Robin control

$$\delta \nabla u \cdot \mathbf{n} + \beta(u - q) = 0, \quad \beta = \mathcal{O}(1)$$

for $\delta \to +0$, but choice of δ is delicate.
Natural choice for singularly perturbed problem (1): $\delta = \epsilon$.

- Boundary layers at outflow part Γ_+ with $|\epsilon \nabla u \cdot \mathbf{n}| \sim 1$
- Boundary layers at characteristic boundaries Γ_0 with $|\epsilon \nabla u \cdot \mathbf{n}| \sim \sqrt{\epsilon}$
- At inflow part Γ_-: $|\epsilon \nabla u \cdot \mathbf{n}| \sim \epsilon$.

  $\rightsquigarrow$ Exclude Dirichlet control at Γ_+, but apply Robin regularization

$$\epsilon \nabla u \cdot \mathbf{n} + \beta(u - q) = 0 \quad \text{on } \Sigma \subseteq \Gamma_- \cup \Gamma_0$$

  with $\beta + \frac{1}{2}\mathbf{b} \cdot \mathbf{n} \geq \beta_0 > 0$ as regularization of Dirichlet condition $u = q$.

Flow in a domain of channel type

Typical situation: Flow in channel $\Omega = (0, L)$

- Flow field $b(x) = ((\frac{H}{2} - |x_2|)^\kappa, 0)^T, \quad \kappa \geq 0$
- Solution u of (1) – temperature field or density of chemical reactant.

Regularization at channel wall

- Robin condition $\epsilon \frac{\partial u}{\partial n} + \beta(u - g) = 0$ with $\beta \geq \beta_0 > 0$ on $\Gamma_0 \setminus \Sigma \subset \Gamma_0$ replaces Dirichlet condition $u = q$
- Insulation condition on $\Gamma_0 \setminus \Sigma$
- Inflow condition $\epsilon \frac{\partial u}{\partial x_1} + \beta(u - g) = 0$ with $\beta + \frac{1}{2}\mathbf{b} \cdot \mathbf{n} \geq \beta_0 > 0$ on Γ_-
- ”Do-nothing” condition on Γ_+

Summary. Outlook

Summary:

- Regularity problem of mixed BVP in polyhedral domains
- Simultaneous distributed and Robin boundary control problem
- Symmetric stabilization via local projection stabilization
- Operations ”discretize” and ”optimize” commute !
- Robin control as regularized Dirichlet control

Some open problems:

- Refined analysis for hybrid meshes required.
- Better resolution of boundary and interior layers
- Extension to optimal control with box-constraints

Summary. Outlook

Summary:

- Regularity problem of mixed BVP in polyhedral domains
- Simultaneous distributed and Robin boundary control problem
- Symmetric stabilization via local projection stabilization
- Operations "discretize" and "optimize" commute !
- Robin control as regularized Dirichlet control

Some open problems:

- Refined analysis for hybrid meshes required.
- Better resolution of boundary and interior layers
- Extension to optimal control with box-constraints

THANK YOU FOR YOUR ATTENTION !

Institut für Numerische und Angewandte Mathematik
Universität Göttingen
Lotzestr. 16-18
D - 37083 Göttingen

Telefon: 0551/394512
Telefax: 0551/393944

Email: trapp@math.uni-goettingen.de URL: http://www.num.math.uni-goettingen.de

Verzeichnis der erschienenen Preprints 2008:

2008-01	M. Körner, A. Schöbel	Weber problems with high-speed curves
2008-02	S. Müller, R. Schaback	A Newton Basis for Kernel Spaces
2008-03	H. Eckel, R. Kress	Nonlinear integral equations for the complete electrode model in inverse impedance tomography
2008-04	M. Michaelis, A. Schöbel	Integrating Line Planning, Timetabling, and Vehicle Scheduling: A customer-oriented approach
2008-05	O. Ivanyshyn, R. Kress, P. Serranho	Huygen's principle and iterative methods in inverse obstacle scattering
2008-06	F. Bauer, T. Hohage, A. Munk	Iteratively regularized Gauss-Newton method for nonlinear inverse problems with random noise
2008-07	R. Kress, N. Vintonyak	Iterative methods for planar crack reconstruction in semi-infinite domains
2008-08	M. Uecker, T. Hohage, K.T. Block, J. Frahm	Image reconstruction by regularized nonlinear inversion - Joint estimation of coil sensitivities and image content
2008-09	M. Schachtebeck, A. Schöbel	IP-based Techniques for Delay Management with Priority Decisions
2008-10	S. Cicerone, G. Di Stefano, M. Schachtebeck, A. Schöbel	Dynamic Algorithms for Recoverable Robustness Problems
2008-11	M. Braack, G. Lube	Stabilized finite elements by local projection for flow problems
2008-12	T. Knopp, X.Q. Zhang, R. Kessler, G. Lube	Calibration of a finite volume discretization and of model parameters for incompressible large eddy-type simulations
2008-13	P. Knobloch, G. Lube	Local projection stabilization for advection-diffusion-reaction problems: One-level vs. two-level approach

2008-14	B. Tews, G. Lube	Optimal control of singularly perturbed advection-diffusion-reaction problems