Residual-based stabilisation of inf-sup stable discretisations of the generalised Oseen problem

Gunar Matthies[1] Gert Lube[2] Lars Röhe[2]

[1] Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstraße 150
44780 Bochum
Germany
email: Gunar.Matthies@ruhr-uni-bochum.de

[2] Institut für Numerische und Angewandte Mathematik
Georg-August-Universität Göttingen
Lotzestrasse 16–18
37083 Göttingen
Germany
email: lube@math.uni-goettingen.de

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We consider stabilised finite element methods for the generalised Oseen problem. The unique solvability based on a modified stability condition and an error estimate are proved for inf-sup stable discretisations of velocity and pressure. The analysis highlights the role of an additional stabilisation of the incompressibility constraint. It turns out that the stabilisation terms of streamline-diffusion (SUPG) type play a less important role. The analysis extends a recent result to general shape-regular meshes and to discontinuous pressure interpolation. Some numerical observations support the theoretical results.

1 Introduction

sec:intro

Let us consider the instationary, incompressible Navier–Stokes problem with homogeneous Dirichlet boundary conditions

$$\begin{aligned} \partial_t u - \nu \triangle u + (u \cdot \nabla) u + \nabla p = \tilde{f} \qquad & \text{in } \Omega \times (0,T), \\ \operatorname{div} u = 0 \qquad & \text{in } \Omega \times (0,T), \\ u = 0 \qquad & \text{on } \partial\Omega \times (0,T], \\ u|_{t=0} = u_0 \qquad & \text{in } \Omega, \end{aligned} \tag{1}$$

Null.1

for the velocity u and the pressure p in the space-time cylinder $\Omega \times (0,T)$ with a polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and a time $T > 0$. The given source term is denoted by $\tilde{f}$. A typical algorithmic approach for solving (1) is to semidiscretise first in time and to apply then a fixed-point iteration within each time step. This leads in each step of this iteration to an auxiliary problem of Oseen type

$$\begin{aligned} L_O(b; u, p) := -\nu \triangle u + (b \cdot \nabla) u + \sigma u + \nabla p = f \qquad & \text{in } \Omega, \\ \operatorname{div} u = 0 \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{2}$$

Null.2

Also the iterative solution of the steady-state Navier–Stokes equations may lead to problems of type (2) with $\sigma = 0$ if a fixed-point iteration is applied.

The basic Galerkin finite element method (FEM) for (2) may suffer from two problems: the dominating advection (and reaction) in the case of $0 < \nu \ll \|b\|_{L^\infty(\Omega)}$, and/or the violation of the discrete inf-sup (or Babuška–Brezzi) stability condition for the velocity and pressure approximations. The streamline-upwind/Petrov–Galerkin method (SUPG), introduced in [3], and the pressure-stabilisation/Petrov–Galerkin method (PSPG), introduced in [10, 11], opened the possibility to treat both problems in a unique framework using rather arbitrary FE approximations of velocity-pressure, including equal-order pairs. Additionally to the Galerkin part, the elementwise residual $L_O(b; u, p) - f$ is tested against the (weighted) non-symmetric SUPG/PSPG parts $(b \cdot \nabla) v + \nabla q$ of $L_O(b; v, q)$. An additional elementwise stabilisation of the divergence constraint $\operatorname{div} u$ in (2), henceforth denoted as grad-div stabilisation, is important for the robustness if $0 < \nu \ll \|b\|_{L^\infty(\Omega)}$, see [5] for equal-order interpolation.

For a unified a-priori analysis of classical residual-based stabilisation (RBS) techniques, we refer to [12]. We emphasise that the design of the stabilisation parameters for equal-order interpolation significantly differs from that for inf-sup stable pairs. In particular, the grad-div stabilisation is much more important in the advection-dominated case if an inf-sup stable interpolation is applied, see also [6, 15].

One of the critical aspects of these RBS techniques for incompressible flows is the strong coupling between velocity and pressure in the stabilising terms. Several attempts have been made to relax this problem, see [2] for an overview. In particular, we mention the promising idea of weakly-consistent, symmetric stabilisation techniques (e.g., via edge stabilisation or local projection).

Within the framework of strongly consistent RBS techniques, one natural idea is to skip the PSPG term in the case of inf-sup stable discretisations of velocity and pressure. We considered this possibility in [6]. The analysis of the so-called *reduced stabilised* scheme is so far restricted to the quasi-uniform case and to continuous pressure approximation. Moreover, the analysis is seemingly not optimal for the case $\nu^2 + \sigma^2 \to +0$, whereas numerical experiments show stable results for this case too.

The goal of the present paper is to refine the analysis in [6] for the reduced stabilised scheme and to relax the assumptions of quasi-uniform meshes and continuous pressure discretisations. We prove an inf-sup stability condition of the scheme which is uniformly valid for $0 < \nu \ll 1$ and an a-priori error estimate. A refined design of the grad-div and SUPG-stabilisation parameters highlights the role of the additional stabilisation of the incompressibility constraint. Moreover, it turns out that the SUPG-stabilisation is less essential. An important technical ingredient is the application of quasi-local interpolation operators preserving the discrete divergence [8]. For brevity, we consider only conforming FEM.

The paper is organised as follows. In Section 2, we introduce notation and the stabilised Galerkin discretisation of the Oseen problem. Then, we analyse the method in Section 3 and discuss the results in Section 4. Finally in Section 5, we consider some open problems.

2 Notation. The discrete problem

c:notation

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded polygonal or polyhedral domain. For a subdomain $G \subset \Omega$, the usual Sobolev spaces $W^{m,p}(G)$ with norm $\|\cdot\|_{m,p,G}$ and semi-norm $|\cdot|_{m,p,G}$ are used. In the case $p = 2$, we have $H^m(G) = W^{m,2}(G)$ and the index p will be omitted. The L^2 inner product on G is denoted by $(\cdot,\cdot)_G$. Note that the index G will be omitted for $G = \Omega$. This notation of norms, semi-norms, and inner products is also used for the vector-valued and tensor-valued case. We set $X := \big(H_0^1(\Omega)\big)^d$, $M := L_0^2(\Omega) := \big\{q \in L^2(\Omega) \,:\, (q,1) = 0\big\}$ and $H(\mathrm{div},\Omega) := \big\{v \in [L^2(\Omega)]^d \,:\, \mathrm{div}\, v \in L^2(\Omega)\big\}$.

The generalised Oseen equations with homogeneous Dirichlet boundary conditions are given by

$$\left.\begin{aligned} -\nu\triangle u + (b\cdot\nabla)u + \sigma u + \nabla p &= f &&\text{in } \Omega,\\ \mathrm{div}\, u &= 0 &&\text{in } \Omega,\\ u &= 0 &&\text{on } \partial\Omega \end{aligned}\right\} \qquad (3)$$

oseen

with constants $\nu > 0$, $\sigma \geq 0$ and a given convection field $b \in H(\mathrm{div},\Omega) \cap \big(L^\infty(\Omega)\big)^d$ with $\mathrm{div}\, b = 0$. For $u, v \in X$, $p, q \in M$, the bilinear forms A, b and linear form L are given by

$$\begin{aligned} A\big((u,p),(v,q)\big) &:= \nu(\nabla u, \nabla v) + \big((b\cdot\nabla)u, v\big) + \sigma(u,v) - b(v,p) + b(u,q),\\ b(v,q) &:= (q, \mathrm{div}\, v),\\ L\big((v,q)\big) &:= (f,v). \end{aligned}$$

Note that the following integration by parts

$$\big((b\cdot\nabla)v, w\big) = -\big((b\cdot\nabla)w, v\big) \tag{4}$$

intparts

holds true for all $v, w \in X$ due to $\operatorname{div} b = 0$.

A weak formulation of the generalised Oseen equations (3) reads:

Find $(u,p) \in X \times M$ such that

$$A\big((u,p),(v,q)\big) = L\big((v,q)\big) \qquad \forall (v,q) \in X \times M. \tag{5}$$

weak

Let $\{\mathcal{T}_h\}$ be a family of shape-regular and exact triangulations of the domain Ω such that

$$\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$$

holds true for all triangulations $\mathcal{T}_h$.

Let X_h be a conforming finite element space based on $\mathcal{T}_h$ for approximating the velocity. The space M_h for approximating the pressure may consist of continuous or generally discontinuous functions. We are interested in inf-sup stable discretisations, i.e., the condition

$$\inf_{q_h\in M_h} \sup_{v_h\in X_h} \frac{(\operatorname{div} v_h, q_h)}{|v_h|_1\,\|q_h\|_0} \geq \beta_0 > 0 \tag{6}$$

infsup

is valid for all $\mathcal{T}_h$ with a positive constant β_0 which is independent of the mesh parameter h. Examples for such pairs are the Taylor–Hood family P_k/P_{k-1}, $k \geq 2$, on simplices and Q_k/Q_{k-1}, $k \geq 2$, on quadrilaterals and hexahedra, see [7] and the references therein. Furthermore, $Q_k/P_{k-1}^{\text{disc}}$, $k \geq 2$, fulfils the inf-sup condition on quadrilaterals and hexahedra, see [7, 14].

We assume that for all cells $K \in \mathcal{T}_h$ the following inverse inequalities

$$\begin{aligned} \|\triangle v_h\|_{0,K} &\leq \mu\, h_K^{-1}\, \|\nabla v_h\|_{0,K} && \forall v_h \in X_h,\\ \frac{1}{\sqrt{d}}\|\operatorname{div} v_h\|_{0,K} \leq \|\nabla v_h\|_{0,K} &\leq \mu\, h_K^{-1}\, \|v_h\|_{0,K} && \forall v_h \in X_h,\\ \|\nabla q_h\|_{0,K} &\leq \mu\, h_K^{-1}\, \|q_h\|_{0,K} && \forall q_h \in M_h, \end{aligned} \tag{7}$$

invineq

are valid with a constant μ which depends only on the shape-regularity parameter of the family of triangulations.

We assume that the discrete velocity space X_h is based on finite elements of order k. One can think of the case there X_h consists of all continuous function those restrictions to a single cell K of the triangulation $\mathcal{T}_h$ belongs to P_k (for simplicial cells) or to Q_k (for quadrilateral and hexahedral cells). The discrete pressure space M_h is assumed to be based on finite elements of order $\ell \geq 1$. This means that the restriction of a function from M_h to a cell $K \in \mathcal{T}_h$ belongs to P_ℓ or Q_ℓ. Note that P_ℓ can be used also on quadrilaterals and hexahedra if no continuity is required in M_h.

The standard finite element interpolation operator $J_h : M \to M_h$ fulfils for all $K \in \mathcal{T}_h$ the estimate

$$|q - J_h q|_{m,K} \le C h_K^{\ell+1-m} \|q\|_{\ell+1,K} \qquad \forall q \in H^{\ell+1}(\Omega) \cap M, \quad m = 0, \ldots, \ell+1,$$

where the constant C is independent of h, see [4]. We choose from [8] for the velocity the quasi-local interpolation operator which preserves the discrete divergence. Hence, we have for the interpolation operator $I_h : X \to X_h$ the estimate

$$|v - I_h v|_{m,K} \le C h_K^{k+1-m} \|v\|_{k+1,\omega(K)} \qquad \forall v \in \big(H^{k+1}(\Omega)\big)^d \cap X, \quad m = 0, \ldots, k+1,$$

where $\omega(K)$ is a suitable neighbourhood of K and C is independent of h, see [8]. Moreover,

$$(\operatorname{div} I_h v, q_h) = (\operatorname{div} v, q_h) \qquad \forall q_h \in M_h,\ \forall v \in X, \tag{8}$$ discdiv

holds true.

Using the finite element spaces X_h and M_h, we can formulate the standard Galerkin discretisation of (5) which reads

Find $(u_h, p_h) \in X_h \times M_h$ such that

$$A\big((u_h, p_h), (v_h, q_h)\big) = L\big((v_h, q_h)\big) \qquad \forall (v_h, q_h) \in X_h \times M_h. \tag{9}$$ Galerkin

In the case of locally dominating convection, one may get solutions of (9) with spurious oscillations which are in general not localised to regions with dominating convection. In order to stabilise the discrete problem, we introduce a modified bilinear form and a modified linear form by

$$\begin{aligned} A_S\big((u,p),(v,q)\big) :=& A\big((u,p),(v,q)\big) + \gamma(\operatorname{div} u, \operatorname{div} v) \\ &+ \sum_{K \in \mathcal{T}_h} \big(-\nu \triangle u + (b \cdot \nabla) u + \sigma u + \nabla p, \delta_K (b \cdot \nabla) v \big)_K, \\ L_S\big((v,q)\big) :=& L\big((v,q)\big) + \sum_{K \in \mathcal{T}_h} \big(f, \delta_K (b \cdot \nabla) v\big)_K \end{aligned}$$

where δ_K is a cell-dependent parameter while γ is a global user-defined parameter. A detailed study of the choice of these parameters will be given later.

The stabilised discrete problem reads

Find $(u_h, p_h) \in X_h \times M_h$ such that

$$A_S\big((u_h, p_h), (v_h, q_h)\big) = L_S\big((v_h, q_h)\big) \qquad \forall (v_h, q_h) \in X_h \times M_h. \tag{10}$$ discstab

Since the additional terms in A_S and L_S vanish in sum for a smooth solution, the stabilised problem is of residual type. Hence, we have the Galerkin orthogonality

$$A_S\big((u - u_h, p - p_h), (v_h, q_h)\big) = 0 \qquad \forall (v_h, q_h) \in X_h \times M_h \tag{11}$$ GalOrtho

where $(u_h, p_h) \in X_h \times M_h$ is the solution of (10) and the solution $(u, p) \in X \times M$ of (5) satisfies additionally the regularity requirement $u \in \big(H^2(\Omega)\big)^d$ and $p \in H^1(\Omega)$.

Remark 1. *It is possible to consider the fully stabilised discrete problem which includes a PSPG term. In this case, the bilinear form A_F and the linear form L_F are defined by*

$$A_F\big((u_h,p_h),(v_h,q_h)\big) = A_S\big((u_h,p_h),(v_h,q_h)\big) + \sum_{K\in\mathcal{T}_h} \big(L_O\big((u_h,p_h),\alpha_K\nabla q\big)_K,$$

$$L_F\big((v_h,q_h)\big) = L_S\big((v_h,q_h)\big) + \sum_{K\in\mathcal{T}_h} \big(f,\alpha_K\nabla q\big)_K,$$

where α_K are user-chosen parameters. Using similar techniques as below, corresponding error estimates and parameter designs can be derived for the fully stabilised scheme. Although the PSPG stabilisation is not needed for inf-sup stable discretisation from the point of stability, the additional term might improve the accuracy of the pressure approximation.

We introduce the norms

$$|[v]|^2 := \nu|v|_1^2 + \sigma\|v\|_0^2 + \gamma\|\operatorname{div} v\|_0^2 + \sum_{K\in\mathcal{T}_h} \delta_K\|(b\cdot\nabla)v\|_{0,K}^2,$$

$$\big|\!\big|\!\big|(v,q)\big|\!\big|\!\big|^2 := |[v]|^2 + \alpha\|q\|_0^2.$$

Note that the norms are well-defined on X and $X\times M$, respectively.

The positive constant α will be chosen later on in the proof of Lemma 2. A lower bound is given in (20). Furthermore, we set

$$b_K := \|b\|_{0,\infty,K}, \qquad b_\infty := \|b\|_{0,\infty}.$$

In this paper, the generic constant C may have different values at different places but it will be always independent of the mesh size h and the parameter ν.

3 Analysis of the method

c:analysis

3.1 Stability and solvability of the discrete problem

:stability

To show that our stabilised discrete problem (10) is uniquely solvable, we will prove for the bilinear form A_S an inf-sup condition on $X_h\times M_h$ where the constant is independent of the mesh size h and parameter ν.

It turns out that our stability analysis requires an upper bound of the SUPG-parameters δ_K which is basically dictated by an upper bound of the advective Galerkin term. We define

$$\varphi := \sqrt{\nu+\sigma C_F^2} + 2b_\infty \min\left(\frac{1}{\sqrt{\sigma}},\frac{C_F}{\sqrt{\nu}}\right) + \sqrt{\gamma d} \tag{12}$$ phi

where C_F is the Friedrichs constant for Ω. We assume that the stabilisation parameters fulfil

$$0\le\gamma, \qquad 0\le\delta_K\le\min\left(\frac{1}{15}\min\left(\frac{1}{\sigma},\frac{C_F^2}{\nu}\right), \frac{1}{30}\frac{h_K^2\beta_0^2}{\mu^2\varphi^2}\right) \tag{13}$$ paramdesi

where μ is the constant from the inverse inequalities (7) and β_0 the inf-sup constant for the pair (X_h,M_h).

lem:infsup **Lemma 2.** *Let the stabilisation parameters fulfil* (13)*. Then, there exists a positive constant β_S independent of the mesh size h and parameter ν such that*

$$\inf_{(v_h,q_h)} \sup_{(w_h,r_h)} \frac{A_S\big((v_h,q_h),(w_h,r_h)\big)}{|||(v_h,q_h)|||\; |||(w_h,r_h)|||} \geq \beta_S > 0 \tag{14}$$

eq:infsup

holds true where the infimum and supremum are taken over $X_h \times M_h$.

Proof. Let (v_h, q_h) be an arbitrary element of $X_h \times M_h$. During the proof, we will use the following abbreviations:

$$X^2 := \sum_{K\in\mathcal{T}_h} \delta_K \|(b\cdot\nabla)v_h\|_{0,K}^2, \qquad Z^2 := \gamma\|\operatorname{div} v_h\|_0^2,$$
$$Y^2 := \sum_{K\in\mathcal{T}_h} \delta_K \|-\nu\triangle v_h + \sigma v_h + \nabla q_h\|_{0,K}^2, \qquad B^2 := \|q_h\|_0^2,$$
$$A^2 := \nu|v_h|_1^2 + \sigma\|v_h\|_0^2,$$

which give immediately that $|[v_h]|^2 = A^2 + X^2 + Z^2$.

The outline of the proof is as follows.

1. We show $A_S\big((v_h,q_h),(v_h,q_h)\big) \geq C_1\,|[v_h]|^2 - \overline{\delta}\,B^2$ with constants C_1 and $\overline{\delta}$. The critical constant $\overline{\delta}$ scales like δ_K/h_K^2, see (16).
2. We get from the inf-sup condition (14) the existence of a function $z_h \in X_h$ such that $A_S\big((v_h,q_h),(-z_h,0)\big) \geq \frac{2}{3}\beta_0\,B^2 - C_2\,|[v_h]|^2$ with C_2 scaling like φ^2, see (18).
3. The function $(w_h,r_h) := (v_h,q_h) + \lambda(-z_h,0) \in X_h \times M_h$ with a suitably chosen $\lambda > 0$ satisfies $A_S\big((v_h,q_h),(w_h,r_h)\big) \geq C_3 |||(v_h,q_h)|||^2$ and $|||(w_h,r_h)||| \leq C_4 |||(v_h,q_h)|||$ which together result in the assertion of this lemma.

Step 1. Using the definition of the bilinear form A_S, we obtain via the Young inequality and integration by parts (see (4))

$$A_S\big((v_h,q_h),(v_h,q_h)\big) \geq |[v_h]|^2 - XY \geq |[v_h]|^2 - \frac{3}{4}X^2 - \frac{1}{3}Y^2.$$

The terms will be estimated separately. Exploiting (7), (13) and $\frac{\nu^2}{\varphi^2} \leq 1$, we get

$$\begin{aligned}
Y^2 \leq& 2\sum_{K\in\mathcal{T}_h} \big(\delta_K\|\nabla q_h\|_{0,K}^2 + \delta_K\|-\nu\triangle v_h + \sigma v_h\|_{0,K}^2\big)\\
\leq& \sum_{K\in\mathcal{T}_h} \frac{2\delta_K\mu^2}{h_K^2}\|q_h\|_{0,K}^2 + 4\Big(\sum_{K\in\mathcal{T}_h} \delta_K\frac{\mu^2}{h_K^2}\nu^2|v_h|_{1,K}^2 + \sum_{K\in\mathcal{T}_h}\delta_K\sigma^2\|v_h\|_{0,K}^2\Big)\\
\leq& \sum_{K\in\mathcal{T}_h} \frac{2\delta_K\mu^2}{h_K^2}\|q_h\|_{0,K}^2 + 4\Big(\sum_{K\in\mathcal{T}_h} \frac{1}{30}\frac{\beta_0^2}{\varphi^2}\nu^2|v_h|_{1,K}^2 + \sum_{K\in\mathcal{T}_h}\frac{1}{15}\sigma\|v_h\|_{0,K}^2\Big)\\
\leq& 2\max_{K\in\mathcal{T}_h}\Big(\frac{\delta_K\mu^2}{h_K^2}\Big)B^2 + \frac{4}{15}A^2.
\end{aligned} \tag{15}$$

eq4

where $\beta_0 \leq 1$ was applied, which is always possible to choose, see (6). Hence, we obtain

$$A_S\big((v_h, q_h), (v_h, q_h)\big) \geq \frac{1}{4}|[v_h]|^2 - \frac{2}{3} \max_{K\in\mathcal{T}_h} \Big(\frac{\delta_K \mu^2}{h_K^2}\Big) B^2. \qquad (16)$$

step1

Step 2. Due to the inf-sup condition (6) for (X_h, M_h), there exists $z_h \in X_h$ such that

$$|z_h|_1 = \|q_h\|_0 = B, \qquad (\operatorname{div} z_h, q_h) \geq \beta_0\, |z_h|_1\, \|q_h\|_0 = \beta_0\, B^2.$$

We have

$$A_S\big((v_h, q_h), (-z_h, 0)\big) \geq \beta_0\, B^2 - \sum_{i=1}^{4} T_i$$

where

$$T_1 := \nu(\nabla v_h, \nabla z_h) + \sigma(v_h, z_h) - ((b\cdot\nabla)z_h, v_h), \qquad T_2 := \gamma(\operatorname{div} v_h, \operatorname{div} z_h),$$
$$T_3 := \sum_{K\in\mathcal{T}_h} \delta_K(-\nu\triangle v_h + \sigma v_h + \nabla q_h, (b\cdot\nabla)z_h)_K, \qquad T_4 := \sum_{K\in\mathcal{T}_h} \delta_K((b\cdot\nabla)v_h, (b\cdot\nabla)z_h)_K.$$

These four terms will be estimated individually. Applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}|T_1| &\leq \big(\nu|v_h|_1^2 + \sigma\|v_h\|_0^2\big)^{1/2}\big(\nu|z_h|_1^2 + \sigma C_F^2|z_h|_1^2\big)^{1/2} + \sum_{K\in\mathcal{T}_h} b_K\|v_h\|_{0,K}|z_h|_{1,K}\\ &\leq \sqrt{\nu + \sigma C_F^2}\, AB + \sum_{K\in\mathcal{T}_h} b_K\|v_h\|_{0,K}|z_h|_{1,K}.\end{aligned}$$

The last term can be estimated in two ways

$$\sum_{K\in\mathcal{T}_h} b_K\|v_h\|_{0,K}|z_h|_{1,K} \leq \sum_{K\in\mathcal{T}_h} \frac{b_K}{\sqrt{\sigma}}(\sqrt{\sigma}\|v_h\|_{0,K})|z_h|_{1,K} \leq b_\infty\frac{1}{\sqrt{\sigma}}AB$$

or

$$\sum_{K\in\mathcal{T}_h} b_K\|v_h\|_{0,K}|z_h|_{1,K} \leq \sum_{K\in\mathcal{T}_h} \frac{b_K}{\sqrt{\nu}}(\sqrt{\nu}\|v_h\|_{0,K})|z_h|_{1,K} \leq b_\infty\frac{C_F}{\sqrt{\nu}}AB.$$

Hence, we get the estimate

$$|T_1| \leq \sqrt{\nu + \sigma C_F^2}AB + b_\infty \min\left(\frac{1}{\sqrt{\sigma}}, \frac{C_F}{\sqrt{\nu}}\right) AB$$

which is governed by the bound of the advective term $((b\cdot\nabla)z_h, v_h)$. Furthermore, we have

$$|T_2| \leq \sqrt{\gamma}\|\operatorname{div} v_h\|_0\sqrt{\gamma}\|\operatorname{div} z_h\|_0 \leq Z\sqrt{\gamma d}\Big(\sum_{K\in\mathcal{T}_h} |z_h|_{1,K}^2\Big)^{1/2} = \sqrt{\gamma d}ZB$$

and

$$
\begin{aligned}
|T_3| &\le \Big(\sum_{K\in\mathcal{T}_h}\delta_K\|-\nu\triangle v_h+\sigma v_h+\nabla q_h\|_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\delta_K\|(b\cdot\nabla)z_h\|_{0,K}^2\Big)^{1/2}\\
&\le Y\Big(\sum_{K\in\mathcal{T}_h}\delta_K b_K^2|z_h|_{1,K}^2\Big)^{1/2}\le\Big(\max_{K\in\mathcal{T}_h}\Big(b_K\sqrt{\delta_K}\Big)\Big)YB.
\end{aligned}
$$

Using (13) and (15), we obtain $Y^2\le\frac{2}{30}\frac{\beta_0^2}{\varphi^2}B^2+\frac{4}{15}A^2$ which gives $Y\le\sqrt{\frac{2}{30}}\frac{\beta_0}{\varphi}B+\frac{2}{\sqrt{15}}A$. Furthermore, we have

$$
\begin{aligned}
|T_3| &\le \frac{2}{\sqrt{15}}\Big(\max_{K\in\mathcal{T}_h}\Big(b_K\sqrt{\delta_K}\Big)\Big)AB+\sqrt{\frac{2}{30}}\frac{\beta_0}{\varphi}\frac{1}{2}\frac{1}{\sqrt{15}}2b_\infty\min\Big(\frac{1}{\sqrt{\sigma}},\frac{C_F}{\sqrt{\nu}}\Big)B^2\\
&\le \frac{2}{\sqrt{15}}\Big(\max_{K\in\mathcal{T}_h}\Big(b_K\sqrt{\delta_K}\Big)\Big)AB+\frac{1}{30}\beta_0B^2.
\end{aligned}
$$

It remains to bound T_4. We obtain

$$
|T_4|\le\Big(\sum_{K\in\mathcal{T}_h}\delta_K\|(b\cdot\nabla)v_h\|_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\delta_K\|(b\cdot\nabla)z_h\|_{0,K}^2\Big)^{1/2}\le\max_{K\in\mathcal{T}_h}\Big(b_K\sqrt{\delta_K}\Big)XB.
$$

We proceed with estimating the max-term via the first argument of the min-term in (13)

$$
\max_{K\in\mathcal{T}_h}\Big(b_K\sqrt{\delta_K}\Big)\le\frac{1}{\sqrt{15}}b_\infty\min\Big(\frac{1}{\sqrt{\sigma}},\frac{C_F}{\sqrt{\nu}}\Big) \tag{17}
$$

maxphi

Note that, due to the upper bound of $|T_1|$, no gain is obtained if the second argument of the min-term in (13) is used. Using (17) and the estimates for $T_1,\dots,T_4$, we end up with

$$
\begin{aligned}
\sum_{i=1}^4 T_i\le\sum_{i=1}^4|T_i| \le&\Big(\sqrt{\nu+\sigma C_F^2}+\frac{17}{15}b_\infty\min\Big(\frac{1}{\sqrt{\sigma}},\frac{C_F}{\sqrt{\nu}}\Big)\Big)AB\\
&+\frac{1}{\sqrt{15}}b_\infty\min\Big(\frac{1}{\sqrt{\sigma}},\frac{C_F}{\sqrt{\nu}}\Big)XB+\sqrt{\gamma d}ZB+\frac{1}{30}\beta_0B^2\\
\le&(A+X+Z)\varphi B+\frac{1}{30}\beta_0B^2.
\end{aligned}
$$

To summarise, we have

$$
\begin{aligned}
A_S\big((v_h,q_h),(-z_h,0)\big)&\ge\beta_0B^2-(A+X+Z)\varphi B-\frac{1}{30}\beta_0B^2\\
&\ge\frac{29}{30}\beta_0B^2-\frac{3\cdot1}{10}\beta_0B^2-\frac{5}{2}\frac{\varphi^2}{\beta_0}(A^2+X^2+Z^2)\\
&=\frac{2}{3}\beta_0B^2-\frac{5}{2}\frac{\varphi^2}{\beta_0}|[v_h]|^2.
\end{aligned} \tag{18}
$$

step2

Step 3. We define $(w_h, r_h) := (v_h, q_h) + \lambda(-z_h, 0)$ with $\lambda > 0$. Using the estimates (16) and (18), we obtain

$$A_S\big((v_h, q_h), (w_h, r_h)\big) \geq \Big(\frac{1}{4} - \frac{5}{2}\frac{\lambda\varphi^2}{\beta_0}\Big)|[v_h]|^2 + \Big(\frac{2}{3}\frac{\lambda\beta_0}{\alpha} - \frac{2}{3}\max_{K\in\mathcal{T}_h}\Big(\frac{\delta_K\mu^2}{\alpha h_K^2}\Big)\Big)\alpha B^2.$$

We choose λ and α such that

$$\frac{1}{4} - \frac{5}{2}\frac{\lambda\varphi^2}{\beta_0} = \frac{1}{30} \quad \text{and} \quad \frac{2}{3}\frac{\lambda\beta_0}{\alpha} - \frac{2}{3}\max_{K\in\mathcal{T}_h}\Big(\frac{\delta_K\mu^2}{\alpha h_K^2}\Big) = \frac{1}{30}. \tag{19}$$

lambda

We obtain

$$\lambda\beta_0 = \frac{13}{150}\frac{\beta_0^2}{\varphi^2} \quad \text{and} \quad \alpha = \frac{26}{15}\frac{\beta_0^2}{\varphi^2} - 20\max_{K\in\mathcal{T}_h}\Big(\frac{\delta_K\mu^2}{\alpha h_K^2}\Big).$$

We can bound α from below and above via (13) as follows

$$\frac{16}{15}\frac{\beta_0^2}{\varphi^2} \leq \alpha \leq \frac{26}{15}\frac{\beta_0^2}{\varphi^2}. \tag{20}$$

alpha

Our choice of λ and α results in

$$A_S\big((v_h, q_h), (w_h, r_h)\big) \geq \frac{1}{30}|||(v_h, q_h)|||^2.$$

Finally, we will show that $|||(w_h, r_h)||| \leq C|||(v_h, q_h)|||$. To this end, we start with

$$|||(w_h, r_h)||| \leq |||(v_h, q_h)||| + \lambda|||(-z_h, 0)|||,$$

and see that it suffices to estimate $|||(-z_h, 0)|||$. We have

$$\begin{aligned} |||(-z_h, 0)|||^2 &= \nu|z_h|_1^2 + \sigma\|z_h\|_0^2 + \gamma\|\operatorname{div} z_h\|_0^2 + \sum_{K\in\mathcal{T}_h}\delta_K\|(b\cdot\nabla)z_h\|_{0,K}^2 \\ &\leq \sum_{K\in\mathcal{T}_h}(\nu + \sigma C_F^2 + \gamma d + b_K^2\delta_K)|z_h|_{1,K}^2 \\ &\leq \varphi^2 B^2 \leq \frac{\varphi^2}{\alpha}|||(v_h, q_h)|||^2 \end{aligned}$$

where we have used (17) to bound $b_K^2\,\delta_K$. Using the above estimate, we obtain

$$|||(w_h, r_h)||| \leq \Big(1 + \frac{\lambda\varphi}{\sqrt{\alpha}}\Big)|||(v_h, q_h)|||.$$

Exploiting the choice of λ in (19) and the lower bound of α in (20), we have

$$Q := 1 + \frac{\lambda\varphi}{\sqrt{\alpha}} \leq 1 + \frac{13}{150}\frac{\beta_0}{\varphi^2}\varphi\sqrt{\frac{15}{16}\frac{\varphi^2}{\beta_0^2}} = 1 + \frac{13}{150}\sqrt{\frac{15}{16}}$$

which results in

$$A_S\big((v_h, q_h), (w_h, r_h)\big) \geq \frac{1}{30Q}|||(v_h, q_h)|||\,|||(w_h, r_h)|||.$$

Hence, the inf-sup constant $\beta_S := 1/(30Q)$ is independent of ν and h. $\square$

ec:apriori

3.2 A preliminary a-priori error estimate

First, we will state and prove a continuity estimate for the bilinear form A_S.

lem:cont

Lemma 3. *Let $u \in \big(H^{k+1}(\Omega)\big)^d \cap X$ and $p \in H^{\ell+1}(\Omega) \cap M$. Moreover, $I_h u$ is the interpolant of u which preserves the discrete divergence, see (8), while $J_h p$ is the standard finite element interpolant of p. Then, for all $(w_h, r_h) \in X_h \times M_h$, the following estimate holds true*

$$\frac{A_S\big((u - I_h u, p - J_h p), (w_h, r_h)\big)}{|||(w_h, r_h)|||} \\ \leq C\Big(\sum_{K\in\mathcal{T}_h}\Big[\nu + \sigma\, h_K^2 + \gamma d + \delta_K b_K^2 + \frac{3\, b_K^2\, h_K^2}{\delta_K\, b_K^2 + \nu + \sigma\, h_K^2}\Big] h_K^{2k}\, \|u\|^2_{k+1,\omega(K)} \\ + \sum_{K\in\mathcal{T}_h}\Big[\delta_K + \frac{2dh_K^2}{\nu + \gamma d}\Big] h_K^{2\ell}\|p\|^2_{\ell+1,K}\Big)^{1/2}.$$

Proof. Let $w := u - I_h u$ and $r := p - J_h p$. As the following estimate of $A_S\big((w, r), (w_h, r_h)\big)$ is straightforward, we only emphasise some important aspects. By separation of symmetric and non-symmetric terms and using the definitions of $|[w]|$ and $|||(w_h, r_h)|||$, we obtain

$$A_S\big((w, r), (w_h, r_h) \leq |[w]|\,|||(w_h, r_h)||| + \Big|\sum_{K\in\mathcal{T}_h}\delta_K\big(-\nu\triangle w + \sigma w + \nabla r, (b\cdot\nabla)w_h\big)_K\Big| \\ + \big|(r_h, \operatorname{div} w)\big| + \big|(r, \operatorname{div} w_h)\big| + \big|\big((b\cdot\nabla)w, w_h\big)\big|.$$

The estimates for the interpolation error result in

$$|[w]| \leq C\sum_{K\in\mathcal{T}_h}\Big[\big(\nu + \sigma\, h_K^2 + \delta_K\, b_K^2 + \gamma d\big) h_K^{2k}\|u\|^2_{k+1,\omega(K)}\Big].$$

Now, the remaining terms are estimated separately. We obtain

$$\Big|\sum_{K\in\mathcal{T}_h}\delta_K\big(-\nu\triangle w + \sigma w + \nabla r, (b\cdot\nabla)w_h\big)_K\Big| \\ \leq C\Big(\sum_{K\in\mathcal{T}_h}\big[(\nu + \sigma h_K^2)h_K^{2k}\|u\|^2_{k+1,\omega(K)} + \delta_K h_K^{2\ell}\|p\|^2_{\ell+1,K}\big]\Big)^{1/2} |||(w_h, r_h)|||$$

where we have used that $\nu\delta_K \leq Ch_K^2$ and $\delta_K\sigma \leq C$ by (12)–(13). Since the interpolation operator I_h preserves the discrete divergence, see (8), we have $(r_h, \operatorname{div} w) = 0$. Note that this term is in general non-zero for standard interpolation operators. An estimate would involve a negative power of α causing additional difficulties. Please note that also the Ritz projection of the Stokes problem would not be sufficient.

The term $|(r, \operatorname{div} w_h)|$ can be handled in two ways

$$\big|(r, \operatorname{div} w_h)\big| \leq\ \gamma^{-\frac{1}{2}}\|r\|_0\sqrt{\gamma}\|\operatorname{div} w_h\|_0$$

$$\leq C\Big(\sum_{K\in\mathcal{T}_h} \gamma^{-1} h_K^{2\ell+2}\|p\|_{\ell+1,K}^2\Big)^{1/2} |||(w_h,r_h)|||$$

or

$$\begin{aligned}\big|(r,\operatorname{div} w_h)\big| &\leq \Big(\sum_{K\in\mathcal{T}_h} d\nu^{-1}\|r\|_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\nu|w_h|_{1,K}^2\Big)^{1/2}\\ &\leq C\Big(\sum_{K\in\mathcal{T}_h} d\nu^{-1}h_K^{2\ell+2}\|p\|_{\ell+1,K}^2\Big)^{1/2}|||(w_h,r_h)|||.\end{aligned}$$

This gives

$$\big|(r,\operatorname{div} w_h)\big| \leq C\Big(\sum_{K\in\mathcal{T}_h}\frac{2d}{\nu+\gamma d}h_K^{2\ell+2}\|p\|_{\ell+1,K}^2\Big)^{1/2}|||(w_h,r_h)|||.$$

There are several ways for estimating the remaining term

$$\begin{aligned}\big|\big((b\cdot\nabla)w,w_h\big)\big| &\leq \sum_{K\in\mathcal{T}_h} b_K|w|_{1,K}\|w_h\|_{0,K} \leq \Big(\sum_K \frac{b_K^2}{\sigma}\,|w|_{1,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\sigma\,\|w_h\|_{0,K}^2\Big)^{1/2}\\ &\leq C\Big(\sum_K \frac{b_K^2}{\sigma}\,h_K^{2k}\|u\|_{k+1,\omega(K)}^2\Big)^{1/2}|||(w_h,r_h)|||\end{aligned}$$

or using integration by parts

$$\begin{aligned}\big|\big((b\cdot\nabla)w,w_h\big)\big| &= \big|\big((b\cdot\nabla)w_h,w\big)\big| \leq \sum_{K\in\mathcal{T}_h} b_K|w_h|_{1,K}\|w\|_{0,K}\\ &\leq \Big(\sum_K\frac{b_K^2}{\nu}\,\|w\|_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\nu\,|w_h|_{1,K}^2\Big)^{1/2}\\ &\leq C\Big(\sum_K\frac{b_K^2}{\nu}\,h_K^{2k+2}\|u\|_{k+1,\omega(K)}^2\Big)^{1/2}|||(w_h,r_h)|||\end{aligned}$$

or

$$\begin{aligned}\big|\big((b\cdot\nabla)w,w_h\big)\big| &= \big|\big((b\cdot\nabla)w_h,w\big)\big|\\ &\leq \Big(\sum_{K\in\mathcal{T}_h}\delta_K^{-1}\|w\|_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\delta_K\big\|(b\cdot\nabla)w_h\big\|_{0,K}^2\Big)^{1/2}\\ &\leq C\Big(\sum_{K\in\mathcal{T}_h}\delta_K^{-1}h_K^{2k+2}\,\|u\|_{k+1,\omega(K)}^2\Big)^{1/2}|||(w_h,r_h)|||.\end{aligned}$$

These three estimates give together

$$\big|\big((b\cdot\nabla)w,w_h\big)\big| \leq C\Big(\sum_{K\in\mathcal{T}_h}\frac{3\,b_K^2\,h_K^2}{\delta_K\,b_K^2+\nu+\sigma\,h_K^2}\,h_K^{2k}\|u\|_{k+1,\omega(K)}^2\Big)^{1/2}|||(w_h,r_h)|||.$$

The combination of all above estimates gives the assertion of the Lemma. □

We are now in a position to derive a preliminary a-priori error estimate using the previous stability and continuity estimates.

m:errorest **Lemma 4.** *Let* $(u,p) \in \big(X \cap H^{k+1}(\Omega)^d\big) \times \big((M \cap H^{\ell+1}(\Omega)\big)$ *and* $(u_h, p_h) \in X_h \times M_h$ *be the solutions of* (5) *and* (10)*, respectively. Moreover, we assume that the assumptions* (12)*–*(13) *are valid. Then, the following estimate holds true*

$$\big|\!\big|\!\big|(u-u_h, p-p_h)\big|\!\big|\!\big|^2 \le C \sum_{K\in\mathcal{T}_h} \bigg(\frac{d h_K^2}{\nu+\gamma d} h_K^{2\ell} \, \|p\|^2_{\ell+1,K} + \Big[\nu + \sigma\, h_K^2 + \delta_K b_K^2 + \gamma d + \frac{b_K^2\, h_K^2}{\delta_K\, b_K^2 + \nu + \sigma\, h_K^2}\Big]\, h_K^{2k}\, \|u\|^2_{k+1,\omega(K)} \bigg). \tag{21}$$ EE

Proof. Using the triangle inequality, we obtain

$$\big|\!\big|\!\big|(u-u_h, p-p_h)\big|\!\big|\!\big| \le \big|\!\big|\!\big|(u-I_h u, p-J_h p)\big|\!\big|\!\big| + \big|\!\big|\!\big|(I_h u-u_h, J_h p-p_h)\big|\!\big|\!\big|$$

where $J_h p$ is the standard finite element interpolant of p and $I_h u$ the interpolant of u which additionally preserves the discrete divergence, see (8). The inf-sup condition for A_S given by Lemma 2 ensures the existence of $(w_h, r_h) \in X_h \times M_h$ such that

$$\begin{aligned}\beta_S \big|\!\big|\!\big|(I_h u-u_h, J_h p-p_h)\big|\!\big|\!\big| &\le \frac{A_S\big((I_h u-u_h, J_h p-p_h),(w_h,r_h)\big)}{\big|\!\big|\!\big|(w_h,r_h)\big|\!\big|\!\big|} \\ &= \frac{A_S\big((I_h u-u, J_h p-p),(w_h,r_h)\big)}{\big|\!\big|\!\big|(w_h,r_h)\big|\!\big|\!\big|}\end{aligned}$$

where we also used the Galerkin orthogonality (11). Application of Lemma 3 yields

$$\begin{aligned}&\beta_S \big|\!\big|\!\big|(I_h u-u_h, J_h p-p_h)\big|\!\big|\!\big| \\ &\qquad \le C \sum_{K\in\mathcal{T}_h} \bigg(\big[\nu + \sigma\, h_K^2 + \delta_K b_K^2 + \gamma d + \frac{3\, b_K^2\, h_K^2}{\delta_K\, b_K^2 + \nu + \sigma\, h_K^2}\big] h_K^{2k}\, \|u\|^2_{k+1,\omega(K)} \\ &\qquad\qquad + \big[\delta_K + \frac{2 d h_K^2}{\nu+\gamma d}\big] h_K^{2\ell} \|p\|^2_{\ell+1,K} \bigg)^{1/2}.\end{aligned}$$

We use the assumptions (12)–(13) for the estimates

$$\frac{h_K^2}{\delta_K} \ge C\varphi^2 \ge C(\nu+\gamma d), \qquad \delta_K \le C \frac{d h_K^2}{\varphi^2} \le C \frac{d h_K^2}{\nu+\gamma d}.$$

This allows a simplification of the $[\cdot]$-factors of the previous estimate.

The interpolation error estimates for I_h and J_h give

$$\big|\!\big|\!\big|(u-I_h u, p-J_h p)\big|\!\big|\!\big|^2 \le C \sum_{K\in\mathcal{T}_h} \Big[\big(\nu + \sigma\, h_K^2 + \gamma d + \delta_K b_K^2\big) h_K^{2k} \|u\|^2_{k+1,\omega(K)} + \alpha\, h_K^2 h_K^{2\ell} \|p\|^2_{\ell+1,K} \Big].$$

We can simplify the right hand side by using (12)–(13), (20) and

$$\delta_K \le C\frac{dh_K^2}{\nu+\gamma d}, \qquad \alpha h_K^2 \le C\frac{dh_K^2}{\varphi^2} \le C\frac{dh_K^2}{\nu+\gamma d}.$$

Putting together all estimates and applying the triangle inequality from the beginning of this proof gives the assertion. □

4 A-priori error estimate. Parameter design

Here we will apply the result of Lemma 4 in order to design the stabilisation parameters δ_K and γ, and to refine the a-priori error estimate.

Our first goal is the design of the grad-div parameter γ. We observe from (21) that a positive, h-independent γ prevents a degeneration of the $[\cdot]$-factor of the p-dependent term if $\nu \to +0$. On the other hand, γ as the scaling parameter of the grad-div stabilisation term must not be too large due to the large kernel of the div-operator. Moreover, for dimensional reasons, γ should scale like other terms in $\varphi^2 \sim \nu + \sigma C_F^2 + b_\infty^2 \min\left(\frac{C_F^2}{\nu}; \frac{1}{\sigma}\right) + \gamma$. The extreme cases of very large or small values of σ motivate a balance of the σ-dependent terms in φ^2, i.e. $\sigma C_F^2 \sim b_\infty^2/\sigma$. This leads to the proposal

$$\gamma = \gamma^*(\nu + b_\infty C_F) = \gamma^*\nu(1+Re_\Omega), \qquad Re_\Omega := \frac{b_\infty C_F}{\nu} \tag{22}$$

with a constant $\gamma^* > 0$ and $C_F \sim \mathrm{diam}(\Omega)$. This implies $\varphi^2 \sim \nu + b_\infty C_F + \sigma C_F^2 + b_\infty^2 \min\left(\frac{C_F^2}{\nu}; \frac{1}{\sigma}\right)$ and

$$0 \le \delta_K \le \frac{\delta^*\ h_K^2}{\nu + b_\infty C_F + \sigma C_F^2 + b_\infty^2 \min\left(\frac{C_F^2}{\nu}; \frac{1}{\sigma}\right)}, \qquad 0 \le \delta^* \le \frac{1}{30}. \tag{23}$$

Here we used that the first argument in the min-term of the upper bound of δ_K in (13) can be omitted for sufficiently small h_K. Moreover, recall that the upper bound of δ_K is basically caused by the advective Galerkin term.

Finally, we observe from (23) that $\delta_K b_K^2 \le b_K^2 \min\left(\frac{h_K^2}{\nu}; \frac{1}{\sigma}\right)$. Combining Lemma 4 with the latter estimates, we obtain the following refined a-priori error estimate.

Theorem 5. *Let* $(u,p) \in \left(X \cap H^{k+1}(\Omega)^d\right) \times \left((M \cap H^{\ell+1}(\Omega)\right)$ *and* $(u_h, p_h) \in X_h \times M_h$ *be the solutions of* (5) *and* (10)*, respectively. Then, with the design conditions* (22) *and* (23)*, the a-priori error estimate reads*

$$\begin{aligned}|\!|\!|(u-u_h, p-p_h)|\!|\!|^2 \;\le\; & C \sum_{K\in\mathcal{T}_h} \Bigg(\left[\nu + b_\infty C_F + \sigma h_K^2 + b_K^2 \min\left(\frac{h_K^2}{\nu}; \frac{1}{\sigma}\right)\right] h_K^{2k}\, \|u\|_{k+1,\omega(K)}^2 \\ & + \frac{1}{\nu + b_\infty C_F} h_K^{2(\ell+1)} \|p\|_{\ell+1,K}^2\Bigg).\end{aligned} \tag{24}$$

Theorem 5 clarifies and generalises several aspects of the result of Theorem 4.1 in [6]. The new result relaxes the assumption of quasi-uniformity of the mesh to shape-regularity and the assumption of continuous pressure approximation to a (potentially) discontinuous ansatz. Finally, the H^2-regularity result for the Stokes problem which is used in [6] can be avoided (as a technical tool).

Let us discuss various aspects of the result of Theorem 5:

i) We emphasise that, for inf-sup stable interpolation of velocity-pressure, the parameter design according to (22)–(23) differs from that for equal-order interpolation, see [5, 9]. Besides the missing PSPG terms, estimate (24) remains valid even if the SUPG-stabilisation is switched off (of course the SUPG-part in the norm $|||\cdot|||$ will vanish). This underlines the important role of the grad-div stabilisation which is nothing but the classical augmented Lagrangian approach to the incompressibility constraint. These facts can be observed in the numerical experiments below, see also [6]. Moreover, note that the design (22) of the grad-div parameter γ is favourable for the efficient solution of the corresponding algebraic systems, see [1, 15].

ii) There are two critical terms on the right hand side term in (24): The first term is σh_K^2. For an implicit time discretisation of the non-stationary Navier–Stokes problem (1), there holds $\sigma \sim \delta t^{-1}$ with time step δt. The error estimate (24) suggests to impose the (reasonable) restriction $\sigma h_K^2 \sim h_K^2/\delta t \leq \mathcal{O}(1)$.

The other critical term is $b_K^2 \min\left(\frac{h_K^2}{\nu}; \frac{1}{\sigma}\right)$. We observe that

$$b_K^2 \min\left(\frac{h_K^2}{\nu}; \frac{1}{\sigma}\right) \leq b_\infty C_F \qquad \text{if} \quad \nu \geq \max_K \frac{b_K^2 h_K^2}{b_\infty C_F} \quad \text{or} \quad \sigma \geq \frac{b_\infty}{C_F}. \tag{25}$$

Under the assumptions of Theorem 5, of (25) and with (time step) restriction $\sigma h_K^2 \leq \mathcal{O}(1)$, the a-priori error estimate reads

$$|||(u-u_h, p-p_h)|||^2 \leq C \sum_{K\in\mathcal{T}_h} \left(\frac{h_K^{2\ell+2}}{\nu + b_\infty C_F} \|p\|_{\ell+1,K}^2 + \left[\nu + b_\infty C_F\right] h_K^{2k} \, \|u\|_{k+1,\omega(K)}^2 \right).$$

iii) The last right hand side term of estimate (21) suggests that an appropriate choice of the SUPG-parameters $\delta_K \geq C h_K^2$ would result in an a-priori estimate which is uniformly valid with respect to $\nu^2 + \sigma^2 \to +0$. Unfortunately, this is not possible due to the upper bound of $\delta_K \leq \delta^* h_K^2/\varphi^2 \leq \delta^* \frac{h_K^2}{b_\infty^2} \max\left(\frac{\nu}{C_F^2}; \sigma\right)$ in (23) where the size of φ^2 is mainly dictated by the estimate of the advective Galerkin term.

So it remains open whether the resulting term $b_K^2 \min\left(\frac{h_K^2}{\nu}; \frac{1}{\sigma}\right)$ in (24) is sharp for $\nu, \sigma \to 0$. In the "stationary" case $\sigma = 0$, it behaves like νRe_K^2 with mesh Reynolds number $Re_K = \frac{b_K h_K}{\nu}$. Although the restriction $\nu Re_K^2 \leq 1$ might be not too restrictive for some flows, the case of $\sigma = 0, \nu \to +0$ is not of large physical relevance as the (Navier–Stokes) flow is typically unsteady for large Reynolds numbers. Therefore, an implicit time-stepping leading to auxiliary Oseen problems is a reasonable approach.

iv) Using $\sigma \sim 1/\delta t$, condition (23) reads:

$$0 \le \delta_K \le \frac{\delta^* h_K^2}{\nu + b_\infty C_F + \frac{C_F^2}{\delta t} + b_\infty^2 \min\left(\frac{C_F^2}{\nu}; \delta t\right)}. \tag{26}$$

For moderate time steps $\delta t \sim 1$, we recover the choice $\delta_K \le \mathcal{O}(h_K^2)$ from [6]. However, the upper bound of δ_K becomes very small with either very small time steps δt or very large time steps $\delta t \ge \frac{C_F^2}{\nu} \gg 1$. This supports the arguments given in i).

v) Let us briefly consider the case of the Stokes problem, i.e., $b \equiv 0$, hence $b_\infty = 0$. This case has been discussed very carefully in [15] for the cases $\sigma \in \{0, 1\}$. The analysis shows that the grad-div stabilisation with $\gamma = \mathcal{O}(1)$ results in improved error bounds for inf-sup stable velocity-pressure approximations. Nevertheless, the error bounds are not optimal for the time-discretised problem with $\delta t \sim \sigma^{-1} \to +0$. Let us remark that the present error analysis is consistent with the result in [15] by replacing condition (22) with $\gamma = \mathcal{O}(1)$.

We conclude the discussion with two examples. To be as close as possible to the Navier–Stokes model, the solution u is chosen as the convective field b. The first example with a smooth and ν-independent solution serves to check some aspects of the a-priori analysis. Then we consider a problem with a ν-dependent solution.

The stabilisation parameters are chosen according to (22)–(23) with δ_K according to the upper bound in (26). It is not possible to discuss the dependence of the scheme with respect to all parameters and data in this paper. In particular, we restrict ourselves to the simplest Taylor–Hood pair $Q2/Q1$ on unstructured, quasi-uniform, quadrilateral meshes. For a more detailed consideration, we refer to [16].

Example 1. We solve the Oseen problem (2) on $\Omega = (0,1)^2$ with $b = u$ and solution

$$u = (u_1(x,y), u_2(x,y))^t = (\sin(\pi x), -\pi y \cos(\pi x))^t, \qquad p(x,y) = \sin(\pi x)\cos(\pi y).$$

First, we look for the optimal grad-div parameter γ^*, exemplarily for $\nu = 10^{-8}$, $\sigma = 1$ and $\delta^* = 10^{-2}$. In Fig. 1, we present the dependence of the velocity error $|[e_u]| := |[u - u_h]|$ and of the pressure error $\|e_p\|_0 := \|p - p_h\|_{0,\Omega}$ on γ^* for a sequence of grids. Although a proper choice of γ^* has no visible influence on $\|e_p\|_0$, it improves $|[e_u]|$ significantly. For the coarsest grid, Table 1 clarifies the remarkable influence of grad-div stabilisation. Moreover, the influence of SUPG-stabilisation is negligible for this example. Please note that the parameter choice for the case of inf-sup stable elements differs completely from the case of equal-order interpolation [5].

Fig. 2 shows the h-convergence for the optimised value of γ^* and for the SUPG-parameters δ_K as above for $\nu = 10^{-6}$ and $\sigma \in \{0, 10^2\}$. Together with the former results, we observe robustness for $\sigma = 0$ (stationary case), for $\sigma = 1$ (moderate time steps) and for $\sigma = 10^2$ (small time steps). The error of the 'streamline derivative' $(\sum_{K \subset \Omega} \delta_K \|(\mathbf{b} \cdot \nabla) e_u\|_{0,K}^2)^{\frac{1}{2}}$ is not shown since it is negligible in comparison to the other terms.

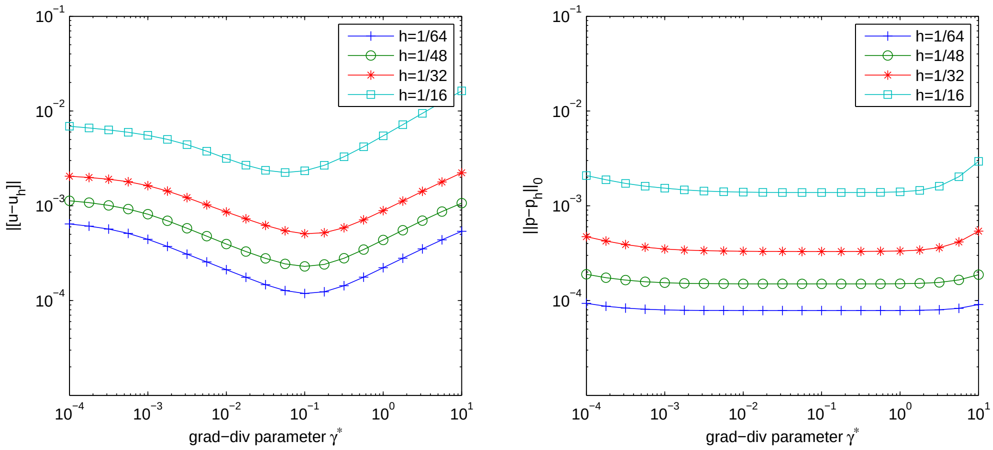

Figure 1: Choice of γ^* for fixed $\nu = 10^{-8}, \sigma = 1$ and $\delta^* = 0.01$ on a sequence of grids. stabwahl

Table 1: Absolute errors for optimal parameters and $\nu = 10^{-6}$, $\sigma = 1$ and $h \approx 1/16$. absolut

	$\|e_u\|_0$	$\|e_u\|_1$	$\|\nabla \cdot u_h\|_0$	$\|e_p\|_0$
unstabilised	0.00515282	0.73055200	0.59034800	0.00261003
supg + grad-div	0.00021771	0.02457250	0.00169816	0.00045265
grad-div	0.00021790	0.02460060	0.00169836	0.00045265
supg($\delta^* = 0.01$)	0.00507245	0.72140500	0.58406400	0.00257348
supg($\delta^* = 5$)	0.00273604	0.33109400	0.31021400	0.00186871

Finally, Fig. 3 shows that the numerical results with $\gamma^* = 0.1$ are basically independent of δ^* for $|[e_u]|$ and $\|e_p\|_0$. SUPG-stabilisation is indeed not necessary for this example. □

Example 2. We solve the Oseen problem (2) on $\Omega = (0,1)^2$ with $b = u$ and solution

$$\begin{aligned} u_1(x) &= \left(1 - \cos\left(\frac{2\pi(e^{R_1x_1}-1)}{e^{R_1}-1}\right)\right) \sin\left(\frac{2\pi(e^{R_2x_2}-1)}{e^{R_2}-1}\right) \frac{R_2}{2\pi} \frac{e^{R_2x_2}}{(e^{R_2}-1)} \\ u_2(x) &= -\sin\left(\frac{2\pi(e^{R_1x_1}-1)}{e^{R_1}-1}\right)\left(1 - \cos\left(\frac{2\pi(e^{R_2x_2}-1)}{e^{R_2}-1}\right)\right) \frac{R_1}{2\pi} \frac{e^{R_1x_1}}{(e^{R_1}-1)} \\ p(x,y) &= R_1R_2 \sin\left(\frac{2\pi(e^{R_1x_1}-1)}{e^{R_1}-1}\right) \sin\left(\frac{2\pi(e^{R_2x_2}-1)}{e^{R_2}-1}\right) \frac{e^{R_2x_1}e^{R_2x_2}}{(e^{R_1}-1)(e^{R_2}-1)}. \end{aligned}$$

The velocity field resembles a counter-clockwise vortex with the centre at $(x_{01}, x_{02}) = \left(\frac{1}{R_1}\log\left(\frac{e^{R_1}+1}{2}\right), \frac{1}{R_2}\log\left(\frac{e^{R_2}+1}{2}\right)\right)$. The parameter are chosen as $R_2 = 0.1$ leading to $x_{02} = 0.5125$ and R_1 such that $x_{01} = 1 - \nu^{\frac{1}{4}}$, i.e. the centre moves with decreasing ν to the right boundary. This leads to a ν-dependent solution with $\|\nabla u\|_0 \sim \nu^{-0.35}$ and $\|p\|_0 \sim \nu^{-0.12}$.

First, we look again for the optimal grad-div parameter γ^*, exemplarily for $\nu = 10^{-3}$, $\sigma = 0$ and $\delta^* = 0.1$. In Fig. 4, we present the dependence of the velocity error $|[e_u]| := |[u - u_h]|$ and of the pressure error $\|e_p\|_0 := \|p - p_h\|_{0,\Omega}$ on γ^* for a sequence of grids. The results are very similar to those of the first example, see Fig. 1.

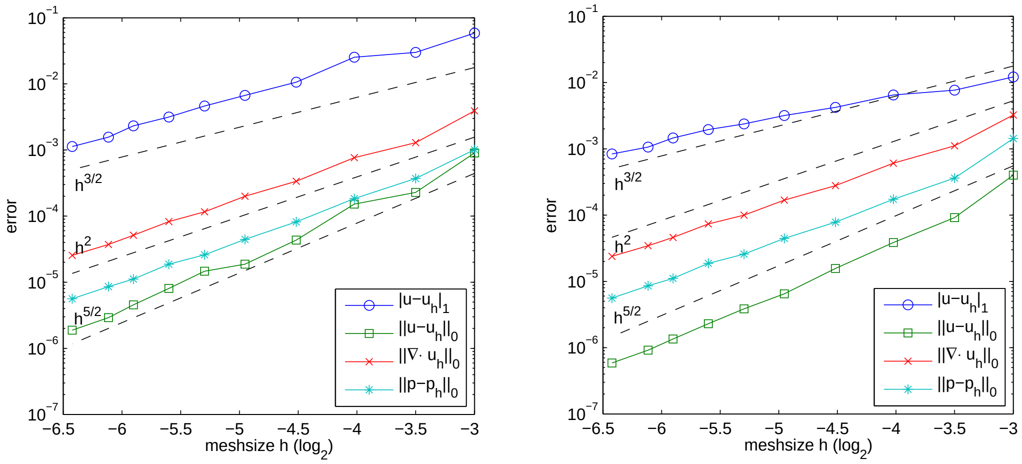

Figure 2: Convergence plots for $\nu = 10^{-6}$, $\sigma = 0$ (left) and $\sigma = 10^2$ (right). konv1

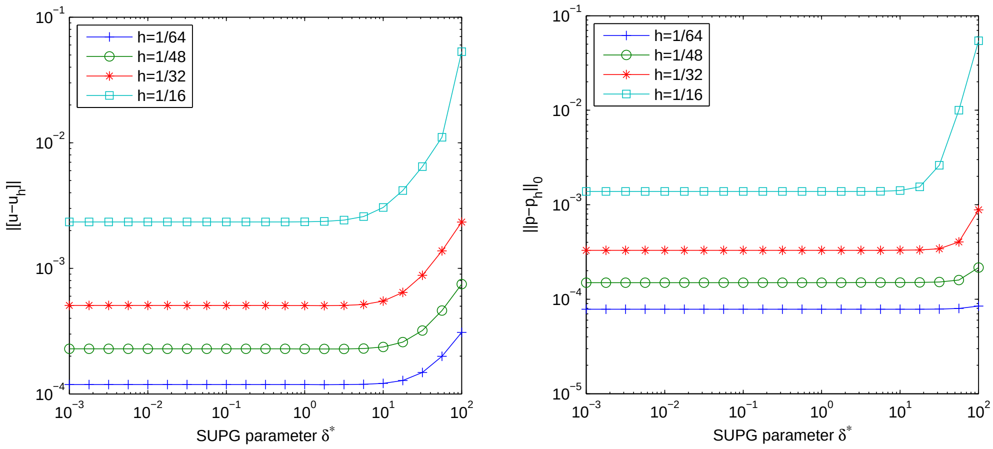

Figure 3: Dependence on δ^* for $\gamma^* = 0.1$ and $\nu = 10^{-8}, \sigma = 1$. delta

Fig. 5 shows the h-convergence for $|[e_u]|$ and $\|e_p\|_0$ (scaled by appropriate Sobolev norms of the solution) for the optimised value of γ^* and δ^* as above for different values of $\nu = 10^{-i}, i = 2, 3, 4, 5, 6$ and $\sigma = 0$. We observe that second order accuracy is reached for the larger values of ν and for the smaller values at least on sufficiently fine grids as full accuracy can only be obtained for a mesh which resolves the boundary layer effects at $x_1 = 1$.

Finally, we consider the robustness of the scheme with optimised parameters for a wide range of values of ν and σ and $h \approx \frac{1}{64}$. Fig. 6 shows a rather weak dependence of the accuracy with respect to both parameters.

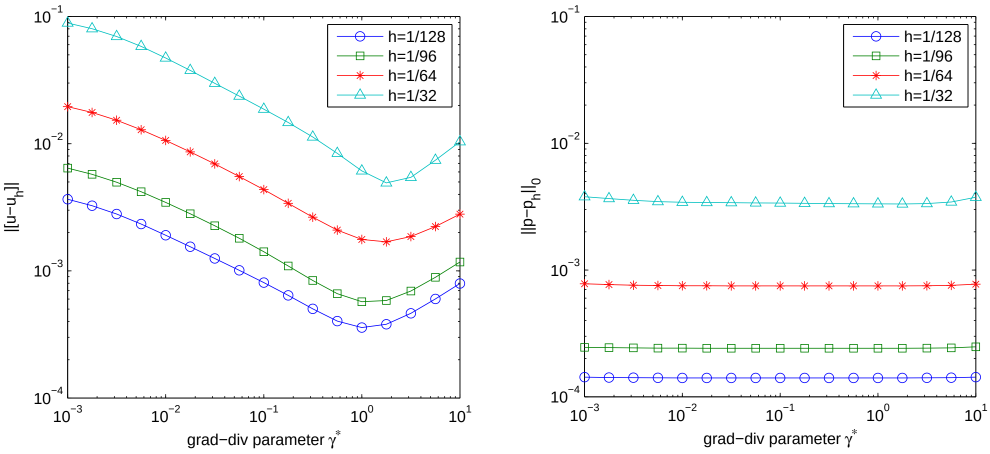

Figure 4: Choice of γ^* for fixed $\nu = 10^{-3}, \sigma = 0$ and $\delta^* = 0.1$. stabwahl_

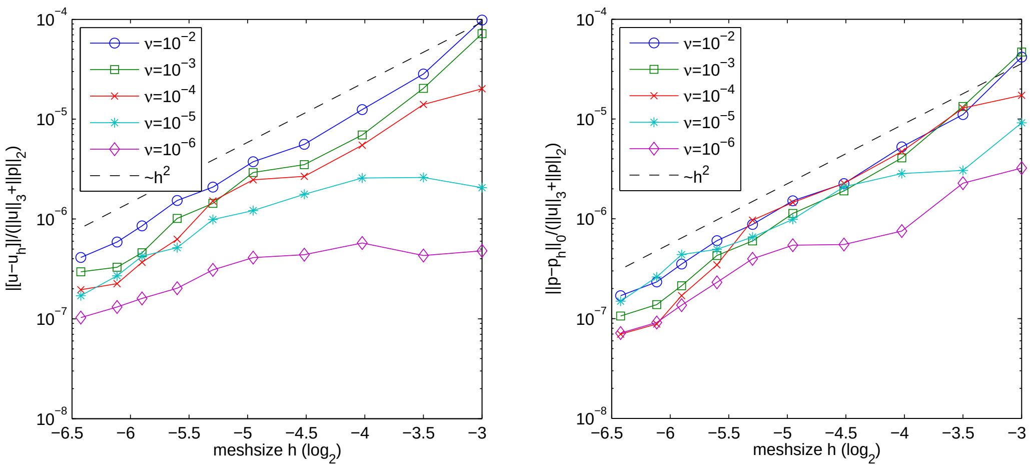

Figure 5: Convergence plots for different values of ν, $\sigma = 0$. konv3

5 Summary. Outlook

ec:outlook

In the present paper, we considered stabilised finite element methods for the generalised Oseen problem. We proved for inf-sup stable discretisations of velocity and pressure the unique solvability based on a modified stability condition and an error estimate. The main results are as follows: First of all, we emphasise the important role of an additional stabilisation of the divergence constraint via grad-div stabilisation. Secondly, the streamline-diffusion (SUPG) stabilisation is obviously less important in the case of inf-sup velocity-pressure pairs. Thirdly, our analysis extends the recent result in [6] on quasi-uniform meshes and continuous pressure approximations to general shape-regular meshes and to discontinuous pressure interpolation. Moreover, we were able to refine the design of the stabilisation parameters given in [6].

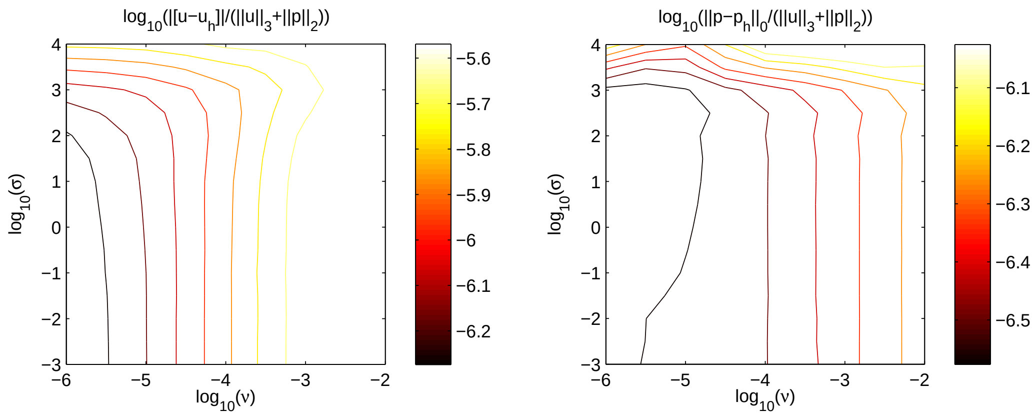

Figure 6: Convergence rates plots for $h \approx \frac{1}{64}$ depending on ν and σ. konv4

Let us finally mention some open problems:

- We didn't discuss the dependence on the polynomial degree of the finite elements. This appears in the stability estimate of Lemma 2 and in the upper bound of δ_K.
- The upper bound of the SUPG-parameter δ_K in formula (23), which stems from the stability analysis, might be not convincing. Let us emphasise that such restriction does not exist for the symmetric stabilisation of local projection type, see e.g. [13].
- The grad-div stabilisation with $\gamma \sim 1$ may lead to problems for iterative solvers of the mixed algebraic problem as the kernel of the div-operator is large. To a certain extent, this is discussed for the Stokes model in [15] and for the Oseen problem in [1].

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