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Local Projection Stabilization of Finite Element Methods for Incompressible Flows

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Abstract A unified analysis for finite element discretizations of linearized incompressible flows using the local projection method with equal-order or inf-sup stable velocity-pressure pairs together with a critical comparison is given.

1 Introduction

A standard numerical approach to the incompressible Navier-Stokes model

$$\partial_t u - \nu \triangle u + (u \cdot \nabla) u + \nabla p = \tilde{f}, \qquad \nabla \cdot u = 0 \qquad \text{in } \Omega \times (0,T) \tag{1}$$

for velocity u and pressure p is to semi-discretize in time first with an A-stable implicit scheme and to apply a fixed-point or Newton-type iteration in each time step. This leads to auxiliary Oseen problems (with $\sigma \geq 0$ from time discretization)

$$-\nu \triangle u + (b \cdot \nabla) u + \sigma u + \nabla p = f, \qquad \nabla \cdot u = 0 \qquad \text{in } \Omega. \tag{2}$$

Residual based stabilization (RBS) methods are the traditional way to cope with spurious solutions of the Galerkin finite element (FE) approximation of (2) caused by violation of the discrete inf-sup stability condition and/or dominating advection. RBS methods are robust and easy to implement, but have severe drawbacks mainly stemming from the strong velocity-pressure coupling in the stabilisation terms.

The key idea of the variational multiscale (VMS) methods [1] is a separation into large, small and unresolved scales. The influence of the unresolved scales has to be modeled. Almost all stabilization methods can be interpreted as VMS methods. In local projection stabilization (LPS) methods [1, 5] the influence of the unresolved scales is modeled by an artificial fine-scale diffusion term.

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Mostly, an equal-order interpolation of velocity-pressure is applied. Other authors prefer discrete inf-sup stable pairs as the natural choice from regularity point of view for fixed $\nu > 0$. Here, a unified theory of LPS methods for equal-order and inf-sup stable pairs for problem (2) is presented. For full proofs, see [4]. Finally, a comparison of both variants is given.

2 Variational Formulation and LPS-Discretization

Standard notations for Lebesgue and Sobolev spaces are used. The $L^2(G)$ inner product in $G \subset \Omega$ is denoted by $(\cdot,\cdot)_G$ with $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$. The notation $a \lesssim b$ is used if there exists a constant $C > 0$ independent of all relevant quantities s.t. $a \le Cb$.

The weak formulation for the Oseen problem (2) with homogeneous Dirichlet data reads: Find $U = (u,p) \in V \times Q := [H_0^1(\Omega)]^d \times L_0^2(\Omega)$, s. t. $\forall V = (v,q) \in V \times Q$:

$$A(U,V) = (\nu\nabla u, \nabla v) + ((b\cdot\nabla)u + \sigma u, v) - (p, \nabla\cdot v) + (q, \nabla\cdot u) = (f,v). \tag{3}$$

Let $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$ be a bounded, polyhedral domain and $\nu \in L^\infty(\Omega)$ with $\nu > 0$, $f \in [L^2(\Omega)]^d$, $b \in [L^\infty(\Omega) \cap H^1(\Omega)]^d$ with $\nabla\cdot b = 0$ and $\sigma \in \mathbb{R}^+$. Usually, b is a FE solution of (2) with $(\nabla\cdot b, q_h) = 0$ for some q_h and $\nabla\cdot b$ does not vanish. A remedy is to write the advective term in skew-symmetric form. The analysis can be extended to problems resulting from Newton iteration including the term $(u\cdot\nabla)b$. Sufficiently small time steps ensure coercivity of $A(\cdot,\cdot)$.

Let $\mathscr{T}_h$ be a shape-regular, admissible decomposition of Ω into d-dimensional simplices or quadrilaterals for $d = 2$ or hexahedra for $d = 3$. h_T is the diameter of a cell $T \in \mathscr{T}_h$ and $h = \max h_T$. Let $\hat{T}$ be a reference element of $\mathscr{T}_h$ and $F_T : \hat{T} \to T$ the standard (affine or bi-/trilinear) reference mapping.

Set $P_{k,\mathscr{T}_h} := \{v_h \in L^2(\Omega) \mid v_h|_T \circ F_T \in \mathbb{P}_k(\hat{T}), T \in \mathscr{T}_h\}$ with the set $\mathbb{P}_k(\hat{T})$ of complete polynomials of degree k on $\hat{T}$ and $Q_{k,\mathscr{T}_h} := \{v_h \in L^2(\Omega) \mid v_h|_T \circ F_T \in \mathbb{Q}_k(\hat{T}), T \in \mathscr{T}_h\}$ with the set $\mathbb{Q}_k(\hat{T})$ of all polynomials on $\hat{T}$ with maximal degree k in each coordinate direction. The FE space of the velocity is given by $\mathbf{V}_{h,k_u} = [Q_{k_u,\mathscr{T}_h}]^d \cap V$ or $\mathbf{V}_{h,k_u} = [P_{k_u,\mathscr{T}_h}]^d \cap V$ with scalar components Y_{h,k_u} of $\mathbf{V}_{h,k_u}$. For simplicity, the analysis is restricted to continuous discrete pressure spaces $Q_{h,k_p} = Q_{k_p,\mathscr{T}_h} \cap C(\overline{\Omega})$ resp. $Q_{h,k_p} = P_{k_p,\mathscr{T}_h} \cap C(\overline{\Omega})$. An extension to discontinuous spaces Q_{h,k_p} is given in [6].

The analysis below takes advantage of the inverse inequalities

$$\exists \mu_{inv} \mid \quad |v|_{1,T} \le \mu_{inv} k_u^2 h_T^{-1} \|v\|_{0,T}, \quad \forall T \in \mathscr{T}_h,\ \forall v_h \in V_{h,k_u}. \tag{4}$$

The Scott-Zhang quasi-interpolant obeys the interpolation properties

$$\exists C > 0 \mid \ \|v - I^u_{h,k_u} v\|_{m,T} \le C h_T^{l-m} k_u^{-(r-m)} \|v\|_{r,\omega_T}, \ \ 0 \le m \le l = \min(k_u+1, r) \tag{5}$$

for $v \in H_0^1(\Omega) \cap H^t(\Omega)$, $t > \frac{1}{2}$ with $v|_{\omega_T} \in H^r(\omega_T)$, $r \geq t$, on the patches $\omega_T := \bigcup_{\overline{T'} \cap \overline{T} \neq \emptyset} T'$. This property can be extended to the vector-valued case with $I^u_{h,k_u} : V \to V_h$. A similar interpolation operator I^p_{h,k_p} satisfying (5) is defined for the pressure.

In LPS-methods the discrete function spaces are split into small and large scales. Stabilization terms of diffusion-type acting only on the small scales are added.

A first variant is to find the large scales on a coarse non-overlapping, shape-regular mesh $\mathscr{M}_h = \{M_i\}_{i \in I}$. $\mathscr{M}_h$ is constructed by coarsening $\mathscr{T}_h$ s. t. each $M \in \mathscr{M}_h$ with diameter h_M consists of one or more neighboring cells $T \in \mathscr{T}_h$. Moreover, suppose that there exists $C \geq 1$ s. t. $h_M \leq C h_T$ for all $T \in \mathscr{T}_h$ with $T \subseteq M \in \mathscr{M}_h$.

Following [5] we define the discrete velocity space D^u_h as a discontinuous FE space on $\mathscr{M}_h$. The restriction to $M \in \mathscr{M}_h$ is denoted by $D^u_h(M) = \{v_h|_M \mid v_h \in D^u_h\}$ The local projection $\pi^u_M : L^2(M) \to D^u_h(M)$ defines the global projection $\pi^u_h : L^2(\Omega) \to D^u_h$ by $(\pi^u_h v)|_M := \pi^u_M(v|_M)$ for all $M \in \mathscr{M}_h$. Denoting the identity on $L^2(\Omega)$ by id, the associated fluctuation operator $\kappa^u_h : L^2(\Omega) \to L^2(\Omega)$ is defined by $\kappa^u_h := id - \pi^u_h$. These operators are applied to vector-valued functions in a component-wise manner.

A discrete space D^p_h and a fluctuation operator κ^p_h are defined similarly.

The second choice consists in choosing lower order discontinuous FE discretizations $D^u_h \times D^p_h$ on $\mathscr{T}_h$ or by enriching $V_{h,k_u} \times Q_{h,k_p}$. The same framework as in the first approach can be used by setting $\mathscr{M}_h = \mathscr{T}_h$.

The LPS scheme reads: find $U_h = (u_h, p_h) \in V_{h,k_u} \times Q_{h,k_p}$ s.t.

$$A(U_h, V_h) + S_h(U_h, V_h) = (f, v_h), \qquad \forall V_h = (v_h, q_h) \in V_{h,k_u} \times Q_{h,k_p}, \tag{6}$$

where the additional stabilization term is given by

$$\begin{aligned} S_h(U_h, V_h) := \sum_{M \in \mathscr{M}_h} \big[& \tau_M (\kappa^u_h((b \cdot \nabla) u_h), \kappa^u_h((b \cdot \nabla) v_h))_M \\ & + \mu_M (\kappa^p_h(\nabla \cdot u_h), \kappa^p_h(\nabla \cdot v_h))_M + \alpha_M (\kappa^u_h(\nabla p_h), \kappa^u_h(\nabla q_h))_M \big]. \end{aligned} \tag{7}$$

An alternative is to replace the first two terms of S_h by the projection of ∇u_h.

The constants τ_M, μ_M and α_M will be determined in Section 3 based on an a priori estimate. Please note that the stabilization S_h acts solely on the fine scales.

In order to control the consistency error of the κ^u_h-dependent stabilization terms, the space D^u_h has to be large enough for the approximation property:

Assumption 1 *The fluctuation operator* κ^u_h *admits for* $0 \leq l \leq k_u$*, the property:*

$$\exists C_\kappa > 0 \mid \qquad \|\kappa^u_h q\|_{0,M} \leq C_\kappa h^l_M k_u^{-l} |q|_{l,M}, \quad \forall q \in H^l(M),\ \forall M \in \mathscr{M}_h. \tag{8}$$

Assumption 1 is valid for the L^2-projection π^u_h. Due to the consistency of the κ^p_h-dependent term in S_h, thus involving D^p_h, such condition is not needed for D^p_h.

The following property of the symmetric and non-negative term $S_h(\cdot,\cdot)$ is valid for all $U \in V \times Q$, see [4], Lemma 2.1:

$$S_h(U,U) \leq C_S |u|_1^2 + C_\kappa^2 \big(\max_M \alpha_M\big) |p|_1^2, \;\; C_S = C_\kappa^2 \max_M \big[\tau_M \|b\|^2_{(L^\infty(M))^d} + \mu_M\big]. \tag{9}$$

Following [5], a special interpolant $j_h^u : H^1(\Omega) \to Y_h$ for the velocity is constructed s.t. the error $v - j_h^u v$ is L^2-orthogonal to D_h^u for all $v \in H_0^1(\Omega)$. A corresponding result can be proved for the pressure too. In order to conserve the standard approximation properties, we additionally assume

Assumption 2 *Let $Y_h(M) := \{v_h|_M \mid v_h \in Y_h, v_h|_{\Omega\setminus M} = 0\}$. There exists β_u, β_p s. t.*

$$\inf_{q_h \in D_h^u} \sup_{v_h \in Y_h(M)} \frac{(v_h, q_h)_M}{\|v_h\|_{0,M}\|q_h\|_{0,M}} \geq \beta_u > 0. \tag{10}$$

$$\inf_{q_h \in D_h^p} \sup_{v_h \in Q_{h,k_p}} \frac{(v_h, q_h)_M}{\|v_h\|_{0,M}\|q_h\|_{0,M}} \geq \beta_p > 0 \tag{11}$$

Remark 1. The space D_h^u must not be too rich w.r.t. (10) but rich enough w.r.t. (8).

Lemma 1. *([4], Lemmata 2, 3) Set $\omega_M := \bigcup_{T\subset M} \omega_T$ for $M \in \mathscr{M}_h$. Under Assumption 2 there are interpolants $j_h^u : V \to V_{h,k_u}$ s.t. for all $v \in [H^l(\Omega)]^d \cap V$:*

$$(v - j_h^u v, q_h) = 0 \qquad \forall q_h \in D_h^u, \tag{12}$$

$$\|v - j_h^u v\|_{0,M} + \frac{h_M}{k_u^2}|v - j_h^u v|_{1,M} \lesssim \left(1 + \frac{1}{\beta_u}\right)\frac{h_M^l}{k^l}\|v\|_{l,\omega_M} \tag{13}$$

and an interpolant $j_h^p : Q \to Q_{h,k_p}$ s.t. for all $v \in Q \cap H^l(\Omega)$:

$$(v - j_h^p v, q_h) = 0, \qquad \forall q_h \in D_h^p, \tag{14}$$

$$\|v - j_h^p v\|_{0,M} + \frac{h_M}{k_p^2}|v - j_h^p v|_{1,M} \lesssim \left(1 + \frac{1}{\beta_p}\right)\frac{h_M^l}{k_p^l}\|v\|_{l,\omega_M}. \tag{15}$$

Remark 2. (13), (15) are optimal w.r.t. h_M but sub-optimal w.r.t. k_u, k_p in $|\cdot|_{1,M}$.

3 A priori Analysis

The stability of the LPS scheme is given for the mesh-dependent norm

$$|[V]|^2 := \|\sqrt{\nu}\nabla v\|_0^2 + \|\sqrt{\sigma}v\|_0^2 + S_h(V,V), \qquad V = (v,q) \in \mathbf{V} \times Q.$$

Then, a "post-processing" argument for the pressure is applied.

Lemma 2. *([4], Lemmata 4 and 5) The following a-priori estimate is valid*

$$\|\sqrt{\nu}\nabla u_h\|_0^2 + \|\sqrt{\sigma}u_h\|_0^2 \leq |[U_h]|^2 = (A + S_h)(U_h, U_h) \leq (f, u_h). \tag{16}$$

There exists a h-independent constant $\gamma > 0$ (depending on the continuous inf-sup constant β and on degree k_u) s. t. (with C_S as in (9) and Poincare constant C_P)

$$\|p_h\|_0 \le \gamma\Big(\sqrt{\nu_\infty}+\sqrt{C_P\sigma}+\min\big(\frac{C_P}{\sqrt{\nu_0}};\frac{1}{\sqrt{\sigma}}\big)b_\infty+\sqrt{C_S}+\max_M \frac{h_M}{\sqrt{\alpha_M}}\Big)|[U_h]| + \frac{\|f\|_{-1}}{\beta}$$

with $\nu_\infty := \|\nu\|_{L^\infty(\Omega)}$, $\nu_0 :=$ *ess inf*$_\Omega \nu(x)$, $b_\infty := \|b\|_{(L^\infty(\Omega))^d}$. *This implies uniqueness and existence of* $(u_h,p_h) \in V_{h,k_u} \times Q_{h,k_p}$ *in (6).*

In LPS methods the Galerkin orthogonality is not fulfilled and a careful analysis of the consistency error has to be done. Subtracting (6) from (3) yields

Lemma 3. *([4], Lemma 6) Let* $U \in V \times Q$ *and* $U_h \in V_{h,k_u} \times Q_{h,k_p}$ *be the solutions of (3) and of (6), respectively. Then, there holds*

$$(A+S_h)(U-U_h,V_h) = S_h(U,V_h), \qquad \forall V_h \in V_{h,k_u} \times Q_{h,k_p}. \tag{17}$$

The consistency error can be estimated using the properties of $S_h(\cdot,\cdot)$.

Lemma 4. *([4], Lemma 7) Let Assumption 1 be fulfilled and* $(u,p) \in V \times Q$ *with* $(b\cdot\nabla)u \in (H^{l_u}(M))^d$, $\nabla\cdot u = 0$, $p \in H^{l_p+1}(M)$ *for all* $M \in \mathscr{M}_h$. *Then, we obtain for* $0 \le l_u, l_p \le k_u$

$$|S_h(U,V_h)| \lesssim \Big(\sum_{M\in\mathscr{M}_h} \tau_M \frac{h_M^{2l_u}}{k_u^{2l_u}}|(b\cdot\nabla)u|^2_{l_u,M} + \alpha_M \frac{h_M^{2l_p}}{k_p^{2l_p}}|p|^2_{l_p+1,M}\Big)^{\frac{1}{2}}|[V_h]|. \tag{18}$$

A combination of the stability and consistency results yields an a-priori estimate.

Theorem 1. *([4], Thm. 1) Let* $U=(u,p)\in V\times Q$ *and* $U_h=(u_h,p_h)\in V_{h,k_u}\times Q_{h,k_p}$ *be the solutions of (3) and of (6). Assume that* $U=(u,p)\in V\times Q$ *is sufficiently regular, i.e.* $p\in H^{l_p+1}(\Omega)$ *and* $u\in[H^{l_u+1}(\Omega)]^d, (b\cdot\nabla)u\in[H^{l_u}(\Omega)]^d$. *Furthermore let the Assumptions 1 and 2 for the coarse velocity space* D_h^u *be satisfied. For the space* D_h^p *we assume that (11) is satisfied. Then, there holds*

$$\begin{aligned}|[U-U_h]|^2 \lesssim \sum_{M\in\mathscr{M}_h}&\Big(\tau_M\big(\frac{h_M}{k_u}\big)^{2l_u}\|(b\cdot\nabla)u\|^2_{l_u,\omega_M} \\ &+\big(1+\frac{1}{\beta_u}\big)^2k_u^2\big(\frac{h_M}{k_u}\big)^{2l_u}C_M^u\|u\|^2_{l_u+1,\omega_M}+\big(1+\frac{1}{\beta_p}\big)^2k_p^2\big(\frac{h_M}{k_p}\big)^{2l_p}C_M^p\|p\|^2_{l_p+1,\omega_M}\Big)\end{aligned} \tag{19}$$

for $1\le l_u\le k_u$ *and* $1\le l_p\le\min\{k_p,k_u\}$ *with*

$$C_M^u := \|\nu\|_{L^\infty(M)} + \frac{h_M^2}{k_u^4}\big(\sigma+\frac{1}{\tau_M}+\frac{1}{\alpha_M}\big)+\mu_M+\|b\|^2_{[L^\infty(M)]^d}\tau_M, \quad C_M^p := \alpha_M + \frac{1}{\mu_M}\frac{h_M^2}{k_p^4}.$$

Under the notation of Lemma 2 we obtain

$$\|p-p_h\|_0 \lesssim \gamma\Big(\sqrt{\nu_\infty}+\sqrt{C_P\sigma}+\min\big(\frac{C_P}{\sqrt{\nu_0}};\frac{1}{\sqrt{\sigma}}\big)b_\infty+\frac{\sqrt{C_S}}{\beta}+\max_M\frac{h_M}{\sqrt{\alpha_M}}\Big)|[U-U_h]|.$$

Now we calibrate the stabilization parameters α_M, τ_M and μ_M w.r.t. h_M, k_u, k_p and problem data by balancing the terms of the right hand side of error estimate (19).

First, equilibrating the τ_M-dependent terms in C_M^u yields $\tau_M \sim h_M/(\|b\|_{(L^\infty(M))^d} k_u^2)$. Similarly, equilibration of the terms in C_M^u and C_M^p involving μ_M and α_M yields $\mu_M \sim h_M^{l_p-l_u+1}/k^{l_p-l_u+2}$, $\alpha_M \sim h_M^{l_u-l_p+1}/k^{l_u-l_p+2}$ where we used $k \sim k_u \sim k_p$.

Corollary 1. *([4], Corollary 2) Let the assumptions of Theorem 1 be valid. For equal-order interpolation $k = k_u = k_p \geq 1$, let $l = l_u = l_p \leq k$ and set $\mu_M = \mu_0 h_M/k^2$, $\alpha_M = \alpha_0 h_M/k^2$, $\tau_M = \tau_0 h_M/(\|b\|_{(L^\infty(M))^d} k^2)$. Then we obtain*

$$|[U-U_h]|^2 \lesssim \sum_{M\in\mathscr{M}} \Big(\big(1+\frac{1}{\beta_p}\big)^2 \frac{h_M^{2l+1}}{k^{2l}} \|p\|^2_{l+1,\omega_M} + \frac{h_M^{2l+1}}{k^{2l+2}} \|\big(\frac{b}{\|b\|_{(L^\infty(M))^d}} \cdot \nabla\big) u\|^2_{l,\omega_M}$$
$$+ \big(1+\frac{1}{\beta_u}\big)^2 \Big[\|\nu\|_{L^\infty(M)} + \sigma \frac{h_M^2}{k^4} + \|b\|_{(L^\infty(M))^d} \frac{h_M}{k^2}\Big] \frac{h_M^{2l}}{k^{2l-2}} \|u\|^2_{l+1,\omega_M} \Big).$$

For inf-sup stable interpolation with $k_u = k_p + 1$, we assume $l_u = l_p + 1 = k_u$ and set $\alpha_M = \alpha_0 h_M^2/k_u^3$, $\mu_M = \mu_0/k_u$, $\tau_M = \tau_0 h_M/(\|b\|_{(L^\infty(M))^d} k_u^2)$. Then we obtain

$$|[U-U_h]|^2 \lesssim \sum_{M\in\mathscr{M}} \Big(\big(1+\frac{1}{\beta_p}\big)^2 \frac{h_M^{2l_u}}{k_u^{2l_u+1}} \|p\|^2_{l_u,\omega_M} + \frac{h_M^{2l_u+1}}{k_u^{2l_u+2}} \|\big(\frac{b}{\|b\|_{L^\infty(M)}} \cdot \nabla\big) u\|^2_{l,\omega_M}$$
$$+ \big(1+\frac{1}{\beta_u}\big)^2 \Big[\|\nu\|_{L^\infty(M)} + \sigma \frac{h_M^2}{k_u^4} + \|b\|_{[L^\infty(M)]^d} \frac{h_M}{k_u^2} + \frac{1}{k_u}\Big] \frac{h_M^{2l_u}}{k_u^{2l_u-2}} \|u\|^2_{l+1,\omega_M} \Big).$$

- For equal-order pairs $V_{h,k} \times Q_{h,k}$ and Taylor-Hood pairs $V_{h,k+1} \times Q_{h,k}$, we obtain the optimal convergence rates $k + \frac{1}{2}$ and $k+1$, respectively, w.r.t. h_M.
- The estimates are not optimal w.r.t. k_u, see Remark 2. Assume that in Lemma 1 there holds $|v - j_h^u v|_{1,M} \lesssim \Big(1+\frac{1}{\beta_u}\Big)\big(\frac{h_M}{k}\big)^{l-1} \|v\|_{l,\omega_M}$ and a similar result for the pressure too. A careful check of the proofs leads to:
  - Equal-order pairs with $k = k_u = k_p$: $\mu_M \sim \alpha_M \sim h_M/k$, $\tau_M \sim \frac{h_M}{\|b\|_{[L^\infty(M)]^d} k_u}$
  - Inf-sup stable pairs with $k_u = k_p + 1$: $\alpha_M \sim \frac{h_M^2}{k_u^2}$, $\mu_M \sim 1$, $\tau_M \sim \frac{h_M}{\|b\|_{(L^\infty(M))^d} k_u}$.

  Then the estimate (19) would be optimal w.r.t. k_u and k_p too with possible exception of the factors depending on β_u, β_p.

Different variants for the choice of the discrete spaces $V_{h,k_u} \times Q_{h,k_p}$ and $D_h^u \times D_h^p$ using simplicial and hexahedral elements are presented in [5] for two variants: a two-level variant with $\mathscr{M}_h = \mathscr{T}_{2h}$ and a one-level variant with $\mathscr{M}_h = \mathscr{T}_h$, thus $h_M = h_K$, with a proper enrichment of $P_{k_u,\mathscr{T}_h}$ by using bubble functions.

Assumption 1 is valid if the local L^2-projection $\pi_M^u : L^2(M) \to D_h^u(M)$ for the velocity and similarly for the pressure is applied, see [5]. In the two-level variant, the constants $\beta_{u/p}$ in Assumption 2 scale like $\mathscr{O}(1/\sqrt{k_{u/p}})$ for simplicial elements and like $\mathscr{O}(1)$ for quadrilateral elements in the affine linear case, see [6].

Please note that the present analysis covers only the case of continuous pressure approximation. An extension to discontinuous discrete pressure approximation, in particular to the case of the case of $Q_k/P_{-(k-1)}$-elements, can be found in [6].

4 Some Numerical Results

A calibration of the LPS parameters requires careful numerical experiments. Some papers validate the design and the convergence rates for the Oseen problem (2) in $\Omega = (0,1)^2$ with the smooth solution $u(x_1,x_2) = \big(\sin(\pi x_1), -\pi x_2 \cos(\pi x_1)\big)$, $p(x_1,x_2) = \sin(\pi x_1)\cos(\pi x_2)$ and data $b = u$, $\sigma = 1$. A study of the one-level variant for equal-order pairs with enrichment of the discontinuous velocity space is given in [7]. The two-level variant is considered in [3] for equal-order and inf-sup stable pairs, see also [6]. Summarizing, all these experiments confirm the calibration of the stabilization parameters w.r.t. h_M and the theoretical a-priori convergence rates.

Here we present some typical results using either Q_2/Q_2 and Q_2/Q_1 pairs for velocity/pressure on unstructured, quasi-uniform meshes for the advection-dominated case $\nu = 10^{-6}$. The coarse spaces of the two-level variant are defined as $D_h^{u/p} := \{v \in [L^2(\Omega)]^d \mid v|_M \in P_{1/1}(M)\}$ and $D_h^{u/p} := \{v \in [L^2(\Omega)]^d \mid v|_M \in P_{1/0}(M)\}$. Table 1 shows comparable results for the best variants of the inf-sup stable Q_2/Q_1

Table 1 Comparison of different variants of stabilization for problem (2) with $\nu = 10^{-6}, h = 1/64$

Pair	τ_0	μ_0	α_0	$\vert u-u_h\vert_1$	$\Vert u-u_h\Vert_0$	$\Vert\nabla\cdot u_h\Vert_0$	$\Vert p-p_h\Vert_0$
Q_2/Q_1	0.0000	0.0000	0.0000	2.56E-1	5.42E-4	2.02E-1	2.31E-4
Q_2/Q_1	0.0562	0.5623	0.0000	1.91E-3	6.20E-6	1.66E-4	8.06E-5
Q_2/Q_1	0.0000	0.5623	0.0000	2.61E-3	7.42E-6	1.72E-4	8.05E-5
Q_2/Q_1	3.1623	0.0000	0.0000	1.87E-2	7.50E-5	1.56E-2	1.08E-4
Q_2/Q_2	0.0000	0.0000	0.0178	1.65E-2	3.48E-5	9.37E-3	6.96E-6
Q_2/Q_2	0.0562	1.0000	0.0178	9.30E-4	2.85E-6	2.14E-4	4.31E-6
Q_2/Q_2	0.0562	0.0000	0.0178	1.77E-3	4.18E-6	1.46E-3	3.25E-6
Q_2/Q_2	0.0000	5.6234	0.0178	3.26E-3	7.20E-6	2.00E-4	7.56E-6

and the equal-order Q_2/Q_2 pairs with the exception of the pressure error. Nevertheless, the importance of the stabilization terms is different. The fine-scale SUPG- and PSPG-type terms are necessary for the equal-order case but not for the inf-sup stable pair. On the other hand, the divergence-stabilization gives clear improvement for the inf-sup stable case and some improvement for the other case. Moreover, the PSPG-type term can be omitted for the inf-sup stable case.

Finally, we apply the LPS stabilization to the lid-driven cavity Navier-Stokes flow (1) with $f = 0$. No-slip data are prescribed with the exception of the upper part of the cavity where $u = (1,0)^T$ is given. A quasi-uniform mesh is used together with the Q_2/Q_1 and Q_2/Q_2 pairs using the two-level LPS variant with scaling parameter τ_0 and μ_0 according to the Oseen case and $\alpha_0 = 0$.

Fig. 1 shows typical velocity profiles for $Re = 5,000$. The results for $h = \frac{1}{64}$ for both variants are in excellent agreement with [2] with well resolved boundary layers. Moreover, the solution for a coarse grid with $h = \frac{1}{16}$ is in good agreement with [2] away from the boundary layers. Similar results are obtained up to $Re = 7.500$ [3]. The results for this nonlinear problem confirm the previous remarks for the

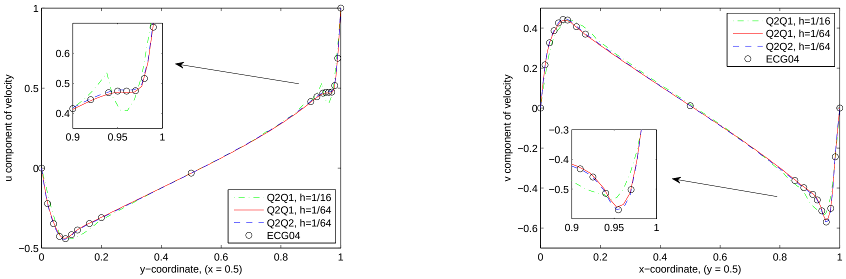

Fig. 1 Lid driven-cavity problem with $Re = 5,000$: Cross-sections of the discrete solutions for Q_2/Q_1 pair with $\tau_0 = \alpha_0 = 0$ and $\mu_0 = 1$ and Q_2/Q_2 pair with $\tau_0 = \alpha_0 = \mu_0 = 1$

linear Oseen problem. For the Q_2/Q_1 element, only the divergence stabilization is necessary whereas for the Q_2/Q_2 pair all stabilization terms are relevant.

5 Summary

A unified a-priori analysis of local projection stabilization (LPS) methods is given for equal-order and inf-sup stable velocity-pressure pairs on isotropic meshes. Numerical results confirm the numerical analysis. Compared to residual-based methods, the error estimates are comparable, but the parameter design is much simpler. A major difference between equal-order and inf-sup stable pairs is that LPS-stabilization is always necessary for equal-order pairs. For inf-sup stable pairs, the necessity of stabilization is much less pronounced. In particular, the grad-div stabilization is much more important than the fine-scale SUPG and PSPG stabilization.

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