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HARDY SPACE INFINITE ELEMENTS FOR SCATTERING AND RESONANCE PROBLEMS*

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Abstract. This paper introduces a new type of infinite elements for scattering and resonance problems. They are derived from a variant of the pole condition as radiation condition. This condition states that a certain transform of the exterior solution belongs to the Hardy space of L^2 boundary values of holomorphic functions on the unit disc if and only if the solution is outgoing. We obtain a symmetric variational formulation of the problem in this Hardy space. Our infinite elements correspond to a Galerkin discretization with respect to the standard monomial orthogonal basis of this Hardy space and lead to simple element matrices. Hardy space infinite elements are particularly well suited for solving resonance problems since they preserve the eigenvalue structure of the problem. We prove super-algebraic convergence for a separated problem. Numerical experiments exhibit fast convergence over a wide range of wave numbers.

Key words. transparent boundary conditions, radiation conditions, pole condition, infinite elements, Hardy spaces, Helmholtz equation

AMS subject classifications. 65N30, 65N12, 35B34, 35J20, 44A10

1. Introduction.

For solving a time-harmonic wave equation on an unbounded domain by finite element methods, appropriate boundary conditions have to be imposed on the artificial boundary of the necessarily finite computational domain. These boundary conditions should be chosen in a way that the solution of the boundary value problem on the computational domain is a good approximation to the restriction of the solution of the wave equation posed on the unbounded domain. Such conditions are called *transparent boundary conditions (TBCs)* and replace the radiation condition at infinity.

The method proposed in this paper works well for scattering problems, but a particular advantage over numerous competing TBCs is the ability to easily treat resonance problems. Such problems appear in molecular physics, acoustics, lasers, and numerous other areas of engineering, natural sciences, and mathematics (cf. [7; 13; 14; 20; 22]). A typical resonance problem for the Neumann-Laplacian in the complement of a smooth, compact domain $K \subset \mathbb{R}^d$ such that $\mathbb{R}^d \setminus K$ is connected consists in finding a nontrivial eigenpair $(u, \lambda) \in H^2_{\text{loc}}(\mathbb{R}^d \setminus K) \times \mathbb{C}$ such that

$$-\Delta u = \lambda u \qquad \text{in } \mathbb{R}^d \setminus K, \tag{1.1a}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial K, \tag{1.1b}$$

$$u \text{ satisfies a radiation condition.} \tag{1.1c}$$

$\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative. For other equivalent definitions of resonances we refer to [21; 22]. In the scattering problem corresponding to (1.1), the number $\lambda \in (0, \infty)$ is given and the homogeneous boundary condition (1.1b) is replaced by an inhomogeneous boundary condition. In the following let $\lambda = \kappa^2$ with $\Re(\kappa) > 0$ and assume that K is contained in the ball $B_a := \{x : \|x\| < a\}$ of radius $a > 0$. One of a several equivalent formulations of the radiation condition (1.1c) is

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that u has an expansion in terms of Hankel functions $H_n^{(1)}$ of the first kind,

$$u(x) = \sum_{l=0}^{\infty} \sum_{m=0}^{M_l} \alpha_{l,m} \, (\kappa|x|)^{1-d/2} H^{(1)}_{l-1+d/2}(\kappa|x|) Y_{l,m} \left(\frac{x}{|x|} \right), \qquad |x| > a, \tag{1.2}$$

where $\{Y_{l,0}, \dots, Y_{l,M_l}\}$ is an orthonormal basis of the l-th eigenspace of the Laplace-Beltrami operator on S^{d-1}. ($Y_{l,m}$ are spherical harmonics $d = 3$ and trigonometric monomials for $d = 2$.) A solution u to (1.1a) satisfying (1.2) is called *outgoing* whereas a solution with a corresponding expansion in terms of Hankel functions of the second kind is called *incoming*. It can be shown that all resonances $\kappa = \sqrt{\lambda}$, $\Re\kappa > 0$ of (1.1) satisfy $\Im(\kappa) < 0$ (cf. [21]). For such values of κ, it follows from the the asymptotic behavior of Hankel functions,

$$|H_l^{(1)}(z)| = \frac{|e^{iz}|}{\sqrt{|z|}} \left(1 + \mathcal{O}\left(\frac{1}{|z|} \right) \right), \quad |H_l^{(2)}(z)| = \frac{|e^{-iz}|}{\sqrt{|z|}} \left(1 + \mathcal{O}\left(\frac{1}{|z|} \right) \right), \quad |z| \to \infty \tag{1.3}$$

that outgoing solutions are exponentially increasing at infinity, and incoming solutions are exponentially decreasing. This implies in particular that incoming, but not outgoing solutions satisfy the Sommerfeld radiation condition

$$r^{(d-1)/2} \left(\frac{\partial u}{\partial r} - i\kappa u \right) \to 0, \qquad \text{as } r = |x| \to \infty \tag{1.4}$$

for $\Im(\kappa) < 0$, i.e. the Sommerfeld condition does not characterize outgoing waves for such values of κ.

The fact that (1.4) is not valid for $\Im(\kappa) < 0$ rules out the simple transparent boundary condition $\partial u/\partial r = i\kappa u$ on ∂B_a for resonance problems as well as higher order local conditions ([6; 11]). Standard infinite elements are based on the series expansion (1.2) or the Wilcox expansion ([3; 4]). Since κ appears in (1.2) in a very nonlinear way inside the argument of the Hankel functions, standard inifinite elements destroy the eigenvalue structure of the problem (1.1). The same holds true for boundary element methods. On the other hand, the PML method preserves the eigenvalue structure, and has been used under the name *complex scaling* for the theoretical study and the numerical computation of resonances in molecular physics since the 1970s ([14; 20]). Despite of the name, Hardy space infinite elements are actually closer to PML than to classical infinite elements (cf. [10]).

In this paper we will use the *pole condition* as radiation condition (cf. [9; 10; 16]). The formulation used in this paper states that a function u is outgoing if and only if a certain transform of u in radial direction belongs to the Hardy space $H^+(S^1)$ on the complex unit circle S^1. Analogously u is incoming if and only if the same transform of u belongs to the orthogonal complement of $H^+(S^1)$ in $L^2(S^1)$. Therefore, we apply the above transform to the variational formulation of the exterior Helmholz equation and incorporate the radiation condition by restricting $L^2(S^1)$ to the correct Hardy space. Hardy space infinite elements correspond to the Galerkin method applied to this variational problem using the standard monomial orthogonal basis of the Hardy space $H^+(S^1)$. For one-dimensional time-dependent problems a similar approach has been studied in [15].

The rest of this paper is organized as follows: We first present a complete treatment of Hardy space infinite elements for one-dimensional problems in §2. In the following section 3 we derive analogous Hardy space infinite elements in arbitrary space dimensions. Then the convergence of this method is analysed using separation arguments in §4. Numerical results are described in §5 before we end this paper by some conclusions including a discussion of pros and cons of the proposed method.

2. One-dimensional Helmholtz equation. In this section we will consider the one-dimensional time harmonic wave equation

$$-u''(r) - \kappa^2 p(r)u(r) = 0, \quad r \geq 0, \tag{2.1a}$$

$$u'(0) = f_0', \tag{2.1b}$$

$$u \quad \text{outgoing} \tag{2.1c}$$

with a given complex wave number $\kappa \in \mathbb{C}$ with positive real part, a boundary value $f_0' \in \mathbb{C}$, and a positive potential $p \in L^\infty((0,\infty))$ satisfying $p(r) = 1$ for $r \geq a$. We will split u into an interior part $u_{\text{int}} := u|_{[0,a]}$ and an exterior part $u_{\text{ext}}(r) := u(r+a)$, $r > 0$. Actually, in one space dimension the Sommerfeld-type transparent boundary condition $u'(a) = i\kappa u(a)$ is exact even for $\Im(\kappa) < 0$, and (2.1) reduces to the simple boundary value problem

$$-u_{\text{int}}'' - p\kappa^2 u_{\text{int}} = 0, \qquad u_{\text{int}}'(0) = f_0', u_{\text{int}}'(a) = i\kappa u_{\text{int}}(a). \tag{2.2}$$

To explain the basic ideas, we will apply Hardy space infinite elements to the problem (2.1) even though this is more complicated then solving (2.2) and requires more degrees of freedom. Note, however, that for the corresponding resonance problem, (2.2) leads to a quadratic eigenvalue problem whereas Hardy space infinite elements will lead to a linear eigenvalue problem.

2.1. Pole condition and Hardy spaces. Since we assumed $p \equiv 1$ on $[a,\infty)$, the exterior part of all solutions to (2.1a) is of the form

$$u_{\text{ext}}(r) = C_1 e^{i\kappa r} + C_2 e^{-i\kappa r}, \quad r \geq 0. \tag{2.3}$$

The term $C_1 e^{i\kappa r}$ corresponds to an outgoing wave, and $C_2 e^{-i\kappa r}$ to an incoming wave. The pole condition distinguishes these two solutions with the help of the Laplace transform $(\mathcal{L}f)(s) := \int_0^\infty e^{-sr} f(r)dr$, $\Re(s) > 0$. Due to the explicit form (2.3), $\widehat{u} := \mathcal{L}u_{\text{ext}}$ is given by

$$\widehat{u}(s) = \frac{C_1}{s - i\kappa} + \frac{C_2}{s + i\kappa}, \ \Re(s) > 0. \tag{2.4}$$

This function has a holomorphic extension to $\mathbb{C} \setminus \{i\kappa, -i\kappa\}$. u is outgoing if and only if $\widehat{u}$ has no pole in the lower complex half-plane and incoming if and only if $\widehat{u}$ has no pole in the upper complex half-plane. This motivates the use of the following Hardy spaces:

Definition 2.1 ($H^-(\mathbb{R})$ and $H^+(\mathbb{R})$). *The Hardy space $H^\pm(\mathbb{R})$ is the set of all functions $f \in L^2(\mathbb{R})$, which are boundary values of a holomorphic function v in $\mathbb{C}^\pm := \{s \in \mathbb{C} : \Im(\pm s) > 0\}$ in the sense that $\int_\mathbb{R} |v(x \pm i\epsilon) - f(x)|^2 dx \xrightarrow{\epsilon \searrow 0} 0$.*

u is outgoing if and only if $\widehat{u}|_\mathbb{R} \in H^-(\mathbb{R})$ and incoming if and only if $\widehat{u}|_\mathbb{R} \in H^+(\mathbb{R})$. Equipped with the standard L^2 inner product, $H^\pm(\mathbb{R})$ are Hilbert spaces (cf. [5]). Moreover, by the Payley-Wiener theorem these spaces are characterized by

$$H^\pm(\mathbb{R}) = \{\widehat{u} \in L^2(\mathbb{R}) : \mathcal{F}^{-1}\widehat{u}(\pm t) = 0 \text{ for almost all } t > 0\} \tag{2.5}$$

in terms of the inverse Fourier transform $(\mathcal{F}^{-1}f)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist} f(s)\,ds$. This yields the orthogonal decomposition $L^2(\mathbb{R}) = H^+(\mathbb{R}) \oplus H^-(\mathbb{R})$. The function v in Definition 2.1 is uniquely determined by f and can be recovered by the Cauchy integral

$$v(s) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(\tilde{s})}{\tilde{s} \pm s}\, d\tilde{s}, \qquad s \in \mathbb{C}^{\pm}. \tag{2.6}$$

Since we are interested in outgoing solutions, we will mainly deal with the space $H^-(\mathbb{R})$. Because of the lack of a convenient orthonormal basis of $H^-(\mathbb{R})$ we will apply a further transform to another closely related Hardy space:

Definition 2.2 ($H^+(S^1)$). *The Hardy space $H^+(S^1)$ is the set of all functions $f \in L^2(S^1)$ which are boundary values of a holomorphic function v in the unit disk $D := \{z \in \mathbb{C} : \ |z| < 1\}$ in the sense that $\int_0^{2\pi} |v(re^{i\theta}) - f(e^{i\theta})|^2 d\theta \overset{r\nearrow 1}{\longrightarrow} 0$. Equipped with the L^2 scalar product $H^+(S^1)$ is a Hilbert space.*

A simple complete orthogonal system of $H^+(S^1)$ is given by the monomials $z^k,\ k = 0, 1, \ldots$ A family of unitary operators identifying the Hilbert spaces $H^-(\mathbb{R})$ and $H^+(S^1)$ can be defined with the help of the Möbius transformations $\varphi_{\kappa_0}(z) := i\kappa_0 \frac{z+1}{z-1}$, $\kappa_0 > 0$, which map the unit disc D to the half space $\mathbb{C}^-$. The parameter κ_0 will act as a tuning parameter in the algorithms to be discussed below. Since $\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_0^{2\pi} \left|(f \circ \varphi_{\kappa_0})(e^{i\theta})\sqrt{\varphi'_{\kappa_0}(e^{i\theta})}\right|^2 d\theta$ and $\varphi'_{\kappa_0}(z) = \frac{-2i\kappa_0}{(z-1)^2}$, the mappings

$$(\mathcal{M}_{\kappa_0} f)(z) := (f \circ \varphi_{\kappa_0})(z)\frac{1}{z-1} \tag{2.7}$$

are isometric from $L^2(\mathbb{R})$ to $L^2(S^1)$ up to the factor $\sqrt{-2i\kappa_0}$, and it can be shown that $\mathcal{M}_{\kappa_0}(H^-(\mathbb{R})) = H^+(S^1)$ (see [5]). Hence, $\sqrt{-2i\kappa_0}\mathcal{M}_{\kappa_0} : H^-(\mathbb{R}) \to H^+(S^1)$ is unitary.

Many of the operators on $H^+(S^1)$ which will appear in our analysis are of the following form:

Definition 2.3 (Toeplitz operator). *Let $f \in L^\infty(S^1)$ be a complex-valued function and let $P : L^2(S^1) \to H^+(S^1)$ denote the orthogonal projection. Then the* Toeplitz operator *$T_f : H^+(S^1) \to H^+(S^1)$ with symbol f is defined by $T_f U := P(fU)$.*

We will need the following classical results on Toeplitz operators: If $f : S^1 \to \mathbb{C}$ is continuous and has no zeros, then T_f is a Fredholm operator, and $\mathrm{ind}(T_f) = -\mathrm{wn}(f)$ where $\mathrm{wn}(f)$ denotes the winding number of f around 0 ([1, Theorem 2.42]). Moreover, if $\mathrm{ind}(T_f) = 0$, then T_f is injective and hence boundedly invertible ([1, Corollary 2.40]).

Let us consider the explicit form of the transform $\widehat{U} := \mathcal{M}_{\kappa_0}\widehat{u}$ of the outgoing solution u of (2.1). With $u_0 := u(a)$ we have

$$u_{\mathrm{ext}}(r) = u_0 e^{i\kappa r} \ \overset{\mathcal{L}|_{\mathbb{R}}}{\longmapsto}\ \widehat{u}(s) = \frac{u_0}{s - i\kappa} \ \overset{\mathcal{M}_{\kappa_0}}{\longmapsto}\ \widehat{U}(z) = \frac{u_0}{i\kappa_0(z+1) - i\kappa(z-1)}. \tag{2.8}$$

Note that $\widehat{U}(1) = u_0/(2i\kappa_0)$. This will be convenient for coupling the transformed exterior to the interior problem. To take advantage of this fact we decompose

$$\widehat{U}(z) = \frac{1}{2i\kappa_0}(u_0 + (z-1)U(z)) \qquad \text{with} \qquad U(z) := \frac{2i\kappa_0\widehat{U}(z) - u_0}{z-1}. \tag{2.9}$$

Since the only singularities of the holomorphic extensions of $\widehat{U}$ and U are simple poles at $\frac{\kappa_0+\kappa}{\kappa_0-\kappa}$ and since $\frac{\kappa_0+\kappa}{\kappa_0-\kappa} \notin \overline{D}$ for $\Re(\kappa/\kappa_0) > 0$, both $\widehat{U}$ and U are analytic on S^1 and belong to $H^+(S^1)$.

2.2. Variational formulation. The formal variational formulation of the differential equation (2.1a) is

$$\int_0^a (u'_{\text{int}}v'_{\text{int}} - \kappa^2 p u_{\text{int}} v_{\text{int}})dr + \int_0^\infty (u'_{\text{ext}}v'_{\text{ext}} - \kappa^2 u_{\text{ext}} v_{\text{ext}})dr = -f'_0 v_{\text{int}}(0). \quad (2.10)$$

The basic identities for transforming the exterior variational problem to the Hardy space are

$$\int_0^\infty f(r)g(r)dr = -\frac{i}{2\pi}\int_{-\infty}^\infty \widehat{f}(-s)\widehat{g}(s)ds = \frac{-i\kappa_0}{\pi}\int_{S^1} \widehat{F}(\overline{z})\widehat{G}(z)|dz| \quad (2.11)$$

with $\widehat{f} = (\mathcal{L}f)|_{\mathbb{R}}$, $\widehat{g} = (\mathcal{L}g)|_{\mathbb{R}}$, $\widehat{F} = \mathcal{M}_{\kappa_0}\widehat{f}$, and $\widehat{G} = \mathcal{M}_{\kappa_0}\widehat{g}$. They will be derived in Lemma A.1 for the more general case $\kappa_0 \in \mathbb{C}$ (cf. Remark 2.8 below). Introducing the bilinear form

$$A(F,G) := \int_{S^1} G(\overline{z})F(z)|dz|, \qquad F,G \in H^+(S^1), \quad (2.12)$$

we have in particular that $\int_0^\infty fg dr = \frac{-i\kappa_0}{\pi} A\left(\widehat{F}, \widehat{G}\right)$.

THEOREM 2.4. *Let $\kappa_0, \Re(\kappa) > 0$ and $X := H^1([0,a]) \oplus H^+(S^1)$. If $u \in H^2_{\text{loc}}([0,\infty))$ is a solution to (2.1), then $(u_{\text{int}}, U)^\top$ with U defined in (2.9) belongs to X and satisfies the variational equation*

$$B\left(\begin{pmatrix} u_{\text{int}} \\ U \end{pmatrix}, \begin{pmatrix} v_{\text{int}} \\ V \end{pmatrix}\right) = -f'_0 v_{\text{int}}(0) \quad (2.13)$$

with

$$B\left(\begin{pmatrix} u_{\text{int}} \\ U \end{pmatrix}, \begin{pmatrix} v_{\text{int}} \\ V \end{pmatrix}\right) := \int_0^a (u'_{\text{int}}v'_{\text{int}} - \kappa^2 p\ u_{\text{int}}v_{\text{int}})dr$$
$$-\frac{i\kappa_0}{4\pi}A(u_0 + (z+1)U, v_0 + (z+1)V) - \frac{i\kappa^2}{4\pi\kappa_0}A(u_0 + (z-1)U, v_0 + (z-1)V)$$

for all $(v_{\text{int}}, V) \in X$ and $v_0 := v_{\text{int}}(a)$. Vice versa, if $(u_{\text{int}}, U)^\top \in X$ is a solution of (2.13), then u_{int} belongs to $H^2([0,a])$ and is the restriction of a solution u to (2.1).

Proof. Assume first that u is a solution to (2.1). It suffices to show that (2.13) holds for all (v_{int}, V) in a dense subset of X. Hence, we start with a test function $v \in C([0,\infty)) \cap H^1([0,a])$ for which v_{ext} has the form

$$v_{\text{ext}}(r) = v_0 e^{ikr}, \qquad \Im(k) > -\Im(\kappa),\ \Re(k) > 0.$$

For such test functions, the product $u \cdot v$ and products of derivatives decay exponentially, and (2.10) can be derived by partial integration. Moreover, for these test functions the identity (2.11) holds both for $f = u_{\text{ext}}$, $g = v_{\text{ext}}$ and for $f = u'_{\text{ext}}$, $g = v'_{\text{ext}}$. In the second case we apply the identities

$$(\mathcal{L}f')(s) = s(\mathcal{L}f)(s) - f_0 \quad (2.14)$$
$$(\mathcal{M}_{\kappa_0}\mathcal{L}|_{\mathbb{R}}f')(z) = i\kappa_0\frac{z+1}{z-1}\ \frac{f_0 + (z-1)F(z)}{2i\kappa_0} - \frac{f_0}{z-1} = \frac{1}{2}\left(f_0 + (z+1)F(z)\right)$$

where f_0 and F are defined in analogy to u_0 and U, to finally arrive at (2.13) with

$$V(z) = \frac{2i\kappa_0(\mathcal{M}_{\kappa_0}\mathcal{L}|_{\mathbb{R}}v_{\text{ext}})(z) - v_0}{z-1} = v_0\frac{k-\kappa_0}{(\kappa_0-k)z+(\kappa_0+k)}. \tag{2.15}$$

Since by virtue of Lemma A.2 the span of such functions is dense in $H^+(S^1)$ and B is continuous on $X \times X$, (2.13) holds for all $(v_{\text{int}}, V)^\top \in X$.

Vice versa, let $(u_{\text{int}}, U)^\top \in X$ be a solution to (2.13). For $v_{\text{int}} = 0$ it follows after multiplication by $-2i\kappa_0$ that

$$\int_{S^1} V(\overline{z}) \left\{-\kappa_0^2\overline{(z+1)}\left[u_0 + (z+1)U(z)\right] - \kappa^2\overline{(z-1)}\left[u_0 + (z-1)U(z)\right]\right\}|dz| = 0 \tag{2.16}$$

for all $V \in H^+(S^1)$. Thus

$$P\left\{\left(-\kappa_0^2|\bullet+1|^2 - \kappa^2|\bullet-1|^2\right)U\right\} = (\kappa_0^2-\kappa^2)u_0 \tag{2.17}$$

with the orthogonal projection P from $L^2(S^1)$ to $H^+(S^1)$. The left hand side of (2.17) is the Toeplitz operator T_m with symbol $m(z) := -\kappa_0^2|z+1|^2 - \kappa^2|z-1|^2$ applied to U. Since $m(z) = -2(\kappa^2+\kappa_0^2) + 2(\kappa^2-\kappa_0^2)\Re(z)$, the graph of m is the straight line connecting $-4\kappa^2$ and $-4\kappa_0^2$. Therefore, T_m is boundedly invertible by the results quoted after Definition 2.3. Hence, (2.17) has a unique solution. By the derivation of (2.13), this solution is given by (2.8) and (2.9), or explicitly $U(z) = u_0\frac{\kappa-\kappa_0}{(\kappa_0-\kappa)z+(\kappa_0+\kappa)}$. Plugging this into (2.13) and using (2.16), we obtain the variational formulation of the boundary value problem (2.2):

$$\int_0^a (v_{\text{int}}'u_{\text{int}}' - \kappa^2 p\ v_{\text{int}}u_{\text{int}})dr = i\kappa u_0 v_0 - v_{\text{int}}(0)f_0'. \tag{2.18}$$

By elliptic regularity results u_{int} belongs to $H^2([0,a])$ and solves (2.2). Hence, it is also part of a solution to (2.1). □

2.3. Gårding-type inequality. It is obvious that the bilinear form B in Theorem 2.4 is bounded and symmetric. Moreover, the interior part $B_{\text{int}}(u_{\text{int}}, v_{\text{int}}) := \int_0^a(u_{\text{int}}'v_{\text{int}}' - \kappa^2 p u_{\text{int}}v_{\text{int}})dr$ satisfies the standard Gårding inequality

$$\Re\left\{B_{\text{int}}\left(u_{\text{int}}, \overline{u_{\text{int}}}\right)\right\} + \beta\|u_{\text{int}}\|_{L^2}^2 \geq \|u_{\text{int}}\|_{H^1}^2, \tag{2.19}$$

with $\beta := (|\kappa|^2+1)\|p\|_{L^\infty} \geq 0$. We want to derive a similar inequality for the whole bilinear form B. Note that we cannot simply choose $V = \overline{U}$ since $\overline{U} \notin H^+(S^1)$ for $U \in H^+(S^1)$ in general. However, a useful conjugation on the Hilbert space $H^+(S^1)$ is given by the mapping $\mathcal{C} : H^+(S^1) \to H^+(S^1)$ defined by

$$(\mathcal{C}F)(z) := \overline{F(\overline{z})}.$$

It is easy to check that $\mathcal{C}$ is well-defined, antilinear and isometric, $\mathcal{C}^2 = I$, i.e. $\mathcal{C}$ is indeed a conjugation. Moreover, it has the useful property that

$$A(F, \mathcal{C}G) = \langle F,\ G\rangle_{L^2(S^1)}. \tag{2.20}$$

Theorem 2.5. *Let $\Re(\kappa^2), \kappa_0 > 0$. Then there exist constants $\alpha, \beta, \gamma > 0$, such that*

$$\Re\left\{(i+\gamma)B\left(\begin{pmatrix}u_{\text{int}}\\ U\end{pmatrix}, \begin{pmatrix}\overline{u}_{\text{int}}\\ \mathcal{C}U\end{pmatrix}\right)\right\} + \beta\|u_{\text{int}}\|_{L^2}^2 \geq \alpha\left\|\begin{pmatrix}u_{\text{int}}\\ U\end{pmatrix}\right\|_X^2.$$

Proof. For the exterior part of the bilinear form $B_{\rm ext} := B - B_{\rm int}$ we obtain from the identity (2.20) that

$$\Re\left\{(i+\gamma)B_{\rm ext}\left(\begin{pmatrix}u_{\rm int}\\ U\end{pmatrix},\begin{pmatrix}\overline{u}_{\rm int}\\ \mathcal{C}U\end{pmatrix}\right)\right\} = \Re\left(\frac{\kappa_0(1-\gamma i)}{4\pi}\right)\|u_0+(z+1)U)\|^2_{L^2(S^1)} + \Re\left(\frac{\kappa^2(1-\gamma i)}{4\pi\kappa_0}\right)\|u_0-(z-1)U\|^2_{L^2(S^1)}$$

for any $\gamma \in \mathbb{R}$. Due to the assumption $\kappa^2 > 0$, we may choose a $\gamma > 0$ such that $\Re(\kappa^2(1-\gamma i)) > 0$. Using the inequality $\|x\|^2+\|y\|^2 \geq \frac{1}{2}\|x-y\|^2$ with $x := u_0+(z+1)U$ and $y := u_0+(z-1)U$ we obtain

$$\Re\left\{(i+\gamma)B_{\rm ext}\left(\begin{pmatrix}u_{\rm int}\\ U\end{pmatrix},\begin{pmatrix}\overline{u}_{\rm int}\\ \mathcal{C}U\end{pmatrix}\right)\right\} \geq \tilde{\alpha}\|U\|^2_{L^2}. \tag{2.21}$$

with $\tilde{\alpha} := \min\left(\Re\left(\frac{\kappa_0(1-\gamma i)}{4\pi}\right), \Re\left(\frac{\kappa^2(1-\gamma i)}{4\pi\kappa_0}\right)\right)$. This together with (2.19) yields the assertion with $\beta := \gamma(|\kappa|^2+1)\|p\|_{L^\infty} > 0$ and $\alpha := \min(\tilde{\alpha},\gamma)$. ◻

Using standard arguments, we obtain the following corollary:

Corollary 2.6. *If the variational equation* (2.13) *has only the trivial solution for $f_0' = 0$, then it has a unique solution for all $f_0' \in \mathbb{R}$, and the solution depends continuously on f_0'.*

By virtue of Theorem 2.4, the variational equation (2.13) is uniquely solvable if and only if κ is not a resonance.

2.4. Galerkin approximation. In the following we will consider the Galerkin approximations to (2.13) using a finite element subspace V_h of $H^1([0,a])$ and the subspace $\Pi_N := \mathrm{span}\{1, z, \ldots, z^N\}$ of $H^+(S^1)$. This leads to the discrete variational problems

$$B\left(\begin{pmatrix}u_h\\ U_N\end{pmatrix},\begin{pmatrix}v_h\\ V_N\end{pmatrix}\right) = -f_0' v_h(0), \qquad \begin{pmatrix}v_h\\ V_N\end{pmatrix} \in X_{h,N} := V_h \oplus \Pi_N. \tag{2.22}$$

Using Theorem 2.5 and the compactness of the embedding $H^1([0,a]) \hookrightarrow L^2([0,a])$, we obtain the following convergence result (cf. [12, Theorem 13.7]).

Theorem 2.7. *Let $\Re(\kappa^2), \kappa_0 > 0$, and assume that κ is not a resonance. Let $(u_{\rm int}, U)^\top \in X$ denote the unique solution to* (2.13). *Then there exist constants $C, N_0, h_0 > 0$ such that the variational problems* (2.22) *have a unique solution $(u_h, U_N)^\top \in X_{h,N}$ for $N \geq N_0$ and $h \leq h_0$, and*

$$\|u-u_h\|^2_{H^1} + \|U-U_N\|^2_{L^2(S^1)} \leq C \inf_{(v_h,V_N)^\top \in X_{h,N}} \left(\|u-v_h\|^2_{H^1} + \|U-V_N\|^2_{L^2(S^1)}\right).$$

Since U is analytic, we have exponential convergence in N, i.e. for some constants $c, \tilde{C} > 0$

$$\inf_{V_N \in \Pi_N} \|U-V_N\|_{L^2(S^1)} \leq \tilde{C}e^{-cN}.$$

Although the derivation of the exterior part of (2.13) is non-standard, its implementation is rather simple: For $F(z) = \sum_{j=0}^\infty \alpha_j z^j$ and $G(z) = \sum_{j=0}^\infty \beta_j z^j$, we have $A(F,G) = 2\pi\sum_{j=0}^\infty \alpha_j\beta_j$. With respect to the monomial basis of Π_N the operators

$$\mathcal{T}_\pm : \mathbb{C} \oplus H^+(S^1) \to H^+(S_1), \qquad \begin{pmatrix}f_0\\ F\end{pmatrix} \mapsto \frac{1}{2}\left(f_0 + (\bullet \pm 1)F\right) \tag{2.23}$$

occurring in (2.13) are approximately represented by the bidiagonal matrices

$$\mathcal{T}_{\pm}^{N} := \frac{1}{2}\begin{pmatrix} 1 & \pm 1 & & & \\ & 1 & \pm 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & \pm 1 \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+2)}. \tag{2.24}$$

The Galerkin approximation (2.22) corresponds to the introduction of an "infinite element" with $N+2$ degrees of freedom, which couples to the interior domain via the unknown u_0. The local element matrix of this infinite element is given by $-2i\kappa_0\left\{\mathcal{T}_{N,+}^{\top}\mathcal{T}_{N,+} + (\kappa/\kappa_0)^2\mathcal{T}_{N,-}^{\top}\mathcal{T}_{N,-}\right\}$.

In the space domain the monomial basis functions correspond to the functions $u_j := (\mathcal{L}|_{\mathbb{R}})^{-1}\left\{\mathcal{M}_{\kappa_0}^{-1}\mathcal{T}_{-}(u_0, z^j)\right\}$, which are given by

$$u_j(r) = e^{i\kappa_0 r}\left\{u_0 + \sum_{n=0}^{j}\begin{pmatrix} j \\ n \end{pmatrix}\frac{(2i\kappa_0 r)^{n+1}}{(n+1)!}\right\}. \tag{2.25}$$

From this formula it is clear that we cannot expect convergence in the exterior space domain with respect to standard norms.

Remark 2.8 (Choice of κ_0). *It follows from* (2.8) *and* (2.9) *or from* (2.25) *that for scattering problems the optimal choice of κ_0 is $\kappa_0 = \kappa$ since in this case $U \equiv 0$, and we obtain the exact transparent boundary condition even with no degrees of freedom in $H^+(S^1)$. For resonance problems, κ_0 should be chosen in the region of the complex plane where resonances are of interest. In this case it is advantageous to choose κ_0 as a complex number with $\Im(\kappa_0) < 0$ and $\Re(\kappa_0) > 0$. All results of this section can be generalized to this case: u is outgoing if and only if $\mathcal{L}u|_{\kappa_0\mathbb{R}}$ belongs to the space $H^-(\kappa_0\mathbb{R}) := \{f(\kappa_0^{-1}\bullet) : f \in H^-(\mathbb{R})\}$. $\mathcal{M}_{\kappa_0}$ maps $H^-(\kappa_0\mathbb{R})$ bijectively to $H^+(S^1)$. In Theorems 2.4, 2.5, 2.7 and Corollary 2.6 we have to replace the conditions on κ and κ_0 by $\Re(\kappa/\kappa_0) > 0$ and $\Re(\kappa^2/\kappa_0) > 0$. These are reasonable assumptions, since κ_0 should be chosen close to the resonances κ of interest, anyways.*

3. Helmholtz equation in higher dimensions. In this section we will treat the Helmholtz equation in higher dimensions in a similar manner as in the previous section in dimension 1. Besides the resonance problem (1.1) we will also study the scattering problem

$$-\Delta u - \kappa^2 u = 0 \qquad \text{in } \mathbb{R}^d \setminus K, \tag{3.1a}$$

$$\frac{\partial u}{\partial \nu} = f \qquad \text{on } \partial K, \tag{3.1b}$$

$$u \text{ satisfies a radiation condition} \tag{3.1c}$$

for given $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ and $f \in H^{-1/2}(\partial K)$. This will be done by considering the Laplace transform of the scaled exterior solution

$$u_{\text{ext}}(r, \hat{x}) := (r+1)^{(d-1)/2}u((r+1)\hat{x}), \quad r > 0,\ \hat{x} \in \Gamma := \partial B_a \tag{3.2}$$

with respect to the radial variable r, i.e.

$$(\mathcal{L}u_{\text{ext}})(s, \hat{x}) := \int_0^\infty e^{-sr}u_{\text{ext}}(r, \hat{x})\,\mathrm{d}r, \quad \Re(s) > 0, \hat{x} \in \Gamma. \tag{3.3}$$

The radial variable is scaled such that $u_{\text{ext}}(r, \hat{x}) \sim \exp(ikar)u_\infty(\hat{x})$ as $r \to \infty$. This scaling is not essential, but simplifies the computations. In particular, we will be able to use part of the analysis of the previous section.

3.1. Pole condition in terms of Hardy spaces. Recall that for Riemannian manifolds A, B the spaces

$$L^2(A; L^2(B)) \sim L^2(A \times B) \sim L^2(A) \otimes L^2(B)$$

are isometrically isomorphic. Consequently, $H^-(\mathbb{R}) \otimes L^2(\Gamma)$ can be considered as a closed subspace of $L^2(\mathbb{R} \times \Gamma)$. It consists of all functions $f \in L^2(\mathbb{R} \times \Gamma)$ for which there exists a function $v \in L^2(\mathbb{C}^- \times \Gamma)$, which is holomorphic in the first variable, such that

$$\int_{\mathbb{R}} \int_{\Gamma} |f(s, \hat{x}) - v(s - i\epsilon, \hat{x})|^2 \,\mathrm{d}\hat{x}\,\mathrm{d}s \overset{\epsilon \to 0}{\longrightarrow} 0.$$

If $v = \mathcal{L}u_{\mathrm{ext}}$ we will shortly write $\mathcal{L}|_{\mathbb{R}} u_{\mathrm{ext}} := f$. Again, v can be recovered from f by a Cauchy integral as in (2.6).

Definition 3.1. *Let u be a complex-valued function on $\mathbb{R}^d \setminus K$, and assume that the Laplace transform $(\mathcal{L}u_{\mathrm{ext}})(s, \bullet)$ is well defined by* (3.2) *and* (3.3) *for all s in some open region $D \subset \mathbb{C}$ and belongs to $L^2(\Gamma)$. We say that u satisfies the* pole condition *if the function $D \to L^2(\Gamma)$, $s \mapsto (\mathcal{L}u_{\mathrm{ext}})(s, \bullet)$ has a holomorphic extension to $\mathbb{C}^-$, and $\mathcal{L}|_{\mathbb{R}} u_{\mathrm{ext}}$ belongs to $H^-(\mathbb{R}) \otimes L^2(\Gamma)$.*

Remark 3.2. *It is easy to see that Definition 3.1 without the condition $\mathcal{L}|_{\mathbb{R}} u_{\mathrm{ext}} \in H^-(\mathbb{R}) \otimes L^2(\Gamma)$ is equivalent to the formulation in [9, Definition 2.1]. Moreover, it was shown in [9, Sec. 9] that the pole condition is equivalent to Sommerfeld's radiation condition for solutions to the Helmholtz equation with $\kappa > 0$. From the results in that section in [9], in particular eq. (9.14) and (9.9b) it can can be seen that the condition $\mathcal{L}|_{\mathbb{R}} u_{\mathrm{ext}} \in H^-(\mathbb{R}) \otimes L^2(\Gamma)$ is also satisfied at least for a sufficiently large.*

Remark 3.3. *In [9] only the case $\kappa > 0$ was considered. However, the pole condition is also a valid radiation condition for $\Im(\kappa) \neq 0$. The singularity of the Laplace transform $\mathcal{L}u_{\mathrm{ext}}$ of an outgoing wave is still a pole with a branch cut located at $i\kappa a$, and hence in the upper half-plane. As mentioned in the introduction, Sommerfeld's radiation condition is not valid for $\Im(\kappa) < 0$, and hence no equivalence result holds true in this case. However, it is actually much simpler to prove equivalence of the pole condition and the radiation condition* (1.2) *since the Hankel function can be recovered from the pole condition approach (see [9, sec. 7]).*

Note that the pole condition is independent of the differential equation. Solutions to the Helmholtz equation will belong to spaces of higher regularity with respect to the second variable.

In analogy to the previous section we consider the Möbius transform $\mathcal{M}_{\kappa_0} \otimes I_{L^2(\Gamma)}$ from $H^-(\kappa_0\mathbb{R}) \otimes L^2(\Gamma)$ to $H^+(S^1) \otimes L^2(\Gamma)$ and write $\widehat{U} := (\mathcal{M}_{\kappa_0} \otimes I_{L^2(\Gamma)})\mathcal{L}|_{\kappa_0\mathbb{R}} u_{\mathrm{ext}}$. Moreover, we define $u_0 := u|_\Gamma$ and

$$U(z, \hat{x}) := \frac{2i\kappa_0 \widehat{U}(z, \hat{x}) - u_0(\hat{x})}{z - 1}, \qquad z \in S^1, \hat{x} \in \Gamma \tag{3.4}$$

in analogy to (2.9).

3.2. Variational formulation. Assume that u is a solution to the scattering problem (3.1) and define $u_{\mathrm{int}} := u|_{\Omega_{\mathrm{int}}}$ with $\Omega_{\mathrm{int}} := B_a \setminus K$ and u_{ext} by (3.2). Then for smooth, rapidly decaying test functions v a straightforward computation yields

$$\int_{\Omega_{\mathrm{int}}} \left\{ \nabla u_{\mathrm{int}} \cdot \nabla v_{\mathrm{int}} - \kappa^2 u_{\mathrm{int}} v_{\mathrm{int}} \right\} \mathrm{d}x + \frac{d-1}{2a} \int_\Gamma u_0 v_0 \,\mathrm{d}\hat{x} + \frac{1}{a} \int_\Gamma \int_0^\infty \partial_r u_{\mathrm{ext}} \partial_r v_{\mathrm{ext}} \,\mathrm{d}r\,\mathrm{d}\hat{x}$$
$$+ a \int_\Gamma \int_0^\infty \left\{ \frac{\nabla_{\hat{x}} u_{\mathrm{ext}} \cdot \nabla_{\hat{x}} v_{\mathrm{ext}}}{(r+1)^2} - \kappa^2 u_{\mathrm{ext}} v_{\mathrm{ext}} - \frac{C_d}{a^2} \frac{u_{\mathrm{ext}} v_{\mathrm{ext}}}{(r+1)^2} \right\} \mathrm{d}r\,\mathrm{d}\hat{x} = -\int_{\partial K} f v_{\mathrm{int}} \,\mathrm{d}s$$

with $C_d := \frac{(d-1)(3-d)}{4}$ and the surface gradient $\nabla_{\hat{x}}$ on Γ.

We first derive the transformation to the Hardy space formally. Due to (2.9), (2.14), and (2.23)) we have

$$i\kappa_0(\mathcal{M}_{\kappa_0}\otimes I)\mathcal{L}|_{\kappa_0\mathbb{R}}u_{\mathrm{ext}} = (\mathcal{T}_-\otimes I)\begin{pmatrix}u_0\\U\end{pmatrix}, \quad (\mathcal{M}_{\kappa_0}\otimes I)\mathcal{L}|_{\kappa_0\mathbb{R}}\partial_r u_{\mathrm{ext}} = (\mathcal{T}_+\otimes I)\begin{pmatrix}u_0\\U\end{pmatrix}.$$

By [9, Theorem 9.1] $(I\otimes\nabla_{\hat{x}})\mathcal{L}u_{\mathrm{ext}}$ is also analytic with respect to the first variable s in $\mathbb{C}^-$ and decays like $|s|^{-1}$ as $|s|\to\infty$. In addition we need to recall the identity

$$\mathcal{L}\left(\frac{f}{\bullet+1}\right)(s) = (\widehat{J}\mathcal{L}f)(s) \quad \text{with} \quad (\widehat{J}\widehat{f})(s) := \int_s^\infty e^{-(\sigma-s)}\widehat{f}(\sigma)d\sigma. \tag{3.5}$$

The inverse operator $\widehat{D} := \widehat{J}^{-1}$ arises from a multiplication with a factor $r+1$, i.e. $(\widehat{D}\mathcal{L}f)(s) = \mathcal{L}\{(\bullet+1)f\}(s) = (-\partial_s+1)\mathcal{L}f(s)$. The Möbius transformed operators are defined by $D := \mathcal{M}_{\kappa_0}\widehat{D}\mathcal{M}_{\kappa_0}^{-1}$ and $J := \mathcal{M}_{\kappa_0}\widehat{J}\mathcal{M}_{\kappa_0}^{-1}$. As

$$\int_\Gamma\int_0^\infty f_1 f_2\,\mathrm{d}r\,\mathrm{d}\hat{x} = \frac{-i\kappa_0}{\pi}A^{\#}\left(\widehat{F}_1,\widehat{F}_2\right)$$

with $A^{\#}(F_1,F_2) := \int_\Gamma\int_{S^1} F_1(\overline{z},\hat{x})F_2(z,\hat{x})\,\mathrm{d}|z|\,\mathrm{d}\hat{x}$ for $\widehat{F}_j = (\mathcal{M}_{\kappa_0}\otimes I)\mathcal{L}|_{\kappa_0\mathbb{R}}f_j$ we obtain

$$\begin{aligned}
&\int_{\Omega_{\mathrm{int}}}\left\{\nabla u_{\mathrm{int}}\nabla v_{\mathrm{int}} - \kappa^2 u_{\mathrm{int}}v_{\mathrm{int}}\right\}\,\mathrm{d}x + \frac{d-1}{2a}\int_\Gamma u_0 v_0\,\mathrm{d}\hat{x}\\
&-\frac{i\kappa_0}{a\pi}A^{\#}\left((\mathcal{T}_+\otimes I)\begin{pmatrix}u_0\\U\end{pmatrix},(\mathcal{T}_+\otimes I)\begin{pmatrix}v_0\\V\end{pmatrix}\right)\\
&-\frac{ai\kappa^2}{\pi\kappa_0}A^{\#}\left((\mathcal{T}_-\otimes I)\begin{pmatrix}u_0\\U\end{pmatrix},(\mathcal{T}_-\otimes I)\begin{pmatrix}v_0\\V\end{pmatrix}\right)\\
&+\frac{ai}{\pi\kappa_0}A^{\#}_{\tan}\left((J\mathcal{T}_-\otimes\nabla_{\hat{x}})\begin{pmatrix}u_0\\U\end{pmatrix},(J\mathcal{T}_-\otimes\nabla_{\hat{x}})\begin{pmatrix}v_0\\V\end{pmatrix}\right)\\
&-\frac{iC_d}{\pi\kappa_0 a}A^{\#}\left((J\mathcal{T}_-\otimes I)\begin{pmatrix}u_0\\U\end{pmatrix},(J\mathcal{T}_-\otimes I)\begin{pmatrix}v_0\\V\end{pmatrix}\right) = -\int_{\partial K} f v_{\mathrm{int}}|_{\partial K}\,\mathrm{d}s.
\end{aligned}\tag{3.6}$$

If $L^2_{\tan}(\Gamma)$ denotes the space of square integrable tangential vector fields on Γ, we define $A^{\#}_{\tan}(F_1,F_2) := \int_\Gamma\int_{S^1} F_1(\overline{z},\hat{x})\cdot F_2(z,\hat{x})\,\mathrm{d}|z|\,\mathrm{d}\hat{x}$.

This bilinear form suggests to introduce the space

$$X^{\#} := \left\{\begin{pmatrix}u_{\mathrm{int}}\\U\end{pmatrix}\in H^1(\Omega_{\mathrm{int}})\oplus H^+(S^1)\otimes L^2(\Gamma) : (J\mathcal{T}_-\otimes\nabla_{\hat{x}})\begin{pmatrix}u_0\\U\end{pmatrix}\in H^+(S^1)\otimes L^2_{\tan}(\Gamma)\right\} \tag{3.7a}$$

with the inner product

$$\begin{aligned}
\left\langle\begin{pmatrix}u_{\mathrm{int}}\\U\end{pmatrix},\begin{pmatrix}v_{\mathrm{int}}\\V\end{pmatrix}\right\rangle_{X^{\#}} &:= \langle u_{\mathrm{int}},v_{\mathrm{in}}\rangle_{H^1(\Omega_{\mathrm{int}})} + \langle U,V\rangle_{H^+(S^1)\otimes L^2(\Gamma)}\\
&+\left\langle(J\mathcal{T}_-\otimes\nabla_{\hat{x}})\begin{pmatrix}u_0\\U\end{pmatrix},(J\mathcal{T}_-\otimes\nabla_{\hat{x}})\begin{pmatrix}v_0\\V\end{pmatrix}\right\rangle_{H^+(S^1)\otimes L^2_{\tan}(\Gamma)}.
\end{aligned}\tag{3.7b}$$

It is easy to see that the bilinear form in (3.6) is bounded with respect to the norm of $X^{\#}$. It is shown in Lemma A.3 that $X^{\#}$ with this inner product is a Hilbert space, and for each $v_{\text{int}} \in H^1(\Omega)$ there exists a vector in $X^{\#}$ containing $v_{\text{int}} \in H^1(\Omega)$ as first component (note that the surface gradient $\nabla_{\hat{x}}$ is not applied to the $H^{1/2}(\Gamma)$-function v_0, but to a sum with other functions). Moreover, it is shown in Lemma A.3 that there exists a dense subset of test functions $(v_{\text{int}}, V)^\top \in X^{\#}$ for which the transforms above are justified. Therefore, we obtain the following result:

THEOREM 3.4. *If u is a solution to the scattering problem* (3.1)*, then* (u_{int}, U) *belongs to the space $X^{\#}$ and satisfies the symmetric variational equation* (3.6).

The converse result will be shown later in Corollary 4.3 using a separation argument.

3.3. Galerkin discretization. Let $V_h \subset H^1(\Omega_{\text{int}})$ be a finite element subspace on the computational domain Ω_{int}, and let $V_h|_\Gamma$ denote the set of traces of functions in V_h on the artificial boundary Γ. Moreover, we use the polynomial subspace $\Pi_N \subset H^+(S^1)$ as in §2. We will use a Galerkin method where the space $X^{\#}$ in Theorem (3.4) is approximated by the finite dimensional subspace

$$X^{\#}_{h,N} := V_h \oplus \Pi_N \otimes V_h|_\Gamma. \tag{3.8}$$

For a given finite element basis of V_h let $\{w_j : j = 0, \dots, N_\Gamma\}$ denote the correspond-

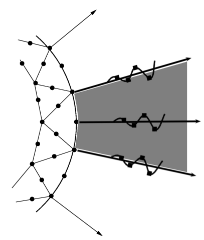

FIG. 3.1. *Hardy space infinite element corresponding to quadratic Lagrange elements*

ing set of nonvanishing traces on Γ. Then we choose the functions $(z, \hat{x}) \mapsto z^n w_j(\hat{x})$ ($j = 0, \dots, N_\Gamma$, $n = 0, \dots, N$) as basis of $\Pi_N \otimes V_h|_\Gamma$. The system matrix with respect to this basis can be assembled element-wise in a finite element fashion as illustrated in Fig. 3.1. Each infinite element couples with the interior finite elements via common degrees of freedom for the Dirichlet values on Γ. Moreover, there is a coupling between neighboring infinite elements. Due to the structure of the bilinear form (3.6), the local element matrices are sums of Kronecker products of matrices. Let M^Γ_{el} and S^Γ_{el} denote the element mass an stiffness matrix on Γ corresponding to the bilinear forms $\int_\Gamma u_0 v_0 \, \mathrm{d}\hat{x}$ and $\int_\Gamma \nabla_{\hat{x}} u_0 \cdot \nabla_{\hat{x}} v_0 \, \mathrm{d}\hat{x}$, respectively. The discrete representation of the operators $\mathcal{T}_\pm$ has already been described in section 2, see eq. (2.24). It remains to discuss the discretization of the operator J. Recall that J is the inverse of a differential operator D, which is given explicitly by

$$(DF)(z) = \frac{(z-1)^2}{2i\kappa_0} F'(z) + \left(\frac{z-1}{2i\kappa_0} + 1 \right) F(z), \quad F \in H^+(S^1). \tag{3.9}$$

To avoid numerical integrations, we use the inverse of the discretization of D

$$D_N := \mathrm{id}_{(N+1)\times(N+1)} + \frac{1}{2i\kappa_0} \begin{pmatrix} -1 & 1 & & & \\ 1 & -3 & 2 & & \\ & 2 & -5 & 3 & \\ & & \ddots & \ddots & \ddots \\ & & & N & -2N-1 \end{pmatrix} \tag{3.10}$$

as discretization of J. Hence, the element matrix of a Hardy space infinite element is given by

$$L_1 \otimes M_\Gamma^{\mathrm{el}} + L_2 \otimes S_\Gamma^{\mathrm{el}} - \kappa^2 L_3 \otimes M_\Gamma^{\mathrm{el}} \tag{3.11}$$

with

$$L_1 = \frac{d-1}{2a}\begin{pmatrix} 1 & \\ & \mathbf{0} \end{pmatrix} - \frac{2i\kappa_0}{a}\mathcal{T}_{N,+}^\top \mathcal{T}_{N,+} - \frac{2C_d i}{\kappa_0 a}\mathcal{T}_{N,-}^\top D_N^{-2}\mathcal{T}_{N,-},$$
$$L_2 = \frac{2ai}{\kappa_0}\mathcal{T}_{N,-}^\top \left(D_a^N\right)^{-2}\mathcal{T}_{N,-}, \qquad \text{and} \qquad L_3 = \frac{2ai}{\kappa_0}\mathcal{T}_{N,-}^\top \mathcal{T}_{N,-}.$$

Note that the eigenvalue structure with respect to κ^2 is preserved for the discretization with Hardy space infinite elements.

4. Convergence analysis for the separated problems. In this section we analyze the convergence of Hardy space infinite elements in the exterior domain (i.e. for the special case $K = B_a$) after a Fourier separation. Implications for the full problem are discussed in §4.4.

4.1. The separated equations. For this end, we choose an orthonormal basis of eigenfunctions $\Phi_n \in L^2(\Gamma)$, $n \in \mathbb{N}_0$ such that $-\Delta_{\hat{x}}\Phi_n = \lambda_n \Phi_n$ for the Laplace-Beltrami operator $\Delta_{\hat{x}}$ on Γ. The functions u_0 and U have expansions with respect to this basis of the form $u_0(\hat{x}) = \sum_{n=0}^\infty u_{0,n}\Phi_n(\hat{x})$, $U(z,\hat{x}) = \sum_{n=0}^\infty U_n(z)\Phi_n(\hat{x})$, and similarly for v_0 and V. Moreover, the Neumann data on $\partial K = \Gamma$, which will be denoted by g instead of f in this section, can be decomposed into the Fourier series $g(\hat{x}) = \sum_{n=0}^\infty g_n \Phi_n(\hat{x})$. Then the variational problem (3.6) decouples into a series of variational problems in $\tilde{X} := \mathbb{C} \oplus H^+(S^1)$

$$B_1\left(\begin{pmatrix} u_{0,n} \\ U_n \end{pmatrix}, \begin{pmatrix} v_0 \\ V \end{pmatrix}\right) + \frac{C_d - a^2\lambda_n}{\kappa_0^2 a} B_2\left(\begin{pmatrix} u_{0,n} \\ U_n \end{pmatrix}, \begin{pmatrix} v_0 \\ V \end{pmatrix}\right) = -g_n v_0, \qquad \begin{pmatrix} v_0 \\ V \end{pmatrix} \in \tilde{X} \tag{4.1}$$

for the Fourier coefficients, where the bilinear forms B_1, B_2 on $\tilde{X}$ are given by

$$B_1\left(\begin{pmatrix} u_0 \\ U \end{pmatrix}, \begin{pmatrix} v_0 \\ V \end{pmatrix}\right) := \frac{d-1}{2a} u_0 v_0$$
$$- \frac{i\kappa_0}{a\pi} A\left(\mathcal{T}_+\begin{pmatrix} u_0 \\ U_n \end{pmatrix}, \mathcal{T}_+\begin{pmatrix} v_0 \\ V_n \end{pmatrix}\right) - \frac{ai\kappa^2}{\pi\kappa_0} A\left(\mathcal{T}_-\begin{pmatrix} u_0 \\ U \end{pmatrix}, \mathcal{T}_-\begin{pmatrix} v_0 \\ V \end{pmatrix}\right),$$
$$B_2\left(\begin{pmatrix} u_0 \\ U \end{pmatrix}, \begin{pmatrix} v_0 \\ V \end{pmatrix}\right) := -\frac{i\kappa_0}{\pi} A\left(J\mathcal{T}_-\begin{pmatrix} u_0 \\ U \end{pmatrix}, J\mathcal{T}_-\begin{pmatrix} v_0 \\ V \end{pmatrix}\right).$$

We use the canonical inner product on $\tilde{X}$ given by the sum of the inner products on $\mathbb{C}$ and $H^+(S^1)$. Defining the operators $K_j : \tilde{X} \to \tilde{X}$ $(j = 1, 2)$ implicitly by

$$\left\langle K_j \begin{pmatrix} u_0 \\ U \end{pmatrix}, \begin{pmatrix} v_0 \\ V \end{pmatrix}\right\rangle_{\tilde{X}} = B_j\left(\begin{pmatrix} u_0 \\ U \end{pmatrix}, \begin{pmatrix} \overline{v_0} \\ \mathcal{C}V \end{pmatrix}\right), \qquad \begin{pmatrix} u_0 \\ U \end{pmatrix}, \begin{pmatrix} v_0 \\ V \end{pmatrix} \in \tilde{X}$$

the variational equations (4.1) can be reformulated as operator equations

$$K_1\begin{pmatrix} u_{0,n} \\ U_n \end{pmatrix} + \frac{C_d - a^2\lambda_n}{a\kappa_0^2} K_2 \begin{pmatrix} u_{0,n} \\ U_n \end{pmatrix} = \begin{pmatrix} -g_n \\ 0 \end{pmatrix}. \tag{4.2}$$

4.2. Uniqueness and smoothness of solutions. Motivated by the Payley-Wiener theorem (2.5) we introduce a transform $\mathcal{Q} : \tilde{X} \to L^2(\mathbb{R}_+)$ by

$$\left(\mathcal{Q}\begin{pmatrix} f_0 \\ F \end{pmatrix}\right)(t) := \frac{-1}{2\pi}\int_{-\infty}^{\infty} e^{ist}\left(\mathcal{M}_{\kappa_0}^{-1}\mathcal{T}_-\begin{pmatrix} f_0 \\ F \end{pmatrix}\right)(\kappa_0 s)\,\mathrm{d}s, \qquad t \geq 0. \tag{4.3}$$

The following result will be used to show uniqueness, but may also be of independent interest.

LEMMA 4.1. *$\mathcal{Q}$ is a norm isomorphism from $\tilde{X}$ to the Sobolev space $H^1(\mathbb{R}_+)$, and $f := \mathcal{Q}(f_0, F)^\top$ satisfies $f(0) = f_0$ and*

$$f'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist}\left(\mathcal{M}_{\kappa_0}^{-1}\mathcal{T}_+\begin{pmatrix} f_0 \\ F \end{pmatrix}\right)(\kappa_0 s)\,\mathrm{d}s, \qquad t \geq 0. \tag{4.4}$$

Proof. Let us first show that the range of $\mathcal{Q}$ is contained in $H^1(\mathbb{R}_+)$. Due to (2.5) we have $f(t) = 0$ for $t < 0$ if we use definition (4.3) also for $t < 0$. Therefore we get $f \in H^1(\mathbb{R}^+)$ if we can show that $w(t) := f(t) + f(-t)$, $t \in \mathbb{R}$ belongs to $H^1(\mathbb{R})$. Introducing $\widetilde{f} := (i\kappa_0)^{-1}\mathcal{M}_{\kappa_0}^{-1}\mathcal{T}_-(f_0, F)^\top$ we have $\widetilde{f}(\kappa_0 s) = (-i\kappa_0)^{-1}(\mathcal{F}f)(s)$ and $-i\kappa_0(\mathcal{F}w)(s) = \widetilde{f}(\kappa_0 s) + \widetilde{f}(-\kappa_0 s)$. Due to (2.14) and the definition (2.23) of $\mathcal{T}_+$ the function $\bullet\widetilde{f} - f_0 = \mathcal{M}_{\kappa_0}^{-1}\mathcal{T}_+(f_0, F)^\top$ belongs to $H^-(\kappa_0\mathbb{R})$. Hence, the function

$$-i\kappa_0^2 s(\mathcal{F}w)(s) = \left(\kappa_0 s\widetilde{f}(\kappa_0 s) - f_0\right) - \left(-\kappa_0 s\widetilde{f}(-\kappa_0 s) - f_0\right), \qquad s \in \mathbb{R}$$

is square integrable, and therefore $\int_{-\infty}^{\infty}(1 + s^2)|(\mathcal{F}w)(s)|^2\, ds < \infty$. This implies that $w \in H^1(\mathbb{R})$. To prove the second assertion first note that

$$\int_0^\infty e^{-ist}f'(t)\,\mathrm{d}t = -f(0) + is\int_0^\infty e^{-ist}f(t)\,\mathrm{d}t = \kappa_0 s\widetilde{f}(\kappa_0 s) - f(0), \quad s \in \mathbb{R}. \tag{4.5}$$

Since we have already shown that $f' \in L^2(\mathbb{R}_+)$, the right hand side is a square integrable function of s by Plancherel's theorem. As $\bullet\widetilde{f} - f_0 = \mathcal{M}_{\kappa_0}^{-1}\mathcal{T}_+(f_0, F)^\top$ also belongs to $L^2(\kappa_0\mathbb{R})$, the constant function $f(0) - f_0$ is square integrable and hence 0. Therefore, $f(0) = f_0$, and applying the inverse Fourier transform to (4.5) yields (4.4). $\mathcal{Q}$ is injective as a composition of injective operators. To prove that $\mathcal{Q}$ is onto, choose an arbitrary $v \in H^1(\mathbb{R}^+)$ and extend it by zero on the negative real axis. Then (2.5) implies that $\mathcal{F}v \in H^-(\mathbb{R})$, and hence $\widehat{V} := (-i\kappa_0)^{-1}\mathcal{M}_{\kappa_0}(\mathcal{F}v)(\kappa_0^{-1}\bullet)$ belongs to $H^+(S^1)$. Moreover, $\left\{\mathcal{M}_{\kappa_0}(\mathcal{F}v')(\kappa_0^{-1}\bullet)\right\}(z) = i\kappa_0\widehat{V}(z) + \frac{2i\kappa_0\widehat{V}(z) - v_0}{z-1}$ with $v_0 := v(0)$ is an element of $H^+(S^1)$. Hence, the function $V(z) := \frac{2i\kappa_0\widehat{V}(z) - v_0}{z-1}$ (cf. (2.9)) belongs to $H^+(S^1)$, and we have $(\mathcal{M}_{\kappa_0}\mathcal{T}_-(v_0, V)^\top)(\kappa_0 s) = -(\mathcal{F}v)(s)$, so $\mathcal{Q}(v_0, V)^\top = v$. The boundedness of $\mathcal{Q}^{-1}$ is follows either directly from the construction above or the open mapping theorem. ∎

Note that the separation index n is the index of an enumeration of the double indices $(l, m) = (l(n), m(n))$ in (1.2). Hence, solutions to (4.2) are given by modified (due to the scaling in (3.2)) and Laplace and Möbius transformed Hankel functions $\mathcal{H}_n^{(1/2)}(r) := r^{1-d/2}H_{l(n)-1+d/2}^{(1/2)}(r)$.

PROPOSITION 4.2. *Let $\Re(\kappa/\kappa_0) > 0$. If ${\mathcal{H}_n^{(1)}}'(\kappa a) \neq 0$, then (4.1) has a unique solution $(u_{0,n}, U_n)^\top \in \tilde{X}$ and $u_{0,n} = \frac{\mathcal{H}_n^{(1)}(\kappa a)}{\kappa{\mathcal{H}_n^{(1)}}'(\kappa a)}g_n$. If ${\mathcal{H}_n^{(1)}}'(\kappa a) = 0$, then (4.1) has a solution if and only if $g_n = 0$.*

Proof. Using lemmas 4.1 and A.1 and the Fourier convolution theorem, it can be shown that (4.1) is equivalent to the variational problem to find $u_n \in H^1(\mathbb{R}_+)$ such that

$$\frac{i}{a\kappa_0}\int_0^\infty \left(-\kappa_0^2 u_n'(t)v'(t) - (\kappa a)^2 u_n(t)v(t) + \frac{a^2\lambda_n - C_d}{(\frac{it}{\kappa_0}+1)^2}u_n(t)v(t)\right)\mathrm{d}t$$
$$+\frac{d-1}{2a}u_n(0)v(0) = -g_n v(0)$$

for all $v \in H^1(\mathbb{R}_+)$. This is the variational formulation of the exterior boundary value problem

$$\kappa_0^2 u_n''(t) - \left((\kappa a)^2 + \frac{C_d - a^2\lambda_n}{(\frac{it}{\kappa_0}+1)^2}\right)u_n(t) = 0, \quad t \geq 0,\ \hat{x} \in \Gamma, \tag{4.6a}$$

$$u_n'(0) = \frac{i}{\kappa_0}\left(ag_n + \frac{d-1}{2}u_n(0)\right), \tag{4.6b}$$

$$u_n \in L^2(\mathbb{R}^+). \tag{4.6c}$$

The general solution of the differential equation (4.6a) is given by

$$u_n(t) = \left(\frac{it}{\kappa_0}+1\right)^{(d-1)/2}\left(c_n^{(1)}\mathcal{H}_n^{(1)}\left(\kappa a\left(\frac{it}{\kappa_0}+1\right)\right) + c_n^{(2)}\mathcal{H}_n^{(2)}\left(\kappa a\left(\frac{it}{\kappa_0}+1\right)\right)\right).$$

Due to the asymptotic behavior (1.3) of the Hankel functions and the assumption $\Re(\kappa/\kappa_0) > 0$, (4.6c) implies that $c_n^{(2)} = 0$. If ${\mathcal{H}_n^{(1)}}'(\kappa a) \neq 0$, then the boundary condition (4.6b) implies $u_{0,n} = \left(\mathcal{H}_n^{(1)}(\kappa a)/(\kappa {\mathcal{H}_n^{(1)}}'(\kappa a))\right) g_n$. Otherwise (4.6b) is satisfied if and only if $g_n = 0$. ∎

As a corollary we obtain the converse of Theorem 3.4:

Corollary 4.3. *If $(u_{\mathrm{int}}, U)^\top \in X^\#$ is a solution to the variation problem* (3.6) *and ${\mathcal{H}_n^{(1)}}'(\kappa a) \neq 0$, then u_{int} is the restriction of a solution to* (3.1).

Proof. Let $(u_{\mathrm{int}}, U)^\top$ be a solution to (3.6) and let $\partial_\nu u \in H^{-1/2}(\Gamma)$ denote the Neumann trace. We rearrange the terms in (3.6) such that only the integrals over Ω_{int} and ∂K are on the left hand side to obtain

$$\int_{\partial K} f v_{\mathrm{int}}|_{\partial K}\,\mathrm{d}s + B_{\mathrm{int}}(u_{\mathrm{int}}, v_{\mathrm{int}}) = B_{\mathrm{ext}}\left(\begin{pmatrix}u_0\\U\end{pmatrix}, \begin{pmatrix}v_0\\V\end{pmatrix}\right).$$

It follows that $B_{\mathrm{ext}}\left((u_0,U)^\top, (v_0,V)^\top\right) = \int_\Gamma \partial_\nu u\, v_0\,\mathrm{d}s$ for all $(v_0,V)^\top$. Now we can apply a Fourier separation on Γ and use Proposition 4.2 to obtain the relation ${\mathcal{H}_n^{(1)}}'(\kappa a)u_{0,n} = \mathcal{H}_n^{(1)}(\kappa a)(\partial_\nu u)_n$ for the Fourier coefficients $(\partial_\nu u)_n := \int_\Gamma \partial_\nu u \overline{\Phi_n}\,\mathrm{d}s$. Therefore, we can define an outgoing exterior solution by (1.2) with the constants $\alpha_{l(n),m(n)} = \frac{\mathcal{H}_n^{(1)}(\kappa a)}{\kappa {\mathcal{H}_n^{(1)}}'(\kappa a)}(\partial_\nu u)_n$ which has the same Cauchy data on Γ as u_{int}. ∎

Lemma 4.4. *We have $U_n \in H^+(S^1) \cap C^\infty(S^1)$.*

Proof. It follows from [9, Prop. 6.6 and Lemma 6.3] that the Fourier coefficients of the Laplace transform, $\hat{u}_n(s) := \langle \mathcal{L}u_{\mathrm{ext}}(x,\cdot), \Phi_n\rangle_{L^2(\Gamma)}$ have an integral representation of the form

$$\hat{u}_n(s) = -\frac{c_n}{i\kappa a - s} - \int_0^\infty \frac{c_n\psi_n(t)}{i\kappa a - t - s}\,\mathrm{d}t, \qquad s \in \mathbb{C}\setminus\{i\kappa a - t : t \geq 0\}$$

with a constant $c_n \in \mathbb{C}$ and a function $\psi_n(t)$ decaying exponentially as $t \to \infty$. This implies that $\hat{u}_n|_{\mathbb{R}} \in H^-(\mathbb{R}) \cap C^\infty(\mathbb{R})$. Hence, $\widehat{U}_n := \mathcal{M}_{\kappa_0}(\hat{u}_n)$ belongs to $H^+(S^1) \cap C^\infty(S^1 \setminus \{1\})$. It remains to study the asymptotic behavior of $\hat{u}_n$ at infinity, or equivalently the behavior of $\widehat{U}_n$ at 1. Expanding the integral kernel in powers of $1/(s - i\kappa_0)$ and using the exponential decay of ψ_n, it can be shown that $\hat{u}_n$ has an asymptotic expansion

$$\hat{u}_n(s) = \sum_{j=1}^{J} \frac{\alpha_j^{(n)}}{(s - i\kappa_0)^j} + o\left(|s - i\kappa_0|^{-J}\right), \qquad |s| \to \infty$$

for any $J \in \mathbb{N}$. By well-known asymptotic formulas for the Laplace transform we have $u_{0,n} = \alpha_1^{(n)}$. Since $\big(\mathcal{M}_{\kappa_0}((\bullet - i\kappa_0)^{-j})\big)(z) = (z-1)^{j-1}/(2i\kappa_0)^j$, it follows that $\widehat{U}_n$ satisfies

$$\widehat{U}_n(z) = \sum_{j=1}^{J} \frac{\alpha_j^{(n)}}{(2i\kappa_0)^j} (z-1)^{j-1} + o\left(|z-1|^{J-1}\right), \qquad \text{as } |z-1| \to 0.$$

Therefore,

$$U_n(z) = \frac{2i\kappa_0 \widehat{U}_n(z) - \alpha_1^{(n)}}{z-1} = \sum_{j=2}^{J} \frac{\alpha_j}{(2i\kappa_0)^{j-1}} (z-1)^{j-2} + o(|z-1|^{J-2}), \quad \text{as } |z-1| \to 0.$$

This implies that U_n is $J-2$ times differentiable at 1. Since J was arbitrary, this together with the properties of $\widehat{U}_n$ shows that $U_n \in H^+(S^1) \cap C^\infty(S^1)$. $\square$

4.3. Convergence. The bilinear form aB_1 essentially coincides with the exterior part B_{ext} of the bilinear form from the one-dimensional case. As in (2.21) we have

$$\Re\left\{(i+\gamma)B_1\left((u_0, U), (\overline{u}_0, \mathcal{C}U)\right)\right\} \geq \alpha \|U\|_X^2 \tag{4.7}$$

for some $\alpha, \gamma > 0$ if $\Re(\kappa_0), \Re(\kappa^2/\kappa_0) > 0$. Therefore, K_1 is boundedly invertible.

Lemma 4.5. *The operator K_2 is compact.*

Proof. K_2 is a rank 1 perturbation of the operator $K_3 : H^+(S^1) \to H^+(S^1)$ given implicitly by

$$(K_3 U, V)_{H^+(S^1)} = -\frac{i\kappa_0}{\pi} \int_{S^1} (\overline{z} - 1) J^2 (z-1) U(z) \overline{V(z)} |dz|. \tag{4.8}$$

Here we have used the boundedness of $J : H^+(S^1) \to H^+(S^1)$ (see (4.9a)) and the symmetry property $A(U, JV) = A(JU, V)$, which follows the representation of $D = J^{-1}$ with respect to the monomial basis. Since the orthogonal projection $P : L^2(S^1) \to H^+(S^1)$ and the operator $H^+(S^1) \to H^+(S^1)$, $U \mapsto J((\bullet - 1)U)$ are bounded, it suffices to show the compactness of $\tilde{K}_4 : H^+(S^1) \to L^2(S^1)$, $(\tilde{K}_4 U)(z) = (\overline{z} - 1)(JU)(z)$, or equivalently the compactness of $K_4 := H^-(\mathbb{R}) \to L^2(\mathbb{R})$, $(K_4 f)(s) := \frac{2i\kappa_0}{s + i\kappa_0} (\widehat{J} f)(s)$. The following inequalities hold for some constants $C > 0$, $f \in H^-(\mathbb{R})$ and $s, s_1, s_2 \in \mathbb{R}$

$$\|\widehat{J} f\|_2 \leq C \|f\|_2, \tag{4.9a}$$

$$|(K_4 f)(s)| \leq \frac{C}{|s + i\kappa_0|} \|f\|_2, \tag{4.9b}$$

$$|(K_4 f)(s_1) - (K_4 f)(s_2)| \leq C \sqrt{|s_1 - s_2|} \|f\|_2. \tag{4.9c}$$

The first inequality is a consequence of Plancherel's Theorem, since $\widehat{J}f = g * f$ with $g(t) := e^{-at}$ for $t \geq 0$ and $g(t) \equiv 0$ for $t < 0$:

$$\|\widehat{J}f\|_2 = \|g * f\|_2 = 2\pi\|\mathcal{F}(g * f)\|_2 = \sqrt{2\pi}\|\mathcal{F}g\ \mathcal{F}f\|_2 \leq \sqrt{2\pi}\|\mathcal{F}g\|_\infty\|\mathcal{F}f\|_2 \leq C\|f\|_2.$$

For the third inequality we assume w.l.o.g. $s_2 > s_1$ and write

$$\left|\frac{(\widehat{J}f)(s_1)}{s_1 + i\kappa_0} - \frac{(\widehat{J}f)(s_2)}{s_2 + i\kappa_0}\right| = \left|\int_{s_1}^{s_2} \frac{e^{a(s_1-\sigma)}}{s_1 + i\kappa_0} f(\sigma)\,\mathrm{d}\sigma + \int_{s_2}^{\infty}\left(\frac{e^{a(s_1-\sigma)}}{s_1 + i\kappa_0} - \frac{e^{a(s_2-\sigma)}}{s_2 + i\kappa_0}\right) f(\sigma)\,\mathrm{d}\sigma\right|$$

The first integral can be estimated with the Cauchy-Schwarz inequality by

$$|I_1| \leq \sqrt{s_2 - s_1} \sup_{\sigma \in [s_1, s_2]} \left|\frac{e^{a(s_1-\sigma)}}{s_1 + i\kappa_0}\right|^2 \left(\int_{s_1}^{s_2} |f(\sigma)|^2 d\sigma\right)^{1/2} \leq C\sqrt{|s_1 - s_2|}\|f\|_2.$$

For I_2 the mean value theorem and the Cauchy-Schwarz inequality yield:

$$|I_2| \leq \widetilde{C}|s_2 - s_1| \sup_{t \in [0,1]} \left|e^{a(t-1)(s_2-s_1)}\right| \left(\int_{s_2}^{\infty} |e^{a(s_2-\sigma)}|^2 d\sigma\right)^{1/2} \|f\|_2$$

and we have shown (4.9c). (4.9b) can be proven in an analogous manner.

In order to show the compactness of K_4 we use the theorem of Arzela-Ascoli. Thus let $(w_n)_{n\in\mathbb{N}}$ be a sequence in $H^-(\mathbb{R})$ with $\|w_n\|_2 \leq 1$ for all $n \in \mathbb{N}$ and $v_n := K_4 w_n$. Due to the Arzela-Ascoli theorem, there exist a subsequence of (v_n) which converges in the supremum norm of a compact subset I of $\mathbb{R}$, since (v_n) is equicontinuous and bounded in I by (4.9b) and (4.9c). Let $I_j := [-j, j] \subset \mathbb{R}$, $v_{n_0(l)} := v_l$ and for every $j \in \mathbb{N}$ $\left(v_{n_j(l)}\right)$ a subsequence of $\left(v_{n_{j-1}(l)}\right)$ converging in the supremum norm of I_j. Thus the diagonal subsequence $v_{n(l)} := v_{n_l(l)}$ converges pointwise in $\mathbb{R}$ and for each I_j in the supremum norm of I_j to a function v. For given $\epsilon > 0$ it remains to show that there exist a $l_0(\epsilon) \in \mathbb{N}$ such that $\|v_{n(l)} - v\|_2 < \epsilon$ for all $l \geq l_0$. This can be seen with (4.9b) since there exist a $j_0(\epsilon) \in \mathbb{N}$ such that

$$\int_{\mathbb{R}\setminus I_{j_0}} \left|v_{n(l)}(s) - v(s)\right|^2 ds \leq 2C \int_{\mathbb{R}\setminus I_{j_0}} \frac{1}{|s + i\kappa_0|^2} ds \leq \frac{\epsilon}{2}.$$

Because of the uniform convergence of $\left(v_{n(l)}\right)$ in I_{j_0}, the subsequence $\left(v_{n(l)}\right)$ of the image sequence $v_n = K_4 w_n$ converges in $L^2(\mathbb{R})$ and the proof is done. □

With these preparations we easily obtain the following super-algebraic convergence result:

THEOREM 4.6. *Assume that* $\kappa_0, \kappa/\kappa_0$, *and* κ^2/κ_0 *have positive real part and that* ${\mathcal{H}_n^{(1)}}'(\kappa a) \neq 0$, *i.e.* κ *is not a resonance of* (4.1). *Then there exist constants* $N_0, C_l > 0$ *such that for* $N \geq N_0$ *there exists a unique solution* $\left(u_{0,n}^{(N)}, U_n^{(N)}\right)^\top$ *in the space* $X_N := \mathbb{C} \oplus \Pi_N$ *to the variational equation*

$$B_1\left(\begin{pmatrix} u_{0,n}^{(N)} \\ U_n^{(N)} \end{pmatrix}, \begin{pmatrix} v_0^{(N)} \\ V^{(N)} \end{pmatrix}\right) + \frac{C_d - a^2\lambda_n}{a\kappa_0^2} B_2\left(\begin{pmatrix} u_{0,n}^{(N)} \\ U_n^{(N)} \end{pmatrix}, \begin{pmatrix} v_0^{(N)} \\ V^{(N)} \end{pmatrix}\right) = -g_n v_0^{(N)} \quad (4.10)$$

for $(v_0^{(N)}, V^{(N)})^\top \in X_N$. *Moreover, for any* $l \in \mathbb{N}$ *the error estimate*

$$\left\|\begin{pmatrix} u_{0,n}^{(N)} \\ U_n^{(N)} \end{pmatrix} - \begin{pmatrix} u_{0,n} \\ U_n \end{pmatrix}\right\|_{\tilde{X}} \leq \frac{C}{N^l} \quad (4.11)$$

holds for some constant C depending on l, n, and κ.

Proof. Due to the coercivity estimate (4.7) the method converges for the bilinear form B_1. Using [12, Theorem 13.7], Proposition 4.2, and Lemma 4.5, it follows that the whole method (4.10) is stable and convergent. From the approximation properties of trigonometric polynomials and Lemma 4.4, it follows that the speed of convergence is super-algebraic. ∎

4.4. Discussion. Since the operators on the left hand side of (4.2) are compact perturbations of Toeplitz operators, we could have appealed to more general convergence results for the Finite Section Method (cf. [1, Chap. 7]) for an alternative proof of Theorem 4.6.

For a fixed finite element subspace on Γ, a separation argument in this subspace and Theorem 4.6 yield super-algebraic convergence to a transformed outgoing solution as $N \to \infty$. However, our results do not exclude the possibility that the constants in the convergence estimate explode as the mesh size tends to 0. To our knowledge this is also the state of the art for usual infinite elements in the space domain (cf. [3; 4]). With uniform estimates in the separation index n one would obtain convergence of the Neumann-to-Dirichlet (or equivalently the Dirichlet-to-Neumann) operators in the natural operator norms, which easily yields a convergence result for the scattering problem (3.1) (cf. [10; 11]). If the convergence is also uniform in κ, one can prove convergence for the resonance problem as shown in a forthcoming paper.

A different possibility to prove convergence is to establish a Gårding-type inequality or an inf-sup-condition for the bilinear form in (3.6) as shown for $d = 1$ in Theorem 2.5. Unfortunately, we have not been able to show either a discrete or a continuous inf-sup-condition for $d \geq 2$ so far, although our conjecture is that both hold true with respect to the norm in (3.7b) and the subspaces in (3.8).

5. Numerical results. In a first example, we considered the scattering of plane incident waves with different wave numbers κ by a kite-shaped domain (Fig. 5.1). The aim is to study the convergence with respect to N, the number of degrees of freedom in the Hardy space $H^+(S^1)$, if the finite element errors are dominated by the Hardy space discretization errors. This rate of convergence is determined by the rate of convergence of the Dirichlet-to-Neumann or Neumann-to-Dirichlet operator on Γ applied to the trace of the exact solution. Therefore, we computed a pair of Cauchy data on Γ as a reference solution by a Nystroem integral equation method (cf. [2, §3.5]). We used the reference Neumann data on spheres of radius 2 and 3.5 as initial data for the Hardy space method (HSM) and compared the Dirichlet data computed by the HSM to the reference Dirichlet data. As basis functions on Γ we used so-called hierarchic shape functions of high polynomial degrees (see [19, Section 3.1.4]). The error plot in Fig. 5.1 clearly exhibits fast convergence with respect to N both for the wave number $\kappa = 5$ and $\kappa = 25$. As for other methods (e.g. PML or standard infinite elements), the error for a fixed number of degrees of freedom in the exterior domain is the smaller the larger the distance of the coupling boundary to the scatterer.

Since a crucial advantage of the HSM is the applicability of the method to resonance problems, we computed as a second example the resonances of a square with a small opening. This was done using the finite element solver NGSOLVE, which is an add-on of the mesh generator NETGEN [18]. In Fig. 5.2 three different eigenfunctions are plotted. Two of them correspond to the real valued eigenvalues of the Laplace operator in a closed square and the third to an exterior surface resonance, the location of which depend mainly on the circumference of the obstacle (cf. [22] and the references therein). This can be seen in Fig. 5.3, where the resonances computed with the HSM

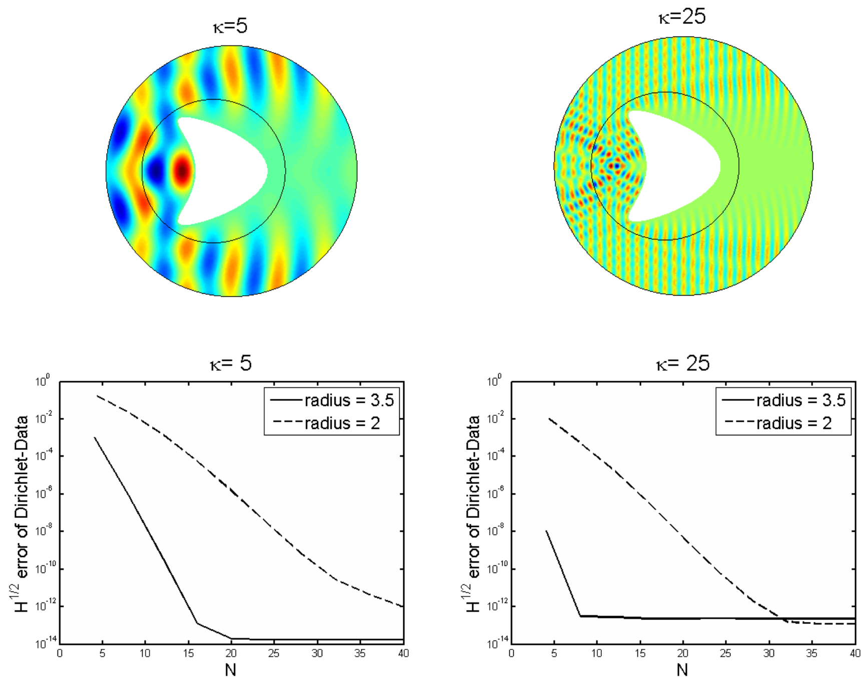

FIG. 5.1. *$H^{1/2}(\Gamma)$-error in the Dirichlet data for different wave numbers and radii as a function of the number N of degrees of freedom in the Hardy space $H^+(S^1)$.*

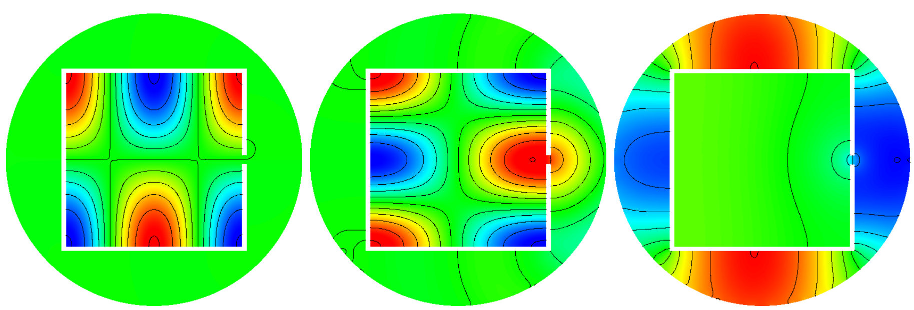

FIG. 5.2. *eigenfunctions of an open square*

($\bullet$), the exact eigenvalues of the closed square ($\Box$) and the exact exterior resonances of a sphere ($\circ$) with the same circumference as the square are plotted. Note that the exterior resonances of a sphere are given by the zero points of the Hankel functions of the first kind. Additionally we used PML ($\Diamond$) as reference solution. The HSM resonances in the third quadrant and the PML resonances in the lower part of the plot are computational artifacts.

6. Conclusions. We have presented a new type of infinite elements based on the pole condition which are derived by transforming the exterior variational formulation of the Helmholtz equation to a Hardy space. They can be coupled with finite elements of arbitrary order in the interior domain and have simple, symmetric element matrices with a tensor product structure. The convergence with respect to the

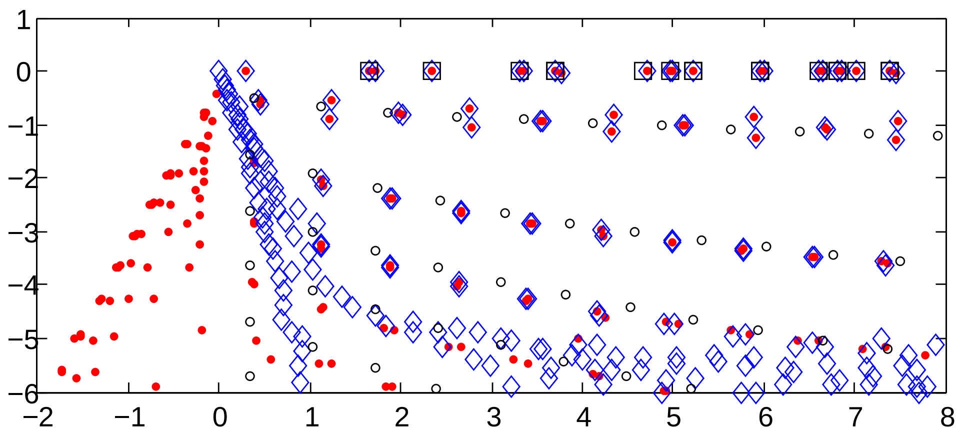

FIG. 5.3. *resonances of an open square (•: HSM, ◇: PML)*

number of degrees of freedom in the transformed radial direction is super-algebraic. Moreover, they are particularly well suited for resonance problems since they preserve the eigenvalue structure. As opposed to other numerical realizations of the pole condition (cf. [8; 17]) it is not possible to recover the exterior solution directly by the HSM.

Let us compare Hardy space infinite elements with PML from a practical perspective: The PML method has the advantage to be easy to implement in standard software package, whereas the HSM requires the implementation of a new (in)finite element. The HSM has the advantage that it is a high order method which can easily be combined with low order codes. Moreover, the only tuning parameter in the HSM is κ_0, and the rule $\kappa_0 \approx \kappa$ yields good results, whereas for PML at least the slope of the path in the complex plane, the width of the layer, and the polynomial degree have to be chosen. Our preliminary numerical experiments suggest that the HSM performs at least as good as PML, but for definite conclusion more thorough numerical studies optimizing the various PML parameters will be necessary.

The HSM is not restricted to the situation studied in this paper, but can be extended to other differential equations and other coupling boundaries, which may be subject of future research.

Appendix A. In this appendix we prove the lemmas needed for the transformation to the Hardy space.

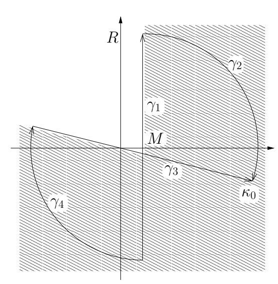

FIG. A.1. *Contour for the proof of Lemma A.1*

Lemma A.1. *Let $M \geq 0$ and $\kappa_0 \in \mathbb{C}$ be given constants with $\Re(\kappa_0) > 0$, and let $f, g : \mathbb{R}_+ \to \mathbb{C}$ be two measurable functions such that $fg \in L^1([0,\infty))$ and the Laplace transforms $\hat{f} := \mathcal{L}f$ and $\hat{g} := \mathcal{L}g$ exist on $\{s \in \mathbb{C} : \Re(s) \geq M\}$. Moreover, assume that $\hat{f}$ and $\hat{g}$ have holomorphic extensions to a neighborhood of $E(M,\kappa_0) := \{s \in \mathbb{C} : \Re(s/(-i\kappa_0)) \geq 0 \vee \Re(s) \geq M\}$ and that $\sup_{s\in E(M,\kappa_0)} |\hat{f}(s)s|, \sup_{s\in E(M,\kappa_0)} |\hat{g}(s)s| < \infty$. Then*

$$\int_0^\infty f(r)g(r)dr = -\frac{i}{2\pi}\int_{\kappa_0\mathbb{R}} \widehat{f}(s)\widehat{g}(-s)ds = \frac{-i\kappa_0}{\pi}\int_{S^1} F(z)G(\overline{z})|dz| \tag{A.1}$$

with $F := \mathcal{M}_{\kappa_0}(\hat{f}|_{\kappa_0\mathbb{R}})$ and $G := \mathcal{M}_{\kappa_0}(\hat{g}|_{\kappa_0\mathbb{R}})$. (The orientation of the contour $\kappa_0\mathbb{R}$ is from left to right.)

Proof. We extend f, g by zero to $f^*, g^* : \mathbb{R} \to \mathbb{C}$ and write the integral as a Fourier transform $(\mathcal{F}\varphi)(s) := \int_{-\infty}^{\infty} e^{-ist}\varphi(t)dt$ evaluated at $s = 0$

$$-\frac{i}{2\pi}\int_0^\infty f(r)g(r)dr = \mathcal{F}\left\{f^*g^*\right\}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\left\{f^*e^{-M\bullet}\right\}(t)\mathcal{F}\left\{g^*e^{M\bullet}\right\}(-t)dt$$

Here $\mathcal{F}\left\{f^*(x)e^{-Mx}\right\}(t) = \widehat{f}(it+M)$ and $\mathcal{F}\left\{g^*(x)e^{Mx}\right\}(-t) = \widehat{g}(-(it+M))$ exist due to our assumptions. The first equation in (A.1) follows by Cauchy's integral theorem for the closed contour $\gamma_1 + \gamma_2 - \gamma_3 + \gamma_4$ shown in Fig. A.1, using the fact that the integrals over γ_2 and γ_4 vanish as $R \to \infty$ due to the assumed decay of $\hat{f}$ and $\hat{g}$:

$$\int_0^\infty f(r)g(r)\,\mathrm{d}r = -\frac{i}{2\pi}\lim_{R\to\infty}\int_{\gamma_1} \widehat{f}(s)\widehat{g}(-s)\,\mathrm{d}s = -\frac{i}{2\pi}\lim_{R\to\infty}\int_{\gamma_3} \widehat{f}(s)\widehat{g}(-s)\,\mathrm{d}s.$$

To prove the second equation we use the substitution of variables $s = \varphi_{\kappa_0}(z)$ and the identities $\varphi'_{\kappa_0}(z) = \frac{-2i\kappa_0}{(z-1)^2}$ and $-\varphi_{\kappa_0}(z) = \varphi_{\kappa_0}(\overline{z})$ for $z \in S^1$ to obtain

$$\begin{aligned} -\frac{i}{2\pi}\lim_{R\to\infty}\int_{\gamma_3} \widehat{f}(s)\widehat{g}(-s)ds &= \frac{-\kappa_0}{\pi}\int_{S^1,\circlearrowright} \frac{\widehat{f}(\varphi_{\kappa_0}(z))}{z-1}\frac{\widehat{g}(-\varphi_{\kappa_0}(z))}{z-1}dz \\ &= \frac{-\kappa_0}{\pi}\int_{S^1,\circlearrowright} F(z)G(\overline{z})\frac{\overline{z}-1}{z-1}dz. \end{aligned}$$

The symbol $\circlearrowright$ indicates clockwise orientation of the contour S^1. Since $\frac{\overline{z}-1}{z-1} = \frac{1/z-1}{z-1} = -\frac{1}{z}$ for $z \in S^1$, and $dz = -iz|dz|$, we obtain the second equation in (A.1). ∎

Lemma A.2. *Let $\kappa_0 \in \mathbb{C}\setminus\{0\}$, let E be an open subset of $\{k \in \mathbb{C} : \Re(k/\kappa_0) > 0\}$, and define $V_k(z) := \frac{k-\kappa_0}{(\kappa_0-k)z+(\kappa_0+k)}$ for $k \in E$. Then $\mathrm{span}\{V_k : k \in E\}$ is dense in $H^+(S^1)$.*

Proof. A straightforwand computation shows that $(\mathcal{M}_1^{-1}V_k)(z) = \frac{i(k-\kappa_0)}{\kappa_0}\frac{1}{s-ik/\kappa_0}$ with the transform $\mathcal{M}_1$ defined in (2.7) (with $\kappa_0 = 1$, not the κ_0 given in the lemma). Since $\mathcal{M}_1 : H^-(\mathbb{R}) \to H^+(S^1)$ is unitary, the statement is equivalent to the density of $Y := \overline{\mathrm{span}\{1/(\bullet - ik/\kappa_0) : k \in E\}}$ in $H^-(\mathbb{R})$. Assume that $f \in Y^\perp$, i.e. $\int_{\mathbb{R}} f(\tilde{s})/(\overline{\tilde{s} - ik/\kappa_0})\,d\tilde{s} = 0$ for all $k \in E$. Then the holomorphic function

$$w(z) := \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(\tilde{s})}{\tilde{s}-z}\,d\tilde{s}, \qquad z \in \mathbb{C}^-$$

vanishes on $\{\overline{ik/\kappa_0} : k \in E\}$, which is an open subset of $\mathbb{C}^-$. Therefore, w vanishes identically in $\mathbb{C}^-$. Due to Definition 2.1 and (2.6), f are the boundary values of w on $\mathbb{R}$, and hence $f = 0$. This shows that $Y^\perp = \{0\}$, i.e. Y is dense in $H^-(\mathbb{R})$. ∎

LEMMA A.3. *Consider the set $X^{\#}$ and the inner product defined in (3.7), and let $\Re(\kappa), \Re(\kappa_0) > 0$.*

1. *$X_{\#}$ is a Hilbert space.*
2. *For each $v_{\rm int} \in H^1(\Omega_{\rm int})$ there exists $V \in H^+(S^1) \otimes L^2(\Gamma)$, such that $(v_{\rm int}, V)^\top \in X_{\#}$.*
3. *There exists an dense subset $\widetilde{X}^{\#} \subset X^{\#}$, such that for all $(v_{\rm int}, V)^\top \in \widetilde{X}^{\#}$ we have $v_{\rm int} \in C^\infty(\overline{\Omega_{\rm int}})$ and there exists a function $v_{\rm ext} \in C^\infty([0,\infty) \times \Gamma)$ such that $(i\kappa_0)^{-1}(\mathcal{M}_{\kappa_0}^{-1}\mathcal{T}_- \otimes I)(v_0, V)^\top = \mathcal{L}|_{\kappa_0\mathbb{R}} v_{\rm ext}$ and the assumptions of Lemma A.1 are fulfilled with $f(r) := \exp(i\kappa r)$ and $g(r) := v_{\rm ext}(r, \hat{x})$ for all $\hat{x} \in \Gamma$ as well as with the first derivatives of $v_{\rm ext}$.*

Proof. 1) A straightforward argument using the closedness of the surface gradient ∇_x shows that $X_{\#}$ is complete.
2) Let $v_{\rm int} \in H^1(\Omega_{\rm in})$ and define $v_0 := u_{\rm int}|_\Gamma$. Since $v_0 \in H^{1/2}(\Gamma)$, the Fourier coefficients of v_0 satisfy $\sum_{n=0}^\infty (1+\lambda_n)^{1/2}|v_{0,n}|^2 < \infty$. Here and in the following we use the notation of section 4. Define $V(z, \hat{x}) := \sum_{n=0}^\infty v_{0,n} V_{k_n}(z)\Phi_n(\hat{x})$ with a sequence (k_n) to be specified later. Since the functions V_k in Lemma A.2 satisfy $V_k(z) = \left(\frac{k/\kappa_0+1}{k/\kappa_0-1} - z\right)^{-1}$, it follows by radial symmetry that $\|V_k\|^2_{L^2(S^1)} = \Xi\left(\frac{|k/\kappa_0+1|}{|k/\kappa_0-1|} - 1\right)$ with $\Xi(t) := \int_{S^1} |1+t-z|^{-2}|dz|$ for $t > 0$. Setting $c := \int_{\pi/6}^{11\pi/6} |1 - \exp(i\theta)|^{-2}\,\mathrm{d}\theta$, we obtain

$$\Xi(t) - c \le \int_{-\pi/6}^{\pi/6} \frac{\mathrm{d}\theta}{|1+t-\exp(i\theta)|^2} \le \int_{-\pi/6}^{\pi/6} \frac{\mathrm{d}\theta}{t^2 + \theta^2/4} = \frac{4\,\mathrm{atan}(\pi/12t)}{t} \le \frac{2\pi}{t},$$

so $\Xi(t) = O(t^{-1})$ as $t \searrow 0$. From the identity $\mathcal{T}_-(1, V_k)^\top = \frac{1}{(\kappa_0 - k)\kappa_0} V_k$ it follows that

$$|\kappa_0|^2|k-\kappa_0|^2 \left\| \mathcal{T}_- \begin{pmatrix} 1 \\ V_k \end{pmatrix} \right\|^2_{L^2(S^1)} = \|V_k\|^2_{L^2(S^1)} = \Xi\left(\frac{|k/\kappa_0+1|}{|k/\kappa_0-1|} - 1\right) = O(k)$$

as $\Re(k) \to \infty$. Now choose k_0 such that $\Re(k_0/\kappa_0) > 0$ and $k_n := k_0 + \sqrt{\lambda_n}$ for $n = 1, 2, \ldots$. Then

$$\begin{aligned} \left\| \begin{pmatrix} v_{\rm int} \\ V \end{pmatrix} \right\|^2_{X^{\#}} - \|v_{\rm int}\|^2_{H^1} &= \sum_{n=0}^\infty |v_{0,n}|^2 \left\{ \|V_{k_n}\|^2_{L^2(S^1)} + \lambda_n \left\| J\mathcal{T}_- \begin{pmatrix} 1 \\ V_{k_n} \end{pmatrix} \right\|^2_{L^2(S^1)} \right\} \\ &\le C \sum_{n=0}^\infty |v_{0,n}|^2 |k_n| \left\{ 1 + \|J\|^2 \frac{\lambda_n}{\kappa_0^2 |k_n - \kappa_0|^2} \right\} \\ &\le C \sum_{n=0}^\infty |v_{0,n}|^2 |k_n| \le C \sum_{n=0}^\infty |v_{0,n}|^2 (1+\lambda_n)^{1/2} < \infty \end{aligned}$$

with a generic constant C. Hence, $(v_{\rm int}, V)^\top \in X^{\#}$.

3. With V as constructed above we have $v_{\rm ext}(r, \hat{x}) = \sum_{n=0}^\infty v_{0,n} \exp(ik_n r)\Phi_n(\hat{x})$ (cf. (2.8), (2.9), (2.15)). If $v_{\rm int} \in C^\infty(\overline{\Omega_{\rm int}})$, then the Fourier coefficients $v_{0,n}$ decay super-algebraically, and the series together with its term by term derivatives converges uniformly on compact subsets. Moreover, $r \mapsto e^{i\kappa r} v_{\rm ext}(r, \hat{x})$ decays exponentially if $\Im(k_n + \kappa) = \Im(k_0 + \kappa) > 0$. This can be arranged by an appropriate choice of k_0. Hence, Lemma A.1 can be applied to $v_{\rm ext}(r, \hat{x})$, and also to its first derivatives. Since everything above remains valid if k_n is chosen in a small ball around $k_0 + \sqrt{\lambda_n}$, the density property follows from Lemma A.2 and the density of $C^\infty(\overline{\Omega_{\rm int}})$ in $H^1(\Omega_{\rm int})$. ∎

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