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Sampling Inequalities for Infinitely Smooth Functions, with Applications to Interpolation and Machine Learning

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Abstract

Sampling inequalities give a precise formulation of the fact that a differentiable function cannot attain large values, if its derivatives are bounded and if it is small on a sufficiently dense discrete set. Sampling inequalities can be applied to the difference of a function and its reconstruction in order to obtain (sometimes optimal) convergence orders for very general possibly regularized recovery processes. So far, there are only sampling inequalities for finitely smooth functions, which lead to algebraic convergence orders. In this paper the case of infinitely smooth functions is investigated, in order to derive error estimates which lead to exponential convergence orders. In particular this approach improves the known error estimates for classical interpolation with inverse Multiquadrics.

Keywords: Gaussians, inverse Multiquadrics, smoothing, approximation, error bounds, radial basis functions, convergence orders

Classification: 41A05, 41A25, 41A63, 65D10, 68T05

1 Introduction

Sampling inequalities give a precise formulation of the fact that a differentiable function cannot attain large values, if its derivatives are bounded and if it is small on a sufficiently dense discrete set. Inequalities of this kind can be used to derive a priori error estimates for various regularized approximation problems as they occur for instance in many machine learning algorithms or PDE solvers [8],[7]. Recently several such *sampling inequalities* for functions $u \in W_p^k(\Omega)$ from certain Sobolev spaces with $1 < p < \infty$ and $k > d/p$ or $p = 1$ and $k \geq d$ on a domain $\Omega \subset \mathbb{R}^d$ were obtained. They usually take the form [10]

$$\|D^\alpha u\|_{L_q(\Omega)} \leq C \left(h^{k-|\alpha|-d(1/p-1/q)_+} |u|_{W_p^k(\Omega)} + h^{-|\alpha|} \|u|_X\|_{\ell_\infty(X)} \right) ,$$

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where $1 \leq q \leq \infty$, or [4]

$$\|D^\alpha u\|_{L_p(\Omega)} \leq C \left(h^{k-|\alpha|} |u|_{W_p^k(\Omega)} + h^{d/p-|\alpha|} \|u|_X\|_{\ell_p(X)} \right),$$

for all $u \in W_p^k(\Omega)$, where

$$h := h_{X,\Omega} := \sup_{x \in \Omega} \min_{x_j \in X} \|x - x_j\|_2$$

denotes the fill distance of the discrete set $X \subset \Omega$. These bounds were for instance used [10] to derive optimal algebraic convergence orders for kernel based smoothed interpolation methods in a finitely smooth setting.
In this paper we derive sampling inequalities for infinitely smooth functions where the convergence orders turn out to depend exponentially on the fill distance h.
We are handling infinitely smooth functions by normed linear function spaces $\mathcal{H}(\Omega)$ on a domain $\Omega \subset \mathbb{R}^d$ which can for a fixed $1 \leq p < \infty$ be uniformly continuously embedded into every classical Sobolev space $W_p^k(\Omega)$. More precisely, for a fixed $p \in [1, \infty)$ and all $k \in \mathbb{N}$ we assume that there are embedding operators $I_k^{(p)}$ and a constant E independent of k such that

$$\begin{aligned} I_k^{(p)} : \mathcal{H}(\Omega) \to W_p^k(\Omega) \quad & \text{with} \\ \|I_k^{(p)}\|_{\{\mathcal{H} \to W_p^k(\Omega)\}} \leq E \quad & \text{for all } k \in \mathbb{N}_0 \,. \end{aligned}$$

There are various examples for spaces with this property, e.g. Sobolev spaces of infinite order as they occur for instance in the study of partial differential equations of infinite order [1], or reproducing kernel Hilbert spaces of Gaussians and inverse multiquadrics (see section 4).
In the case of infinitely smooth functions the shape of the domain Ω crucially influences our sampling inequalities. For general Lipschitz domains Ω which satisfy an interior cone condition we use a polynomial reproduction [9] which accepts slight oversampling to bound the Lebesgue-constants. This results in a good behaviour of the term with the discrete norm. In this case we get that for sufficiently small fill distance h there are generic constants $c > 0$ such that with $\tilde{q} \in \{q, \infty\}$ the inequality

$$\|D^\alpha u\|_{L_q(\Omega)} \leq c e^{c \log(ch)/\sqrt{h}} \|u\|_{\mathcal{H}(\Omega)} + c h^{-|\alpha|} \|u|_X\|_{\ell_{\tilde{q}}(X)}$$

holds for all $u \in \mathcal{H}(\Omega)$. The best approximation orders for the first term can be obtained on compact cubes since we can use then a polynomial reproduction from [3]. Unfortunately this approach is limited to cubes and cannot cope with derivatives on the left hand side of our sampling inequalities. Nevertheless we obtain that there are generic constants $c > 0$, such that the inequality

$$\|u\|_{L_q(\Omega)} \leq e^{c \log(ch)/h} \|u\|_{\mathcal{H}(\Omega)} + c^{1/h} \|u|_X\|_{\ell_{\tilde{q}}(X)} \,,$$

holds for all $u \in \mathcal{H}(\Omega)$ with $\tilde{q} \in \{q, \infty\}$ if the fill distance h is sufficiently small.
Our main examples, however, deal with reconstruction problems in Hilbert spaces. Therefore in the second part we will focus on the native Hilbert spaces of Gaussian and inverse

Multiquadric kernels. In this case we suppose u to be an error function $u = f - Rf$, where f denotes the function we would like to reconstruct and Rf is the reconstruction. In order to obtain optimal order error bounds one needs two properties of the reconstruction, namely

$$\|Rf\|_{\mathcal{H}} \leq C\|f\|_{\mathcal{H}} \quad \text{and} \quad \|(Rf - f)|_X\|_{\ell_p(X)} \leq g(f, h),$$

where g determines the expected approximation order. This can be used to show that the theory presented here improves the well-known error estimates for the standard interpolation problem in the native Hilbert space of the inverse Multiquadrics and reproduces them for Gaussian kernels.

2 Estimates on general Lipschitz domains

Following [5] we first obtain estimates on local domains $\mathcal{D} \subset \mathbb{R}^d$ and use a covering argument to get global results. We assume a domain $\mathcal{D}$ that is is star shaped with respect to a ball $B_r(x_c)$ and that is contained in a ball $B_R(x_c)$. In this case we know [5] that $\mathcal{D}$ satisfies an interior cone condition as well. We denote the associated chunkiness parameter with

$$\gamma = \frac{\delta_{\mathcal{D}}}{\rho_{\max}},$$

where $\rho_{\max} = \sup\{\rho : \mathcal{D} \text{ is star shaped wtih respect to a ball of radius } \rho\}$ and $\delta_{\mathcal{D}}$ denotes the diameter of $\mathcal{D}$.
Let $\left\{a_j^{(\alpha)} : j = 1, \ldots, N\right\}$ be a polynomial reproduction of degree k with respect to a discrete set $X = \{x_1, \ldots, x_N\} \subset \mathcal{D}$, i.e.,

$$D^\alpha q(x) = \sum_{j=1}^{N} a_j^{(\alpha)}(x) q(x_j)$$

holds for every $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$, all $x \in \mathcal{D}$ and all $q \in \mathbb{P}_k^d(\mathcal{D})$ where $\mathbb{P}_k^d$ denotes the space of all d-variate polynomials of degree not exceeding k. Then we have

$$\begin{aligned}
|D^\alpha u(x)| &\leq |D^\alpha u(x) - D^\alpha p(x)| + |D^\alpha p(x)| \\
&\leq \|D^\alpha u - D^\alpha p\|_{L_\infty(\mathcal{D})} + \sum_{j=1}^{N} \left|a_j^{(\alpha)}(x)\right| |p(x_j)| \\
&\leq \|D^\alpha u - D^\alpha p\|_{L_\infty(\mathcal{D})} + \sum_{j=1}^{N} \left|a_j^{(\alpha)}(x)\right| \|p|_X\|_{\ell_\infty(X)} \\
&\leq \|D^\alpha u - D^\alpha p\|_{L_\infty(\mathcal{D})} \\
&+ \sum_{j=1}^{N} \left|a_j^{(\alpha)}(x)\right| \left(\|u - p\|_{L_\infty(\mathcal{D})} + \|u|_X\|_{\ell_\infty(X)}\right) \qquad (1)
\end{aligned}$$

for arbitrary $u \in W_p^k(\mathcal{D})$ and any polynomial $p \in \mathbb{P}_k^d(\mathcal{D})$. As a polynomial approximation we use averaged Taylor polynomials. They are defined as

$$Q^k u(x) := \sum_{|\alpha|<k} \frac{1}{\alpha!} \int_B D^\alpha u(y)(x - y)^\alpha \phi(y) dy,$$

where B is a ball relative to which $\mathcal{D}$ is star shaped and having radius $\geq 1/2\rho_{\max}$ and $\phi \in C^\infty$ is a bump function supported on $\bar{B}$ satisfying both $\int_B \phi(y)dy = 1$ and $\max \phi \leq C_d \operatorname{diam}(B)^{-d}$. For the remainder

$$R^k := u - Q^k u$$

there is the following bound from [2], where the explicit constants can be found in [5].

Lemma 2.1 *For $u \in W_p^k(\mathcal{D})$ with $1 < p < \infty$ and $k > |\alpha| + d/p$ or in the case $p = 1$ and $k \geq |\alpha| + d$ we get*

$$\|D^\alpha u - D^\alpha Q^k u\|_{L_\infty(\mathcal{D})} \leq C_{d,\theta} \frac{d^{k-|\alpha|}}{(k-|\alpha|)!} \delta_{\mathcal{D}}^{k-|\alpha|-d/p} |u|_{W_p^k(\mathcal{D})}$$

where the constant $C_{d,\theta}$ depends only on the space dimension d and the angle θ.

Proof: We use the identity [2]

$$D^\beta Q^k u = Q^{k-|\beta|} D^\beta u ,$$

for all $|\beta| \leq k$.This leads to

$$\begin{aligned}
\|D^\alpha u - D^\alpha Q^k u\|_{L_\infty(\mathcal{D})} &= \|D^\alpha u - Q^{k-|\alpha|} D^\alpha u\|_{L_\infty(\mathcal{D})} \\
&\leq C_d (1+\gamma)^d \frac{d^{k-|\alpha|}}{(k-|\alpha|)!} \delta_{\mathcal{D}}^{k-|\alpha|-d/p} |D^\alpha u|_{W_p^{k-|\alpha|}(\mathcal{D})} \\
&\leq C_{d,\theta} \frac{d^{k-|\alpha|}}{(k-|\alpha|)!} \delta_{\mathcal{D}}^{k-|\alpha|-d/p} |u|_{W_p^k(\mathcal{D})} .
\end{aligned}$$

Here we used the fact [5] that the chunkiness parameter γ can be bounded by $1 \leq \gamma \leq \csc\left(\frac{\theta}{2}\right)$. □

We shall use the following local polynomial reproduction from [9].

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^d$ satisfy an interior cone condition with angle $\theta \in (0, \pi/2)$ and radius r, $\ell \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq \ell$. Then there are constants $c_0, c_1^{(\alpha)}, c_2 > 0$, such that for all $X = \{x_1, \ldots, x_N\} \subset \Omega$ with $h_{X,\Omega} \leq h_0 := c_0/\ell^2$ and all $x \in \Omega$ there exist numbers $\tilde{a}_1^{(\alpha)}(x), \ldots, \tilde{a}_N^{(\alpha)}(x)$ with*

1. $\sum_{j=1}^N p(x_j) \tilde{a}_j^{(\alpha)}(x) = D^{(\alpha)} p(x)$ *for* $x \in \Omega$ *and* $p \in \mathbb{P}_\ell^d(\Omega)$
2. $\sum_{j=1}^N |\tilde{a}_j^{(\alpha)}(x)| \leq c_1^{(\alpha)} h_{X,\Omega}^{-|\alpha|}$ *for all* $x \in \Omega$,
3. $\tilde{a}_j^{(\alpha)}(x) = 0$, *if* $\|x - x_j\|_2 > c_2^{(\alpha)} h_{X,\Omega}$ *and* $x \in \Omega$.

The condition $\theta \in (0, \pi/2)$ implies $\sin\theta \in (0,1)$, i.e., $\left(\frac{1}{2(1+\sin\theta)}\right)^{|\alpha|} \leq \frac{1}{2(1+\sin\theta)}$ for all $\alpha \in \mathbb{N}_0^d$. Therefore we can choose all the constants independent of α, i.e., there exist constants c_θ depending only on θ such that [9]

$$c_1^{(\alpha)} \leq c_\theta 2^{-|\alpha|} \leq c_\theta, \quad c_2 := c_\theta \ell^2 . \tag{2}$$

Inserting the bounds of Lemma 2.1 and Theorem 2.2 into (1) leads to the following local estimate.

Theorem 2.3 *Suppose $\mathcal{D}$ satisfies an interior cone condition with angle θ and radius r, let $\alpha \in \mathbb{N}_0^d$ such that $k > |\alpha| + d/p$ for $1 < p < \infty$, or $k \geq d$ if $p = 1$. Then*

$$\begin{aligned} \|D^\alpha u\|_{L_\infty(\mathcal{D})} &\leq \frac{C_{d,\theta} d^k}{(k-|\alpha|)!} \delta_{\mathcal{D}}^{k-d/p} \left(\delta_{\mathcal{D}}^{-|\alpha|} + h^{-|\alpha|} \right) |u|_{W_p^k(\mathcal{D})} + \\ &+ C_{d,\theta} h^{-|\alpha|} \|u|_X\|_{\ell_\infty(X)} \end{aligned}$$

holds for all $u \in W_p^k(\mathcal{D})$.

Corollary 2.4 *Under the assumptions from Theorem 2.3 we get for $1 \leq q \leq \infty$*

$$\begin{aligned} \|D^\alpha u\|_{L_q(\mathcal{D})} &\leq vol(\mathcal{D})^{1/q} \|D^\alpha u\|_{L_\infty(\mathcal{D})} \leq \delta_{\mathcal{D}}^{d/q} \|D^\alpha u\|_{L_\infty(\mathcal{D})} \\ &\leq c_{d,\theta} \frac{d^k}{(k-|\alpha|)!} \delta_{\mathcal{D}}^{k-d\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\delta_{\mathcal{D}}^{-|\alpha|} + h^{-|\alpha|} \right) |u|_{W_p^k(\mathcal{D})} + \\ &+ C_{d,\theta} \delta_{\mathcal{D}}^{d/q} h^{-|\alpha|} \|u|_X\|_{\ell_\infty(X)} . \end{aligned}$$

Now we consider a ‘global‘ domain $\Omega \subset \mathbb{R}^d$ that is bounded, has a Lipschitz boundary and satisfies an interior cone condition with maximum radius R and angle $\phi \in (0, \pi/2)$. To cover Ω with smaller star shaped domains $\{\mathcal{D}_t\}$ we use a construction described in [5]. There is a constant $Q_{\theta,R}$ such that for $h \leq \frac{Q_{\theta,R}}{k^2}$ there is a covering with the following properties.

- Each set $\mathcal{D}_t$ is star shaped with respect to a ball $B_t \subset B_R(t) \cap \Omega$.
- Each set D_t satisfies an interior cone condition with radius r and angle θ where r and θ can be expressed explicitely by R and ϕ.
- There are constants $D_\phi, \tilde{D}_\phi$ such that $\tilde{D}_\phi \cdot hk^2 \leq \delta_{\mathcal{D}_t} \leq D_\phi \cdot hk^2$.
- There is a constant $M_1 = M_1(\theta, d) > 0$ such that $\sum_{t \in T_r} \chi_{\mathcal{D}_t} \leq M_1$.
- There is a constant $M_2(\theta, d) > 0$ such that $\#T_h < M_2(hk^2)^{-d}$.

Theorem 2.5 *Let $\alpha \in \mathbb{N}_0^d$ and $k \in \mathbb{N}$ be fixed with $|\alpha| < k$, $k > d/p$ for $1 < p < \infty$ or $k \geq d$ for $p = 1$ and set $C_{\min} := \min\left\{\frac{c_0}{2}, Q_{\theta,R}\right\}$ with the constant c_0 from Theorem 3.1. Suppose a discrete set $X \subset \Omega$ with fill distance $h \leq C_{\min}/k^2$. Then for all $u \in W_p^k(\Omega)$ the inequality*

$$\begin{aligned} \|D^\alpha u\|_{L_q(\Omega)} &\leq \frac{c^k}{(k-|\alpha|)!} h^{-|\alpha|} (hk^2)^{k-|\alpha|-d\left(\frac{1}{p}-\frac{1}{q}\right)_+} |u|_{W_p^k(\Omega)} \\ &+ c h^{-|\alpha|} (hk^2)^{d/q} \|u|_X\|_{\ell_q(X)} \end{aligned} \quad (3)$$

holds for $1 \leq q \leq \infty$ with generic positive constants c which may depend only on d, R, ϕ, p, q and α.

Proof: For $u \in W_p^k(\Omega)$ we can use the decomposition described above, Corollary 2.4 and the estimate $\delta_{\mathcal{D}}^{-|\alpha|} \leq C_{\phi,\alpha} h^{-|\alpha|}$ which gives

$$\begin{aligned}
\|D^\alpha u\|_{L_q(\Omega)} &= \left(\int_\Omega |D^\alpha u(x)|^q \, dx \right)^{1/q} \leq \left(\sum_{\mathcal{D}_t} \|D^\alpha u\|^q_{L_q(\mathcal{D}_t)} \right)^{1/q} \\
&\leq \frac{C_{d,\theta} d^k}{(k-|\alpha|)!} \left(D_\Phi h k^2\right)^{k-d\left(\frac{1}{p}-\frac{1}{q}\right)} C_{\phi,\alpha} h^{-|\alpha|} \left(\sum_{\mathcal{D}_t} |u|_{W_p^k(\mathcal{D}_t)} \right)^{1/q} + \\
&+ \quad \left(D_\Phi h k^2\right)^{d/q} h^{-|\alpha|} \left(\sum_{\mathcal{D}_t} \|u|_{X\cap\mathcal{D}_t}\|_{\ell_\infty(X\cap\mathcal{D}_t)} \right)^{1/q} \\
&\leq \frac{C^k_{d,\theta,\alpha,p,q}}{(k-\alpha|)!} h^{-|\alpha|} \left(hk^2\right)^{k-d\left(\frac{1}{p}-\frac{1}{q}\right)-d\left(\frac{1}{q}-\frac{1}{p}\right)_+} \left(\sum_{\mathcal{D}_t} |u|_{W_p^k(\mathcal{D}_t)} \right)^{1/q} + \\
&+ \quad C h^{-|\alpha|} \left(hk^2\right)^{d/q} \left(\sum_{\mathcal{D}_t} \|u|_{X\cap\mathcal{D}_t}\|_{\ell_\infty(X\cap\mathcal{D}_t)} \right)^{1/q} \\
&\leq \quad \frac{C^k}{(k-|\alpha|)!} h^{-|\alpha|} \left(hk^2\right)^{k-d\left(\frac{1}{p}-\frac{1}{q}\right)_+} \|u\|_{W_p^k(\Omega)} + \\
&+ \quad C h^{-|\alpha|} (hk^2)^{d/q} \|u|_X\|_{\ell_q(X)} \ .
\end{aligned}$$

□

Corollary 2.6 *Under the assumptions from Theorem 2.5 we get with an analogous calculation*

$$\begin{aligned}
\|D^\alpha u\|_{L_q(\Omega)} \quad &\leq \quad \frac{c^k}{(k-|\alpha|)!} h^{-|\alpha|} (hk^2)^{k-|\alpha|-d\left(\frac{1}{p}-\frac{1}{q}\right)_+} |u|_{W_p^k(\Omega)} \\
&+ \quad c h^{-|\alpha|} \|u|_X\|_{\ell_\infty(X)} \qquad (4)
\end{aligned}$$

We shall now relate h and k to derive exponential estimates.

Theorem 2.7 *There are constants $c, h_0 > 0$ depending on d, p, q, R, Φ, α such that for all data sets $X \subset \Omega$ with fill distance $h \leq h_0$, the inequality*

$$\|D^\alpha u\|_{L_q(\Omega)} \leq e^{c \log(ch)/\sqrt{h}} \|u\|_{\mathcal{H}(\Omega)} + c h^{-|\alpha|} \, \|u|_X\|_{\ell_q(X)}$$

holds for all $u \in \mathcal{H}(\Omega)$ and all $1 \leq q \leq \infty$.

Proof: We use Stirling's formula to estimate

$$\frac{1}{(k-|\alpha|)!} \leq \frac{k^{|\alpha|}}{k!} \leq \frac{k^{|\alpha|} e^k}{k^k} \ .$$

Since $\|u\|_{W_p^k(\Omega)} \leq \|u\|_{\mathcal{H}(\Omega)}$ holds for all $k \in \mathbb{N}$ we can bound the first term of (3) for arbitrary $k \in \mathbb{N}$ by

$$(\tilde{c}hk)^k (h^2 k)^{-|\alpha|} (hk^2)^{-d\left(\frac{1}{p}-\frac{1}{q}\right)_+} \|u\|_{\mathcal{H}(\Omega)} .$$

We set $B = \min\{c_{\min}, 1/\tilde{c}\}$ and choose $k \in \mathbb{N}$ such that $\frac{B}{2k^2} \leq h \leq \frac{B}{k^2}$ holds. Then the first term can be bounded by

$$ck^{-k} h^{-3|\alpha|/2} \|u\|_{\mathcal{H}(\Omega)} \leq e^{c\log(ch)/\sqrt{h}} \|u\|_{\mathcal{H}(\Omega)}$$

where the constants $c > 0$ may depend on d, p, q, R, ϕ, α.
With this choice the second term of (3) can be bounded by

$$ch^{-|\alpha|} (hk^2)^{d/q} \left\| u|_X \right\|_{\ell_\infty(X)} \leq ch^{-|\alpha|} \left\| u|_X \right\|_{\ell_\infty(X)} .$$

□

Corollary 2.8 *If we use Corollary 2.6 instead of Theorem 2.5 we get*

$$\|D^\alpha u\|_{L_q(\Omega)} \leq e^{c\log(ch)/\sqrt{h}} \|u\|_{\mathcal{H}(\Omega)} + ch^{-|\alpha|} \left\| u|_X \right\|_{\ell_\infty(X)}$$

3 Estimates on cubes

To derive estimates for function values on a compact cube we can use the following local polynomial reproduction from [3].

Theorem 3.1 *Let Ω be a cube in $\mathbb{R}^d$. There exist constants $c_0, c_2 > 0$ depending only on Ω such that for every $\ell \in \mathbb{N}$ and every $X = \{x_1, \ldots, x_N\} \subset \Omega$ with $h_{X,\Omega} \leq c_0/\ell$ we can find functions $a_j : \Omega \to \mathbb{R}$ satisfying*

- $\sum_{j=1}^N a_j(x) p(x_j) = p(x)$ *for all $x \in \Omega$ and all $p \in \pi_\ell(\mathbb{R}^d)$,*
- $\sum_{j=1}^N |a_j(x)| \leq e^{2d\gamma_d(\ell+1)}$ *for all $x \in \Omega$,*
- $a_j(x) = 0$ *if $\|x - x_j\|_2 > c_2 \ell h_{X,\Omega}$.*

The numbers γ_d are defined recursively by $\gamma_1 = 2$ and $\gamma_d = 2d(1 + \gamma_{d-1})$.

Lemma 2.1 gives in the special case $\alpha = 0$ the bound

$$\left\| u - Q^k u \right\|_{L_\infty(\mathcal{D})} \leq \frac{C_{d,\theta} d^k}{k!} \delta_{\mathcal{D}}^{k-d/p} |u|_{W_p^k(\mathcal{D})} .$$

If we insert this estimate and the bound from Theorem 3.1 into (1) we find

$$\|u\|_{L_\infty(\mathcal{D})} \leq \frac{c^k \delta_{\mathcal{D}}^{k-d/p}}{k!} |u|_{W_p^k(\mathcal{D})} + c^k \left\| u|_X \right\|_{\ell_\infty(X)} ,$$

and

$$\|u\|_{L_q(\mathcal{D})} \leq \frac{c^k \delta_{\mathcal{D}}^{k-d\left(\frac{1}{p}-\frac{1}{q}\right)}}{k!} |u|_{W_p^k(\mathcal{D})} + c^k \delta_{\mathcal{D}}^{d/q} \left\| u|_X \right\|_{\ell_\infty(X)} .$$

To derive global estimates we use the obvious covering of the big cube Ω with axially parallel small cubes $\mathcal{D}_t$ with the following properties. A similar approach can be found in [4].

- There are constants $D, \tilde{D} > 0$ such that $\tilde{D}hk \le \delta_{D_t} \le Dhk$.
- There is a constant $M_1 = M_1(\theta, d) > 0$ such that $\sum_{t \in T_r} \chi_{\mathcal{D}_t} \le M_1$.
- There is a constant $M_2(\theta, d) > 0$ such that $\#T_h < M_2(hk)^{-d}$.

As in the previous section we find the following global estimate.

Theorem 3.2 *Under the assumptions from above there exists a positive constant c which depends only on p, q, the side length R of the cube Ω and the space dimension d such that*

$$\|u\|_{L_q(\Omega)} \le \frac{c^k}{k!}(hk)^{k-d\left(\frac{1}{p}-\frac{1}{q}\right)_+} |u|_{W_p^k(\Omega)} + c^k (hk)^{d/q} \|u|_X\|_{\ell_q(X)}$$

holds for all data sets $X \subset \Omega$ and for all $u \in W_p^k(\Omega)$ with $k > d/p$ for $1 < p < \infty$ or for $k \ge d$ for $p = 1$ and all $1 \le q \le \infty$.

As in the case of domains obeying a cone condition we get the following corollary.

Corollary 3.3 *Under the assumptions from Theorem 3.2 we obtain*

$$\|u\|_{L_q(\Omega)} \le \frac{c^k}{k!}(hk)^{k-d\left(\frac{1}{p}-\frac{1}{q}\right)_+} |u|_{W_p^k(\Omega)} + c^k \|u|_X\|_{\ell_\infty(X)} \ .$$

Now we can derive the following exponential orders.

Theorem 3.4 *There exist constants $c, h_0 > 0$ such that for all data sets X with fill distance $h \le h_0$*

$$\|u\|_{L_q(\Omega)} \le e^{c \log(ch)/h} \|u\|_{\mathcal{H}(\Omega)} + c^{1/h} \|u|_X\|_{\ell_q(X)}$$

holds for all $u \in \mathcal{H}$ with (1) and all $1 \le q \le \infty$.

Proof: Since $\|u\|_{W_p^k(\Omega)} \le \|u\|_{\mathcal{H}(\Omega)}$ for all $u \in \mathcal{H}(\Omega)$ Theorem 3.2 gives

$$\|u\|_{L_q(\Omega)} \le \frac{C^k}{k!}(hk)^{k-d\left(\frac{1}{p}-\frac{1}{q}\right)_+} \|u\|_{\mathcal{H}(\Omega)} + C^k (hk)^{d/q} \|u|_X\|_{\ell_q(X)} \ .$$

Using Stirling's formula we can bound the first term by

$$\frac{C^k}{k!}(hk)^{k-d\left(\frac{1}{q}-\frac{1}{p}\right)_+} \le \frac{C^k}{k!} h^k k^k h^{-d\left(\frac{1}{q}-\frac{1}{p}\right)_+} \le (Ch)^k h^{-d\left(\frac{1}{q}-\frac{1}{p}\right)_+} \ .$$

If we set $B := \min\left\{\frac{c_0}{2}, \frac{1}{C}\right\}$ with the constant c_0 from the local polynomial reproduction 3.1 and choose $k \in \mathbb{N}$ such that $\frac{B}{2k} \le h \le \frac{B}{k}$ we can further estimate

$$\frac{C^k}{k!}(hk)^{k-d\left(\frac{1}{p}-\frac{1}{q}\right)_+} \le k^{-k} h^{-d\left(\frac{1}{p}-\frac{1}{q}\right)_+} \le e^{c \log(ch)/h} \ . \tag{5}$$

By the choice of k there exists a constant c such that the second term is bounded by

$$C_{d,R}^k (hk)^{d/q} \|u|_X\|_{\ell_q(X)} \le c^{1/h} \|u|_X\|_{\ell_q(X)} \ . \tag{6}$$

Adding (5) and (6) establishes the claim. □

Corollary 3.5 *Under the assumptions from Theorem 3.4 we obtain*

$$\|u\|_{L_q(\Omega)} \le e^{c \log(ch)/h} \|u\|_{\mathcal{H}(\Omega)} + c^{1/h} \|u|_X\|_{\ell_\infty(X)} \ .$$

4 Kernels and Native Spaces

In this section we will provide two famous examples of function spaces of infinitely smooth functions. In the case of the Gaussian radial basis function

$$K(x) = e^{-c\|x\|_2^2} \tag{7}$$

the native Hilbert space is definded via

$$\mathcal{N}_K(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d) \cap L_1(\mathbb{R}^d) \ : \|f\|_{\mathcal{N}_K} < \infty \right\}$$

where the norm is given by

$$\|f\|_{\mathcal{N}_K}^2 := \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 e^{\frac{\|x\|_2^2}{4c^2}} d\omega.$$

The Sobolev spaces on $\mathbb{R}^d$ are defined via

$$W_2^k(\mathbb{R}^d) := \left\{ f \in L_2(\mathbb{R}^d) \ : \hat{f}(\cdot)(1 + \|\cdot\|_2^2)^{k/2} \in L_2(\mathbb{R}^d) \right\} \ .$$

This shows for all $k \geq 0$

$$\mathcal{N}_K(\mathbb{R}^d) \subset W_2^k(\mathbb{R}^d) \ .$$

Using [9, Theorem 10.46], every $f \in \mathcal{N}_K(\Omega)$ has an extension $Ef \in \mathcal{N}_K(\mathbb{R}^d)$ with $\|Ef\|_{\mathcal{N}_K(\mathbb{R}^d)} \leq \|f\|_{\mathcal{N}_K(\Omega)}$. Thus we have for $f = Ef|_\Omega \in \mathcal{N}_K(\Omega) \subset W_2^k(\Omega)$ for all non-negative k

$$\|f\|_{W_2^k(\Omega)} \leq \|Ef\|_{W_2^k(\mathbb{R}^d)} \leq \|Ef\|_{\mathcal{N}_K(\mathbb{R}^d)} \leq \|f\|_{\mathcal{N}_K(\Omega)} \ . \tag{8}$$

Similar considerations apply to the Inverse Multiquadrics

$$K(x) = \left(c^2 + \|x\|_2^2\right)^{-\beta}$$

with Fourier transform [9, Theorem 6.1]

$$\hat{K}(\omega) = \frac{2^{1-\beta}}{\Gamma(\beta)} \left(\frac{\|\omega\|_2}{c} \right)^{\beta - d/2} \mathcal{K}_{d/2-\beta}\left(c\|\omega\|_2\right) \ ,$$

where $\mathcal{K}_\sigma$ denotes the modified Bessel function of the third kind (sometimes called Mcdonald's function). They are defined as

$$\mathcal{K}_\sigma(z) := \int_0^\infty e^{z \cosh(t)} \cosh(\sigma t) dt$$

for $z \in \mathbb{C}$ with $|\arg(z)| < \pi/2$. For $\sigma \in \mathbb{R}$ they are bounded from below by [9, Corollary 5.12]

$$\begin{aligned} \mathcal{K}_\sigma(x) &\geq \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} \ , \quad x > 0 \text{ and } |\sigma| \geq 1/2 \\ \mathcal{K}_\sigma(x) &\geq \frac{\sqrt{\pi} 3^{|\sigma|-1/2}}{2^{|\sigma|+1}\Gamma(|\sigma|+1/2)} \frac{e^{-x}}{\sqrt{x}} \ , \quad x \geq 1 \text{ and } |\sigma| < 1/2 \ . \end{aligned}$$

The essentially same argument as above leads to

$$\mathcal{N}_K(\mathbb{R}^d) \subset W_2^\tau(\mathbb{R}^d)$$

and

$$\|f\|_{W_2^\tau(\Omega)} \leq \|f\|_{\mathcal{N}_K(\Omega)} \,. \tag{9}$$

5 Applications to Smoothed Interpolation

Now we shall apply our general results in the case of -possibly regularized- kernel based interpolation. To start with, we briefly summarize the problem. One is given centers $X = \{x_1, \dots x_N\} \subset \mathbb{R}^d$ and data $(f_1, \dots f_N)^T \in \mathbb{R}^N$ generated by an unknown function $u \in \mathcal{N}_K$. One has to solve the system

$$(\mathbf{K} + \lambda \mathrm{I}_d)\, b = f|_X \ , \tag{10}$$

with $\mathbf{K} = (K(x_i - x_j))_{i,j=1\dots N}$ to build an approximant

$$s_{\lambda,X,K}(f)(\cdot) := \sum_{x_j \in X} b_j K(\cdot, x_j).$$

We point out that the classical interpolant is a special case, namely for choosing $\lambda = 0$. It is known [10] that

$$\begin{aligned} \|s_{\lambda,X,K}(f)\|_{\mathcal{N}_K} &\leq 2\|f\|_{\mathcal{N}_K} \\ \|s_{\lambda,X,K}(f)|_X - f|_X\|_{\ell_\infty(X)} &\leq \sqrt{\lambda}\|f\|_{\mathcal{N}_K} \end{aligned}$$

holds.

Theorem 5.1 *If Ω is a cube then there exists a constant $h_0 > 0$ and a generic constant $c > 0$ such that for all data sets $X \subset \Omega$ with fill distance $h \leq h_0$ we get*

$$\|f - s_{\lambda,X,K}(f)\|_{L_q(\Omega)} \leq 3\left(e^{c\log(ch)/h} + \sqrt{\lambda} c^{1/h}\right)\|f\|_{\mathcal{N}_K}$$

for all $f \in \mathcal{N}_K(\Omega)$ and $1 \leq q \leq \infty$.

Remark 5.2 *In the case $\lambda = 0$, i.e., the standard interpolation we have approximation order $e^{c\log(ch)/h}$. That is exactly the well known order for interpolation with Gaussian kernels and improves the known results for IMQs [9].*

Proof of Theorem 5.1:

$$\begin{aligned} \|f - s_{\lambda,X,K}(f)\|_{L_q(\Omega)} &\leq e^{c\log(\tilde{c}h)/h}\|f - s_{\lambda,X,K}(f)\|_{\mathcal{N}_K} \\ &\quad + \bar{c}^{1/h}\|s_{\lambda,X,K}(f)|_X - f|_X\|_{L_\infty(X)} \\ &\leq 3e^{c\log(\tilde{c}h)/h}\|f\|_{\mathcal{N}_K} + \sqrt{\lambda}\bar{c}^{1/h}\|f\|_{\mathcal{N}_K} \\ &\leq 3\left(e^{c\log(\tilde{c}h)/h} + \sqrt{\lambda}\bar{c}^{1/h}\right)\|f\|_{\mathcal{N}_K} \end{aligned}$$

□

Corollary 5.3 *For the choice*

$$\lambda \leq e^{\frac{2(c\log(\tilde{c}h)-\log(\bar{c}))}{h}}$$

we get

$$\|f - s_{\lambda,X,K}(f)\|_{L_q(\Omega)} \leq 3e^{c\log(\tilde{c}h)/h}\|f\|_{\mathcal{N}_K} \ .$$

Theorem 5.4 *If the domain Ω satisfies an interior cone condition, there exist constants A, B, C, c_θ, h_0 such that for all data sets $X \subset \Omega$ with fill distance $h \leq h_0$ and $1 \leq q \leq \infty$*

$$\|D^\alpha(f - s_{\lambda,X,K}(f))\|_{L_q(\Omega)} \leq \left(Ae^{B\log(Ch)/\sqrt{h}} + \sqrt{\lambda}c_\theta h^{-|\alpha|}\right)\|f\|_{\mathcal{N}_K(\Omega)} \ .$$

Here the constants A, B, C, c_θ depend only on θ, α, d, q.

Corollary 5.5 *For the choice*

$$\lambda \leq Ae^{-2B\log(Ch)/\sqrt{h}}h^{2|\alpha|}$$

we get

$$\|f - s_{\lambda,X,K}(f)\|_{L_q(\Omega)} \leq 3Ae^{B\log(Ch)/\sqrt{h}}\|f\|_{\mathcal{N}_K} \ .$$

This shows that we can improve the condition number of the system (10) at least to the value of $\lambda = Ae^{-2B\log(Ch)/\sqrt{h}}h^{2|\alpha|}$ instead of e^{-1/q^2} for the Gaussian or $e^{-1/q}$ for the Inverse Multiquadric and still get good approximation orders. We point out that we get rid of the seperation distance $q_X := \frac{1}{2}\min_{1\leq i,j\leq N}\|x_j - x_i\|_2$, which can spoil the condition number in case of badly distributed points.

6 Applications to Support Vector Machines

We shall consider the following optimization problem ν-SVR in Hilbert space formulation. The function

$$V_\epsilon(x) = |x|_\epsilon + \epsilon\nu$$

is related to Vapnik's ϵ-intensive loss function

$$|x|_\epsilon = \begin{cases} 0 & if\,|x| \leq \epsilon \\ |x| - \epsilon & if\,|x| > \epsilon \end{cases}$$

but has an additional term with a positive parameter ν. The associated optimization problem takes the form

$$\min_{\substack{f\in\mathcal{N}_K \\ \epsilon\in\mathbb{R}^+}} \frac{1}{N}\sum_{j=1}^{N}|f(x_j) - y_j|_\epsilon + \epsilon\nu + \frac{1}{2C}\|f\|^2_{\mathcal{N}_K} \ . \qquad (11)$$

The solution of this problem can be computed by solving a finite dimensional optimization problem. In the case of exact data, where $f \in \mathcal{N}_K$, i.e.

$$f(x_j) = y_j \quad \text{for } j = 1, \ldots, N, \tag{12}$$

the solution (f^*, ϵ^*) satisfies the following estimates

$$\begin{aligned} \|f^*\|_{\mathcal{N}_K} &\leq \|f\|_{\mathcal{N}_K} \\ \|f^*|_X - y\|_{\ell_\infty(X)} &\leq \frac{N}{2C}\|f\|^2_{\mathcal{N}_K} - \epsilon^* \cdot (N\nu - 1) \, . \end{aligned}$$

Theorem 6.1 *Let Ω be a cube and let (f^*, ϵ^*) be a solution of (11) . Then the generalization error can be bounded by*

$$\|f - f^*\|_{L_q(\Omega)} \leq \tilde{C} \left(e^{c \log(\tilde{c}h)/h} \|f\|_{\mathcal{N}_K} + \frac{N}{2C}\|f\|^2_{\mathcal{N}_K} + (1 - N\nu)_+ \epsilon^* \right) .$$

Corollary 6.2 *In case of quasi-uniform exact data we can choose the problem parameters as*

$$C = \frac{N\|f\|_{\mathcal{N}_K}}{e^{c\log(\tilde{c}h)/h}} \quad \textit{and} \quad \nu = \frac{1}{N} \, ,$$

to get

$$\|f - f^*\|_{L_q(\Omega)} \leq C e^{c\log(\tilde{c}h)/h} \|f\|_{\mathcal{N}_K} \, . \tag{13}$$

Now we shall consider general domains satifying an interior cone condition. Similarily to the case above we get

Theorem 6.3 *Let (f^*, ϵ^*) be a solution of (11) . Then the approximation error can be bounded by*

$$\|f - f^*\|_{L_q(\Omega)} \leq E \left(A e^{B\log(Dh)/\sqrt{h}} \|f\|_{\mathcal{N}_K} + \frac{N}{2C}\|f\|^2_{\mathcal{N}_K} - (N\nu - 1)\epsilon^* \right) .$$

Here the constants A, B, D, E depend on θ, α, d, q. In case of quasi-uniform data we can choose the parameters as

$$C = \frac{N\|f\|_{\mathcal{N}_K}}{A e^{B\log(Ch)/\sqrt{h}}} \quad \textit{and} \quad \nu \geq \frac{1}{N} \, ,$$

to get the estimate

$$\|f - f^*\|_{L_q(\Omega)} \leq C A e^{B\log(Ch)/\sqrt{h}} \|f\|_{\mathcal{N}_K} \, . \tag{14}$$

We now shall consider the case of inexact given data, i.e.

$$f(x_j) = y_j + r_j \quad \text{for } j = 1, \ldots, N, \quad \text{and } f \in \mathcal{N}_K \tag{15}$$

where we have some additive error $r = (r_1, \ldots, r_N)$. There are no assumptions concerning the error distribution. Again we have to bound both the native space norm and the discrete norm. Following [6] we have

$$\begin{aligned} \|f^*\|_{\mathcal{N}_K} &\leq \sqrt{\frac{2C}{N}\sum_{j=1}^{N}|r_j|_\epsilon + 2C\nu\epsilon + \|f\|^2_{\mathcal{N}_K}} \\ \|f^* - y\|_{\ell_\infty(X)} &\leq \sum_{j=1}^{N}|r_j|_\epsilon + \nu N\epsilon - \epsilon^* \cdot (N\nu - 1) + \frac{N}{2C}\|f\|^2_{\mathcal{N}_K}. \end{aligned}$$

Theorem 6.4 *Under the assumption (15) we have for all* $\epsilon > 0$

$$\begin{aligned} \|f - f^*\|_{L_q(\Omega)} \leq \tilde{C}\Bigg(& e^{c\log(\tilde{c}h)/h}\left(\|f\|_{\mathcal{N}_K} + \sqrt{\frac{2C}{N}\sum_{j=1}^{N}|r_j|_\epsilon + 2C\nu\epsilon + \|f\|^2_{\mathcal{N}_K}}\right) \\ &+ \sum_{j=1}^{N}|r_j|_\epsilon + \nu N\epsilon - \epsilon^*(N\nu - 1) + \frac{N}{2C}\|f\|^2_{\mathcal{N}_K}\Bigg) . \end{aligned}$$

□

Corollary 6.5 *If we choose*

$$C = \frac{N\|f\|^2_{\mathcal{N}_K}}{2\delta e^{c\log(\tilde{c}h)/h}} \quad \textit{and} \quad \epsilon = \delta, \quad \nu = \frac{1}{N}$$

we get in case of quasi-uniform data for any non-trivial solution

$$\|f - f^*\|_{L_2(\Omega)} \leq C\left(e^{c\log(\tilde{c}h)/h}\|f\|_{\mathcal{N}_K} + \delta\right) . \tag{16}$$

Note that bounds like (16) allow excellent bounds on the number of training samples required in the worst possible case to get required prediction quality.
Now we shall also here consider a general Lipschitz domain Ω. Analogously to the case above we get

Theorem 6.6 *Under the assumption (15) we have for all* $\epsilon > 0$

$$\begin{aligned} \|f - f^*\|_{L_q(\Omega)} \leq \tilde{C}\Bigg(& Ae^{B\log(Ch)/\sqrt{h}}\left(\|f\|_{\mathcal{N}_K} + \sqrt{\frac{2C}{N}\sum_{j=1}^{N}|r_j|_\epsilon + 2C\nu\epsilon + \|f\|^2_{\mathcal{N}_K}}\right) \\ &+ \sum_{j=1}^{N}|r_j|_\epsilon + \nu N\epsilon - \epsilon^*(N\nu - 1) + \frac{N}{2C}\|f\|^2_{\mathcal{N}_K}\Bigg) . \end{aligned}$$

Here the constants A, B, C *depend on* θ, α, d, q. *Again we get for the choice*

$$\begin{aligned} C &= \frac{N\|f\|^2_{\mathcal{N}_K}}{2\delta Ae^{B\log(Ch)/\sqrt{h}}} \\ \epsilon &= \delta, \quad \nu = \frac{1}{N} \end{aligned}$$

in case of quasi-uniform data for any non-trivial solution (f^*, ϵ^*)

$$\|f - f^*\|_{L_2(\Omega)} \leq C \left(A e^{B \log(Ch)/\sqrt{h}} \|f\|_{\mathcal{N}_K} + \delta \right). \tag{17}$$

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