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A Class of Infinite Potential Games *

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Abstract

We investigate potential functions for a class of infinite games, the path player games. Existence of potential functions is ensured only for a few classes of games. However, potential functions turn out to be interesting for existence and computation of equilibria in games. We show that path player games possess ordinal potential functions. Furthermore, by extending the benefit functions, even exact potential functions can be found. In addition, we propose a modified definition of potential functions, so called *restricted potential functions* which is valid for generalized equilibria. We show that restricted potential functions can be found for a generalized version of the path player game. From the existence of potential functions, we derive two ways to compute equilibria. The first approach uses that optimal solutions of a linear restricted optimization problem are equilibria. Afterwards, the computation of exact and approximate equilibria by a greedy approach is discussed. In addition, we present classes of path player games, where exact equilibria can be found within a finite number of improvement steps. Finally, we will present an alternative proof of the existence of pure-strategy equilibria in path player games.
Keywords: network games, potential games, generalized Nash equilibrium games, computation of equilibria. **JEL-Code:** C72

1 Introduction

Potential functions in game theory have been first mentioned by Rosenthal [19] who used a potential function to prove the existence of pure-strategy equilibria in congestion games. Monderer and Shapley introduced in [14] several definitions of potentials and presented ways to characterize games with potential functions.

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Path player games are a game theoretic model that describes the flow in a network from the viewpoint of the path owner. In this approach, see [18, 23], we consider paths in a network as players. These players choose the amount of flow that uses the path, which can be interpreted as offering bandwidth to the flow. The payoff is given e.g. by the income a path owner receives from the fee the flow has to pay for using the bandwith. The path player game is the basic model of the line planning game, which is currently under consideration in the framework of the European project *ARRIVAL* [1], see [22] for first results. Here, the line design of a public transportation network, like in railway systems, has to be found. That means, the lines themselves and their frequency have to be determined, see the referenced paper for details. In this framework, the lines are players that want to minimize their delay with respect to given capacity constraints and customer's demand. The model in this paper is a building block for the larger process of setting up traffic networks for real-world problems. It turns out that line planning games are potential games as well, which allows to develop efficient methods for computation of equilibria. Thus, this model can be used within the larger framework of the European project *ARRIVAL* to finally achieve robust and large-scale line plans.

As path player games are infinite games, our special interest is on infinite potential games. Monderer and Shapley present in [14] sufficient conditions for the existence of approximate equilibria and equilibria in infinite potential games. The former is satisfied by path player games, the latter is not due to the non-continuous character of the payoffs in those games. Kukushkin [9] proved the existence of equilibria in a revised version of infinite ordinal potential games with compact strategy sets. Norde and Tijs [16] are considering games where one or more players have infinite strategy sets and present sufficient conditions for weakly determinedness of these games. A game is weakly determined if it provides (ϵ, k)-equilibria, i.e. equilibra, where each player is "reasonably" satisfied. That means, he either cannot improve his benefit more than an ϵ or is gaining already a benefit higher than k. In [28], Voorneveld extends the results by Norde and Tijs, which were given for potential games to generalized ordinal potential games.

Several research has been done linking the field of potential games with other fields in game theory: Potential functions in cooperative games are considered in [3, 5]. [26, 25, 24] use the relation of potential games and the Shapley value of a cooperative game. Games with incomplete information and robustness of equilibria in potential games are studied in [27, 15]. Evolutionary processes are considered in [20] where infinite player sets are given. In [2] evolutionary dynamics are considered under stochastic perturbations. Mallozzi et al. [11] combine the concept of hierarchical games, like Stackelberg games, with potential functions and introduce a hierarchical potential game. Kukushkin [10] presents potential functions for games with perfect information and the relation to subgame perfect equilibrium. Brânzei et al. [4] investigate the relation of potential games and submodular games.

Variations of the classical definition of potential games have been presented in the following publications: Voorneveld proposes in [29] *best-response potential games*, a class of games containing the class of ordinal potential games. Best-responses potential functions provide a sufficient condition for the existence of equilibria. A recent approach is presented in Monderer [13], where multiple player types are considered in *q-potential games*.

In this paper, we consider a class of infinite games, *path player games*, introduced by Puerto et al. in [18]. Path player games are generalized Nash equilibrium (GNE) games (see e.g. [7]), which have the crucial property that the strategy sets of the players are not fixed, but mutually dependent on the strategies chosen by the other players. By these dependencies, feasible and infeasible solutions can be described. To be able to analyze GNE games, we introduce a new definition of a potential function, the *restricted potential function*, that takes the dependency of strategy sets into account, whereas the classical definition did not. We study the path player game with dependent strategy sets and show that the newly introduced restricted potential function exist in that case. Furthermore, we also study the case of non-dependent strategy sets in path player games by allowing infeasible solutions which get punished by a negative payoff. In this case, we present an ordinal potential function, which is a weaker type of a potential. Moreover, we strengthen this result in a second approach to an exact potential by extending the payoff function. The obtained potential functions are used to develop a method for the computation of equilibria in path player games by solving an optimization problem. The strength of this approach is that it is independent of the type of payoff. Nevertheless, to obtain all equilibria, we study another approach, which takes advantage of the existence of potential functions. In this approach, we create (best-reply) improvement sequences, which, if they are maximal, end up with an equilibrium. We are able to describe instances, where the best-reply improvement sequences are finite, although we study a problem with an infinite number of strategies. For all other cases, approximate equilibria can be obtained by using ϵ-improvement sequences.

We start with a short introduction to path player games and the notations of potential functions in Section 2. In Section 3, we define a restricted potential function which is existing for the generalized version of the path player game. In the same section, we present a linear restricted optimization problem that has feasible equilibria as optimal solutions. For the original path player game we propose an ordinal potential in Section 4. Furthermore, an extension of the original benefit function enables an exact potential for the path player game, which we will analyze in Section 5. Computation of equilibria using the greedy approach of improvement sequences is discussed in Section 6. The paper finishes with a summary and an outlook to future work.

2 Preliminaries

We start with an introduction to the main properties of path player games, presented in [18]. In these games a network with a single source s and a single sink t is considered. The players are represented by the paths $P \in \mathcal{P}$, connecting s and t. The private information of each player is given by the flow $f_P \geq 0$ assigned to the path. The cost functions assigned to the edges are given by $c_e(f_e)$ and are assumed to be continuous for $f_e \geq 0$. The flow on an edge is given by $f_e = \sum_{P:e\in P} f_P$. A player receives a certain *benefit*[1] $b_P(f)$, which is given either by a sum over cost on edges or by a punishment of $-M$ if the flow f is *infeasible*, i.e. exceeds a given threshold r.

$$b_P(f) = \begin{cases} c_P(f) & if \quad \sum_{P\in\mathcal{P}} f_P \leq r \\ -M & if \quad \sum_{P\in\mathcal{P}} f_P > r \end{cases} .$$

Therefore, for a profile of strategies $f = (f_P)_{P\in\mathcal{P}}$ the payoff function for player P is $b_P(f)$ for any $P \in \mathcal{P}$. Existence of pure-strategy equilibria for path player games is proved in [18]. Furthermore, Pareto-dominance of flows is investigated in [21]. It is interesting that in path player games, in fact all relations between the set of non-dominated solutions and equilibria are possible. In particular, in the paper mentioned, an example similar to the Prisoner's Dilemma is presented, where each equilibrium is dominated and each non-dominated solution is non-stable. But also classes of games with nice behavior are described, where the set of equilibria and the set of non-dominated solution are equal, or at least, each non-dominated solution is an equilibrium.

The following notation will be used in this text: The vector $f_{-P} \in \mathbb{R}_+^{|\mathcal{P}|-1}$ is derived from the flow f, by removing strategy f_P. With

$$d_P = r - \sum_{P_k\in\mathcal{P}\setminus\{P\}} f_{P_k} \tag{1}$$

we denote the *decision limit* of a player P. The decision limit is an upper bound on the flow f_P that yields a feasible flow f if it is satisfied. With

$$f_P^{max} = \{f_P \geq 0 : f_P \text{ maximizes } \; b_P(f_{-P}, f_P)\} \tag{2}$$

we denote the best reaction set of P, i.e. the set of strategies that maximizes the benefit of player P with respect to the strategies chosen by the competitors.

In a game, a *potential function* is given by a player-independent, not necessarily unique function that represents for each player the change of payoff due to the change of his strategy. Formally, we call a function $\Pi : \mathbb{R}_+^{|P|} \to \mathbb{R}$ an *exact potential* for a game Γ if for every $P \in \mathcal{P}$, for every $f_{-P} \in \mathbb{R}_+^{|\mathcal{P}|-1}$ and for every $x,\, z \in \mathbb{R}_+$ it holds:

$$b_P(f_{-P}, x) - b_P(f_{-P}, z) = \Pi(f_{-P}, x) - \Pi(f_{-P}, z) \; . \tag{3}$$

Γ is called an *exact potential game* if it admits an exact potential function.

A weaker form of a potential function is defined as follows: A function $\Pi : \mathbb{R}_+^{|P|} \to \mathbb{R}$ is called an *ordinal potential* for Γ if for every $P \in \mathcal{P}$, for every $f_{-P} \in \mathbb{R}_+^{|\mathcal{P}|-1}$ and for every $x,\, z \in \mathbb{R}_+$

$$b_P(f_{-P}, x) - b_P(f_{-P}, z) > 0 \quad \Leftrightarrow \quad \Pi(f_{-P}, x) - \Pi(f_{-P}, z) > 0 \; .$$

[1] In the original definition, the benefit function covers a third aspect, the *security payment*. Security payment will not be considered in this material.

Analogously, Γ is called an *ordinal potential game* if it admits an ordinal potential. By definition of an equilibrium (see e.g. [17]) it is true that for an ordinal potential game Γ, a flow f^* is an equilibrium if and only if for every $P \in \mathcal{P}$ and for every f_P it holds that $\Pi(f^*_{-P}, f^*_P) \geq \Pi(f^*_{-P}, f_P)$.

It follows that if we can find a maximum for the ordinal potential Π, then the equilibrium for Γ exists in pure strategies. In [14] that fact is used to draw the conclusion that each finite ordinal potential game has a pure-strategy equilibrium. Thus, knowing a (ordinal) potential helps to identify equilibria in a game. It is therefore interesting to have ways to check if a game is a potential game and to determine the potentials themselves. For this purpose strategy sequences are introduced next.

Definition 2.1. *A* strategy sequence[2]*, or simply* sequence $\varphi = (f^0, f^1, \ldots, f^k, \ldots)$ *in a game is given as an ordered sequence of flows* f^k *such that for every* $k \geq 1$ *there is a unique player* $P(k)$ *with* $f^k = (f^{k-1}_{-P(k)}, f^k_{P(k)})$ *and* $f^k_{P(k)} \neq f^{k-1}_{P(k)}$*. We call* $P(k)$ *the* active player *and the movement from* f^{k-1} *to* f^k *the* k^{th} step *in* φ*.*
A sequence is called improvement sequence *if for every* $k \geq 1$ *it holds that*

$$b_{P(k)}(f^k) > b_{P(k)}(f^{k-1}) \ . \tag{4}$$

A game Γ *satisfies the* finite improvement property (FIP) *if every improvement sequence is finite. For a sequence* φ*, we call* f^0 *its* initial flow *and, if* φ *is finite,* f^N *its* terminal flow*. Furthermore, we say that a finite* φ *is* connecting f^0 *and* f^N*. The* length *of a finite sequence* $\varphi = (f^0, f^1, \ldots, f^N)$ *is given by* $l(\varphi) = N$*.*

A finite improvement sequence that ends because no improvement step is possible anymore (a so-called *maximal sequence*), provides an equilibrium as terminal flow. If a game satisfies FIP we can use the improvement sequences to determine equilibria. In [14] the authors show that every finite ordinal potential game satisfies FIP, as by (4) for each improvement sequence, the potential values of the flows have to increase strictly in each step. As the set of strategies is finite, each improvement sequence has to be finite.
Unfortunately, although we can show in the following argumentation that path player games have potential functions, we can not apply that result as path player games are infinite. The improvement of benefit in a step may become arbitrarily small, which may lead to infinite improvement sequences. Further approaches to create sequences that yield equilibria or approximate equilibria are discussed in Section 6.

The *cost of a finite sequence* $\varphi = (f^0, f^1, \ldots, f^N)$ is given by

$$I(\varphi) = \sum_{k=1}^{N} \left[b_{P(k)}\left(f^k\right) - b_{P(k)}\left(f^{k-1}\right)\right] \ . \tag{5}$$

Note that for a sequence $\varphi^{-1} = (f^N, f^{N-1}, \ldots, f^0)$ which is the reverse of φ it holds that:

$$I(\varphi) = -I(\varphi^{-1}) \ . \tag{6}$$

[2] In [14], a "strategy sequence" is called "path". As this term is here already occupied for the paths that the players own in the network, we introduce the term "sequence" instead.

A sequence is called *closed* if $f^0 = f^N$ and a closed sequence is called *simple* if $f^\ell \neq f^k$ for all $\ell \neq k$ and $0 \leq \ell, k \leq N-1$.

With the next example, we show that path player games in their original notation are in general no exact potential games.

Example 2.1. *Consider the path player game illustrated in Figure 1. We set* $\bar{f} = (0,0)$ *and* $f = (0.5, 0.5)$*. Consider the sequences* $\varphi_1 = ((0,0),(0,0.5),(0.5,0.5))$ *and* $\varphi_2 = ((0,0),(2,0),(2,0.5),(0.5,0.5))$*. We have:* $I(\varphi_1) = (0.5-0) + (0.5-0) = 1$ *and*

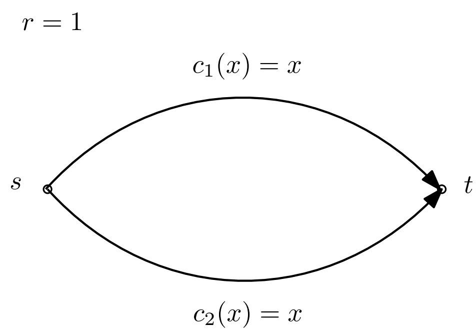

Figure 1: Example 2.1

$I(\varphi_2) = (-M-0) + (-M-(-M)) + (0.5-(-M)) = 0.5$ *and thus* $I(\varphi_1) \neq I(\varphi_2)$*. The closed sequence that is obtained by connecting* φ_1 *and* φ_2^{-1} *has cost* $1 - 0.5 = 0.5$*. By a result from [14] it is known that in an exact potential game, every simple closed sequence of length four has cost zero. Thus, the presented game is not an exact potential game.*

As far as we know the study of potentials for generalized Nash games is new and no reference can be traced back in the literature. Nevertheless, in the following sections, we develop approaches to determine potential functions in path player games. We start the investigation of potential functions just considering feasible flows. This is motivated by the fact that if we consider the complete set of flows $f \in \mathbb{R}_+^{|\mathcal{P}|}$, the situation gets much more complicated due to the non-continuity of the benefit function. Thus, in terms of introducing the results, it is more convenient to start with the investigation of feasible flows and to extend this concept later on to the complete set of flows. By considering only feasible flows, we obtain a different type of game, namely a *generalized path player game*, which is characterized by strategy sets that are restricted by the chosen strategies of the competitors. We introduce this concept more in detail in Section 3. As the classical definition of potential function is not applicable to restricted strategy sets, we propose a new type of potential function, called *exact restricted potential function*. We show that generalized path player games possess exact restricted potential functions. Furthermore, we develop an algorithmic approach to compute feasible equilibria by solving an optimization problem. In Section 4, we consider the complete strategy set, including the infeasible flows. Due to the structure of the benefit function, an exact potential is not existing here. Nevertheless, we prove the existence of ordinal potential functions, a weaker form of potentials. In a third approach in Section 5 we extend the benefit function such that even an exact potential function for the complete strategy set can be found. Finally, in Section 6, computation of equilibria by using a greedy approach is discussed. We present classes of path player games, where equilibria can be found within a finite sequence of greedy improvement steps.

3 Exact Potential Function for the Generalized Path Player Game

In this section, we introduce a new definition of potential functions, valid for games with mutually dependent players' set of strategies. With this new notion, we are able to obtain a (restricted) potential function for path player games. We allow a player to choose f_P only from $[0, d_P]$. (Recall that d_P was defined in (1).) Note, that d_P is dependent on the strategies of the competitors f_{-P}. By feasibility, the benefit is then given by $b_P(f) = c_P(f)$. Games where the strategy sets of the players are dependent on the strategies of the competitors are called *generalized Nash equilibrium games* (see e.g. [7, 6]). See [23] for a generalization of path player games to *games on polyhedra*, a special type of generalized Nash equilibrium games. Consequently, we call a path player game, where the strategy set of each player is restricted to $[0, d_P]$ a *generalized path player game.*

Definition 3.1. *In a generalized path player game, a flow f^* is a* generalized equilibrium *if and only if for all paths $P \in \mathcal{P}$ and for all $f_P \in [0, d_P(f_{-P})]$, it satisfies that*

$$b_P(f^*_{-P}, f^*_P) \geq b_P(f^*_{-P}, f_P) \ .$$

The investigation of generalized path player games is not only interesting for the sake of finding potential functions. It makes also sense if we are interested in feasible equilibria, as the set of feasible equilibria in a path player game and the set of generalized equilibria in the corresponding generalized path player game are coincide:

Lemma 3.2. *Consider a path player game Γ and the corresponding generalized path player game $\hat{\Gamma}$. A flow f^* is a feasible equilibrium in Γ if and only if f^* is a generalized equilibrium in $\hat{\Gamma}$.*

The proof is obvious and therefore omitted. Since we are going to restrict ourselves to feasible flows, the definition of potentials needs an adjustment. In the definition of a classical potential it is not considered that strategy sets may be mutually players' dependent.

Definition 3.3. *A function $\Pi : f \to \mathbb{R}$ is an* exact restricted potential *for a generalized path player game Γ if for every $P \in \mathcal{P}$, for every f_{-P} with $\sum_{P_k \in \mathcal{P} \setminus \{P\}} f_{P_k} < r$ and for every x, $z \in [0, d_P]$ it satisfies:*

$$b_P(f_{-P}, x) - b_P(f_{-P}, z) = \Pi(f_{-P}, x) - \Pi(f_{-P}, z) \ . \tag{7}$$

Consequentely, a path player game Γ is called an *exact restricted potential game* if it admits an exact restricted potential. This definition is extended in [23] to games on polyhedra.

Definition 3.4. *A sequence* $\varphi = (f^0, \ldots, f^N)$ *is called* feasible *if* f^k *is feasible for every* $0 \leq k \leq N$.

To prove the main result of this section we need the following Lemma.

Lemma 3.5. *Let* $\mathbb{F} = \{f : f_P \geq 0 \ \forall \ P \in \mathcal{P} \ \wedge \ \sum_{P \in \mathcal{P}} f_P \leq r\}$ *be the set of feasible flows in a generalized path player game* Γ. *The following statements are equivalent:*

$$\Gamma \text{ is an exact restricted potential game} \tag{8}$$

$$I(\varphi) = 0 \text{ for every finite closed feasible sequence } \varphi \tag{9}$$

$$I(\varphi) = 0 \text{ for every finite simple closed feasible sequence } \varphi \tag{10}$$

$$I(\varphi) = 0 \text{ for every simple closed feasible sequence } \varphi \text{ of length four} \tag{11}$$

An exact restricted potential of Γ is given by fixing a feasible flow $\bar{f}$ and defining $\Pi(f) = I(\varphi)$ $\forall$ feasible f where φ is a feasible sequence connecting $\bar{f}$ and f. Note that $\Pi(f)$ is well-defined, as $I(\varphi_1) = I(\varphi_2)$ holds for φ_1 and φ_2 having the same initial and terminal flow.

Proof. To prove the thesis of this lemma we follow an argument similar to the one in [14] but applied to sets (the strategies of the different players are linked by constraints) of strategies that are mutually dependent and to the exact restricted potential functions. First, note that (9) implies (10) which implies (11). It remains to show that (8) is equivalent to (9) and (11) implies (9).

(8) $\Rightarrow$ (9) Let Π be an exact restricted potential for Γ. Consider a feasible closed sequence φ. By the definition of an exact restricted potential, the cost of a sequence is given by

$$I(\varphi) = \sum_{k=1}^{N} \left[\Pi\left(f^k\right) - \Pi\left(f^{k-1}\right)\right] = \Pi\left(f^N\right) - \Pi\left(f^0\right) = 0 \ .$$

(8) $\Leftarrow$ (9) We assume that $I(\varphi) = 0$ holds for any finite closed feasible sequence φ. Now consider two feasible flows $\bar{f}, f \in \mathbb{F}$. Any finite feasible sequence which has $\bar{f}$ as initial and terminal flow and which contains f has cost 0 by assumption. Thus, using Equation (6) it can be seen that all feasible sequences connecting $\bar{f}$ and f have to have equal cost.
We define: For $f \in \mathbb{F}$, we set $\Pi(f) = I(\varphi)$, for all feasible φ connecting a fixed feasible $\bar{f}$ with f. It remains to show that $\Pi(f)$ is an exact restricted potential in Γ. Consider any $P \in \mathcal{P}$ and any f_{-P} with $\sum_{P_k \in \mathcal{P} \setminus \{P\}} f_{P_k} < r$. Furthermore consider $x, z \in [0, d_P]$. Let $\varphi_1 = (\bar{f}, f^1, \ldots, (f_{-P}, x))$ be a feasible sequence connecting $\bar{f}$ and (f_{-P}, x). Set $\varphi_2 = (\bar{f}, f^1, \ldots, (f_{-P}, z))$. It follows that $\Pi(f_{-P}, x) - \Pi(f_{-P}, z) = I(\varphi_1) - I(\varphi_2) = b_P(f_{-P}, x) - b_P(f_{-P}, z)$, from which we conclude that Π is an exact restricted potential.

Part c) (11) $\Rightarrow$ (9):

We suppose that $I(\varphi) = 0$ for every simple closed feasible sequence φ of length $l(\varphi) = 4$.

Assume there is a finite closed feasible sequence with non-zero cost and let us consider such a sequence $\varphi = (f^0, \ldots, f^N)$ with minimal length $l(\varphi) > 4$. As the sequence is closed, there is a step q with $f^q_{P(q)} - f^{q-1}_{P(q)} < 0$, i.e. player $P(q)$ is decreasing his flow. Without loss of generality, let q be the first step: $q = 1$ and set the active player in the first step $P(q) = P_1$. Because of $f^0 = f^N$ there has to be a $2 \leq j \leq N$ such that $P(j) = P_1$ with $f^q_{P(q)} - f^{q-1}_{P(q)} > 0$, i.e. player P_1 has to be active a second time where he increases his flow. For $j = 2$ we obtain a contradiction to the minimality of $l(\varphi)$, due to $I(f^0, f^2, \ldots, f^N) = I(\varphi)$. A similar contradiction can be obtained for $j = N$. Hence, we assume $2 \leq j \leq N-1$. Consider the flow

$$\hat{f}_j = \left(f^{j-1}_{-\{P_1, P(j+1)\}}, f^{j-1}_{P_1}, f^{j+1}_{P(j+1)}\right)$$

which is obtained by proceeding step $j+1$ before step j. Note that $P(j+1) \neq P_1$ holds as otherwise we would have a contradiction to minimality of $l(\varphi)$. Because of

$$f^{j-1}_P = f^{j+1}_P \;\forall\; P \notin \{P_1, P(j+1)\}$$

and

$$f^{j-1}_{P_1} < f^j_{P_1} = f^{j+1}_{P_1}$$

we can conclude that $\hat{f}$ is a feasible flow:

$$\begin{aligned}
\sum_{P \in \mathcal{P}} \hat{f}_P &= \sum_{P \in \mathcal{P} \setminus \{P_1, P(j+1)\}} f^{j-1}_P + f^{j-1}_{P_1} + f^{j+1}_{P(j+1)} \\
&< \sum_{P \in \mathcal{P} \setminus \{P_1, P(j+1)\}} f^{j+1}_P + f^{j+1}_{P_1} + f^{j+1}_{P(j+1)} \\
&= \sum_{P \in \mathcal{P}} f^{j+1}_P \\
&\leq r \;,
\end{aligned}$$

as f^{j+1} is feasible. It can be verified that the simple feasible sequence of length four $(f^{j-1}, f^j, f^{j+1}, \hat{f}^j)$ is closed and has by assumption cost zero. Thus, the feasible sequences (f^{j-1}, f^j, f^{j+1}) and $(f^{j-1}, \hat{f}^j, f^{j+1})$ have equal length, and consequently $l(\varphi) = l(\hat{\varphi})$ with $\hat{\varphi} = (f^0, \ldots, f^{j-1}, \hat{f}^j, f^{j+1}, \ldots, f^N)$ is true. Note that in $\hat{\varphi}$ it holds $P(j+1) = P_1$. By iteration of this replacement process we will obtain a finite closed feasible sequence φ^* with $I(\varphi^*) = I(\varphi) \neq 0$ and $P(N) = P(1) = P_1$, which leads to a contradiction of the minimality assumption of $l(\varphi)$. We conclude that $I(\varphi) = 0$ holds for each finite closed feasible sequence φ. □

A different representation of the cost functions will be needed for the following proof. The cost of a path P is given by the sum of the costs on the edges belonging to this path. We distinguish in the following two types of edges, the ones belonging exclusively to P, and the ones shared with other paths. Let $E_P^{exc} = \{e : e \in P \wedge e \notin P_k \ \forall \ P_k \neq P\}$ be the set of *exclusively used edges of* P, and $E_P^{com} = \{e : e \in P \ \wedge \ \exists \ P_k \neq P : e \in P_k\}$ the set of *common used edges*. As $f_e = f_P$ holds for exclusively used edges, we obtain the *cost of a path in extensive form*:

$$c_P(f) = \sum_{e \in P} c_e(f_e) = \sum_{e \in E_P^{exc}} c_e(f_P) + \sum_{e \in E_P^{com}} c_e \left(f_P + \sum_{P_k : e \in P_k, P_k \neq P} f_{P_k} \right) \tag{12}$$

We present now the main theorem of this section.

Theorem 3.6. *Generalized path player games are exact restricted potential games.*

Proof. We show that each generalized path player game satisfies property (11) of Lemma 3.5.

Consider any two active players P_i and P_j that create a simple closed feasible sequence φ of length $l(\varphi) = 4$ by alternatively choosing a new strategy and returning to the first strategy afterwards. We denote the set of strategies of the remaining players with $f_{-\{P_i P_j\}}$. The sequence φ is given by:

$$\begin{aligned} \varphi \ = \ & ((f_{-\{P_iP_j\}}, f_{P_i}, f_{P_j}), (f_{-\{P_iP_j\}}, \bar{f}_{P_i}, f_{P_j}), (f_{-\{P_iP_j\}}, \bar{f}_{P_i}, \bar{f}_{P_j}), \\ & (f_{-\{P_iP_j\}}, f_{P_i}, \bar{f}_{P_j}), (f_{-\{P_iP_j\}}, f_{P_i}, f_{P_j})) \ . \end{aligned}$$

As φ is feasible, we have $b_P(f^k) = c_P(f^k)$ for all P and for all $0 \leq k \leq 3$.
We obtain:

$$I(\varphi) = \big(c_{P_i}(f^1) - c_{P_i}(f^0)\big) + \big(c_{P_j}(f^2) - c_{P_j}(f^1)\big) + \big(c_{P_i}(f^3) - c_{P_i}(f^2)\big) + \big(c_{P_j}(f^0) - c_{P_j}(f^3)\big) \ .$$

To determine $I(\varphi)$ we need only to consider the cost functions of P_i and P_j, and as $f_{-\{P_iP_j\}}$ is fixed, the influence of $f_{-\{P_iP_j\}}$ in the common used edges $E_{P_i}^{com}$ and $E_{P_j}^{com}$ can be neglected. Thus, we modify the sets of common and exclusively used edges as follows: $\bar{E}_{P_i}^{exc} = \{e \in E : e \in P_i \ \wedge \ e \notin P_j\}$ and $\bar{E}_{P_i}^{com} = \{e \in E : e \in P_i \ \wedge \ e \in P_j\}$. The values $\bar{E}_{P_j}^{exc}$ and $\bar{E}_{P_j}^{com}$ are defined analogously. As we are only considering the two players P_i and P_j it holds: $\bar{E}_{P_i}^{com} = \bar{E}_{P_j}^{com} =: \bar{E}^{com}$. Thus, to investigate the costs, apart from the exclusive edges of P_i and P_j, we only consider that edges which P_i and P_j have in common, while those edges which are used common with the remaining paths carrying invariant flow, are represented in $\bar{E}_{P_i}^{exc}$ (and $\bar{E}_{P_j}^{exc}$ resp.). The path cost in extensive form (see (12)) is thus rewritten:

$$c_P(f) = \sum_{e \in \bar{E}_P^{exc}} c_e(f_P) + \sum_{e \in \bar{E}^{com}} c_e \left(f_P + \sum_{P_k : e \in P_k, P_k \neq P} f_{P_k} \right) \tag{13}$$

We denote $S_{-\{P_iP_j\}}(e) = \sum_{P : e \in P, P \notin \{P_i, P_j\}} f_P$ and obtain:

$$
\begin{aligned}
I(\varphi) = \quad & \left(\sum_{e \in \bar{E}^{exc}_{P_i}} c_e(\bar{f}_{P_i}) + \sum_{e \in \bar{E}^{com}} \left(\bar{f}_{P_i} + f_{P_j} + S_{-\{P_i P_j\}}(e)\right) \right) \\
& - \left(\sum_{e \in \bar{E}^{exc}_{P_i}} c_e(f_{P_i}) + \sum_{e \in \bar{E}^{com}} \left(f_{P_i} + f_{P_j} + S_{-\{P_i P_j\}}(e)\right) \right) \\
& + \left(\sum_{e \in \bar{E}^{exc}_{P_j}} c_e(\bar{f}_{P_j}) + \sum_{e \in \bar{E}^{com}} \left(\bar{f}_{P_i} + \bar{f}_{P_j} + S_{-\{P_i P_j\}}(e)\right) \right) \\
& - \left(\sum_{e \in \bar{E}^{exc}_{P_j}} c_e(f_{P_j}) + \sum_{e \in \bar{E}^{com}} \left(\bar{f}_{P_i} + f_{P_j} + S_{-\{P_i P_j\}}(e)\right) \right) \\
& + \left(\sum_{e \in \bar{E}^{exc}_{P_i}} c_e(f_{P_i}) + \sum_{e \in \bar{E}^{com}} \left(f_{P_i} + \bar{f}_{P_j} + S_{-\{P_i P_j\}}(e)\right) \right) \\
& - \left(\sum_{e \in \bar{E}^{exc}_{P_i}} c_e(\bar{f}_{P_i}) + \sum_{e \in \bar{E}^{com}} \left(\bar{f}_{P_i} + \bar{f}_{P_j} + S_{-\{P_i P_j\}}(e)\right) \right) \\
& + \left(\sum_{e \in \bar{E}^{exc}_{P_j}} c_e(f_{P_j}) + \sum_{e \in \bar{E}^{com}} \left(f_{P_i} + f_{P_j} + S_{-\{P_i P_j\}}(e)\right) \right) \\
& - \left(\sum_{e \in \bar{E}^{exc}_{P_j}} c_e(\bar{f}_{P_j}) + \sum_{e \in \bar{E}^{com}} \left(f_{P_i} + \bar{f}_{P_j} + S_{-\{P_i P_j\}}(e)\right) \right) \\
= \quad & 0 \; .
\end{aligned}
$$

□

Definition 3.7. *Let* $n = |\mathcal{P}|$ *be the number of players in a generalized path player game. Consider a flow* f *and a fixed flow* $\bar{f}$*. A sequence* φ *of length* $l(\varphi) = \nu \leq n+1$ *with* $\varphi = \left(\bar{f}, f^1, \ldots, f^{\nu-1}, f\right)$ *is called* direct sequence from $\bar{f}$ to f*, if for all* $k = 1, \ldots, \nu$ *it holds:*

$$f^k_{P(\ell)} = f_{P(\ell)} \ \forall\ \ell \leq k \quad \textit{and} \quad f^k_{P(\ell)} = \bar{f}_{P(\ell)} \ \forall\ \ell > k \ .$$

In a direct sequence, in each step a unique player $P(k)$ changes its flow from $\bar{f}_{(P(k))}$ to $f_{P(k)}$. After ν steps, f is obtained.

Lemma 3.8. *If* f *is a feasible flow then each direct sequence connecting the zero flow* $\bar{f} = \mathbb{0}_{|\mathcal{P}|}$ *and* f *is a feasible sequence.*

Proof. As f is feasible it holds $\sum_{P\in\mathcal{P}} f_P \leq r$. For each step $k = 1, \ldots, \nu$ it holds

$$\sum_{P\in\mathcal{P}} f_P^k = \sum_{\ell=1}^{k} f_{P(\ell)} + \sum_{\ell=k+1}^{n} 0 \leq \sum_{\ell=1}^{n} f_{P(\ell)} = \sum_{P\in\mathcal{P}} f_P \leq r \ .$$

□

Note that Lemma 3.8 does not hold for arbitrary feasible $\bar{f}$, which is illustrated in the next example. Nevertheless, there always exists a feasible sequence connecting two feasible flows $\bar{f}$ and f. We obtain such a sequence by ordering the players such that those players, which want to reduce their flow, get active first.

Example 3.1. *Consider the game represented by Figure 1. Set* $\bar{f} = (0.25, 0.75)$ *and* $f = (0.5, 0.5)$*. The direct sequence* $\varphi = ((0.25, 0.75), (0.5, 0.75), (0.5, 0.5))$ *is not feasible as the flow* f^1 *is infeasible. A feasible direct sequence is given by* $\varphi = ((0.25, 0.75), (0.25, 0.5), (0.5, 0.5))$*, see Figure 2.*

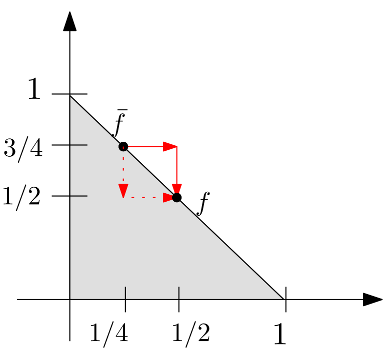

Figure 2: Feasible and infeasible direct sequences

By fixing $\bar{f} = \mathbb{0}_{|\mathcal{P}|}$ and using Lemmas 3.5 and 3.8 we can derive the following result:

Lemma 3.9. *Consider a generalized path player game with* n *players. An exact restricted potential* Π *is given by*

$$\Pi(f) = I(\varphi) = \left[b_{P(k)} \begin{pmatrix} f_{P(1)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} - b_{P(k)} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right] + \sum_{k=2}^{n} \left[b_{P(k)} \begin{pmatrix} f_{P(1)} \\ \vdots \\ f_{P(k-1)} \\ f_{P(k)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} - b_{P(k)} \begin{pmatrix} f_{P(1)} \\ \vdots \\ f_{P(k-1)} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right] .$$

The next theorem provides a shorter representation of the potential function.

Theorem 3.10. *Consider a generalized path player game with n players. An exact restricted potential Π is given by*

$$\Pi(f) = \sum_{e\in E} c_e(f_e) \ .$$

Proof. We show that in an exact restricted potential game, each feasible sequence φ connecting a fixed flow $\bar{f}$, say $\bar{f} = \mathbb{O}_{|\mathcal{P}|}$, and some feasible flow f has cost

$$I(\varphi) = \sum_{e\in E} [c_e(f_e) - c_e(0)] \ .$$

Consider a generalized path player game with n players, a feasible flow f and a direct sequence φ from $\bar{f} = \mathbb{O}_{|\mathcal{P}|}$ to f. It is sufficient to prove the claim for direct sequences only, as all feasible sequences connecting $\bar{f}$ and f have equal costs.
As f is feasible, it holds by Lemma 3.8 that φ is feasible too. Denote
$I_k(\varphi) = b_{P(k)}(f^k) - b_{P(k)}(f^{k-1}) = c_{P(k)}(f^k) - c_{P(k)}(f^{k-1})$. Then

$$\begin{aligned} I_k(\varphi) &= \sum_{e\in E^{exc}_{P(k)}} c_e(f_{P(k)}) + \sum_{e\in E^{com}_{P(k)}} c_e\left[f_{P(k)} + \sum_{P:e\in P\wedge P\neq P(k)} f^k_P\right] \\ &\quad - \left(\sum_{e\in E^{exc}_{P(k)}} c_e(0) + \sum_{e\in E^{com}_{P(k)}} c_e\left[\sum_{P:e\in P\wedge P\neq P(k)} f^k_P\right]\right) \ . \end{aligned}$$

And thus

$$\begin{aligned} I(\varphi) &= \sum_{k=1}^{n} I_k(\varphi) \\ &= \sum_{k=1}^{n} \sum_{e\in E^{exc}_{P(k)}} \left[c_e(f_{P(k)}) - c_e(0)\right] \\ &\quad \underbrace{+\sum_{k=1}^{n} \sum_{e\in E^{com}_{P(k)}} \left[c_e\left(f_{P(k)} + \sum_{P:e\in P\wedge P\neq P(k)} f^k_P\right) - c_e\left(\sum_{P:e\in P\wedge P\neq P(k)} f^k_P\right)\right]}_{(\#)} \end{aligned}$$

We reorder (#) by first summing up over the edges. For a given edge e, consider the players P that use e but not exclusively: $\{P : e \in E^{com}_P\}$. Assume that these players get active in the following order $\{P_{i(1)}, \dots, P_{i(\ell)}\}$, which is a subsequence of $\{P(k)\}_{k=1,\dots,n}$. With E^c we denote the set of edges that are used by more than one player, with E^e

we denote the set of edges that are used by exactly one player.

$$\begin{aligned}
(\#) &= \sum_{e\in E^c}\sum_{k:e\in E^{com}_{P(k)}}\left[c_e\left(f_{P(k)}+\sum_{P:e\in P\wedge P\neq P(k)} f^k_P\right)-c_e\left(\sum_{P:e\in P\wedge P\neq P(k)} f^k_P\right)\right]\\
&= \sum_{e\in E^c}\sum_{m=1}^{\ell}\left[c_e\left(\sum_{q=1}^{m} f_{P_{i(q)}}\right)-c_e\left(\sum_{q=1}^{m-1} f_{P_{i(q)}}\right)\right] \qquad (14)\\
&= \sum_{e\in E^c}\left[c_e\left(\sum_{q=1}^{\ell} f_{P_{i(q)}}\right)-c_e(0)\right]\\
&= \sum_{e\in E^c}\left[c_e(f_e)-c_e(0)\right] \qquad (15)
\end{aligned}$$

Equation (14) is true as in the direct sequence after the m^{th} step, $f^m_{P_{i(q)}} = f_{P_{i(q)}}$ for $q \leq m$ and $f_{P_{i(q)}} = 0$ for $q > m$ holds. In addition, (15) holds as $f_e = \sum_{P:e\in P} f_P$.
By using that

$$\sum_{k=1}^{n}\sum_{e\in E^{exc}_{P(k)}}\left[c_e\left(f_{P(k)}\right)-c_e(0)\right]=\sum_{e\in E^e}\sum_{k:e\in E^{exc}_{P(k)}}\left[c_e\left(f_{P(k)}\right)-c_e(0)\right]=\sum_{e\in E^e}\left[c_e\left(f_e\right)-c_e(0)\right]$$

we are able to conclude:

$$I(\varphi) \;=\; \sum_{e\in E}\left[c_e(f_e)-c_e(0)\right] \ . \qquad (16)$$

By Lemma 3.5 it holds that (16) is a an exact restricted potential function. As $c_e(0)$ is constant for all $e \in E$, it holds that $\Pi(f) = \sum_{e\in E} c_e(f_e)$ is an exact restricted potential function, as well. □

Note that it would be an alternative (and shorter) proof for Theorem 3.10 to check that $\Pi(f) = \sum_{e\in E} c_e(f_e)$ satisfies (7). Nevertheless, the proof just presented contains more information, as it illustrates the construction of the potential function.

The next theorem follows immediately from the previous result and Lemma 3.2. It provides an algorithmic approach for finding equilibria in arbitrary path player games. The strength of this approach is that it is independent of the choice of the payoff function.

Theorem 3.11. *In a path player game, feasible equilibria are given by optimal solutions of the following problem:*

$$\max \Pi(f) = \sum_{e\in E} c_e(f_e) \quad \textit{subject to} \quad f \in \mathbb{F} \ .$$

4 Ordinal Potential Functions for Path Player Games

To determine a potential, we reduced in the previous section the strategy space to the feasible strategies. If we are considering the original path player game on the complete strategy space $f \in \mathbb{R}_+^{|\mathcal{P}|}$, the infeasiblity penalty complicates the situation. By allowing infeasible strategies and thus infeasible sequences, we get problems with the cost of a sequence each time the sequence leaves the feasible strategy space $\mathbb{F}$. (see Example 2.1). Nevertheless, we are able to present an ordinal potential for path player games. For the proof, the next proposition is necessary.

Proposition 4.1. *Let $f^x = (f_{-P}, x)$ and $f^z = (f_{-P}, z)$ be two feasible flows that differ only in the component f_P. Then*

$$\sum_{e \in E} c_e\left(f_e^x\right) - \sum_{e \in E} c_e\left(f_e^z\right) = c_P\left(f^x\right) - c_P\left(f^z\right) \quad .$$

Proof.

$$\begin{aligned}
& \sum_{e \in E} c_e(f_e^x) - \sum_{e \in E} c_e(f_e^z) \\
= & \sum_{e \in E} c_e(f_e^x) \\
= & \sum_{e \in P} c_e(f_e^x) \qquad (17) \\
= & \; c_P(f^x) - c_P(f^z) \quad .
\end{aligned}$$

Equation (17) is true as f^x and f^z are different only with respect to path P. □

Theorem 4.2. *Path player games are ordinal potential games. An ordinal potential function is given by*

$$\Pi(f) = \begin{cases} \sum_{e \in E} c_e(f_e) & if \quad \sum_{P \in \mathcal{P}} f_P \leq r \\ -M & if \quad \sum_{P \in \mathcal{P}} f_P > r \end{cases} \quad . \qquad (18)$$

Proof. Consider any player P and any two flows (f_{-P}, x) and (f_{-P}, z). We distinguish four cases:
Case 1: (f_{-P}, x) infeasible, (f_{-P}, z) infeasible:

$$b_P(f_{-P}, x) - b_P(f_{-P}, z) = -M - (-M) = \Pi(f_{-P}, x) - \Pi(f_{-P}, z) \quad .$$

Case 2: (f_{-P}, x) infeasible, (f_{-P}, z) feasible:

$$b_P(f_{-P}, x) - b_P(f_{-P}, z) = -M - c_P(f_{-P}, z) < 0 \quad ,$$

as M is a large number. By the same argument,

$$\Pi(f_{-P}, x) - \Pi(f_{-P}, z) = -M - \sum_{e \in E} c_e(f_e) < 0 \quad ,$$

holds.
Case 3: (f_{-P},x) feasible, (f_{-P},z) infeasible:

$$b_P(f_{-P},x) - b_P(f_{-P},z) = c_P(f_{-P},x) - (-M) > 0 \ ,$$

as M is a large number. Also,

$$\Pi(f_{-P},x) - \Pi(f_{-P},z) = \sum_{e \in E} c_e(f_e) - (-M) > 0 \ ,$$

holds.
Case 4: (f_{-P},x) feasible, (f_{-P},z) feasible:

$$\Pi(f^x) - \Pi(f^z) = \sum_{e \in E} c_e(f_e^x) - \sum_{e \in E} c_e(f_e^z) =$$
$$c_P(f^x) - c_P(f^z) = b_P(f^x) - b_P(f^z) \ .$$

The final equation holds by Proposition 4.1.
Resulting from these four cases we can conclude that

$$b_P(f_{-P},x) - b_P(f_{-P},z) > 0 \quad \Leftrightarrow \quad \Pi(f_{-P},x) - \Pi(f_{-P},z) > 0.$$ □

For two classes of path player games, namely for games on path-disjoint networks and for games with strictly increasing costs, we are able to present a simpler description of an ordinal potential function. A network is path-disjoint, if there is no edge that belongs to more than one path. In path player games on path-disjoint networks, it holds that $c_P(f) = c_P(f_{-P}, f_P) = c_P(\cdot, f_P)$.

Theorem 4.3. *In a path player game that is played on a path-disjoint network or in a path player game where the cost functions $c_e(f_e)$ are strictly increasing, an ordinal potential function is given by:*

$$\Pi(f) = \begin{cases} \sum_{P \in \mathcal{P}} c_P(f) & if \quad \sum_{P \in \mathcal{P}} f_P \leq r \\ -M & if \quad \sum_{P \in \mathcal{P}} f_P > r \end{cases} \ . \tag{19}$$

Proof. Consider any player P and any two flows (f_{-P},x) and (f_{-P},z). We distinguish the same four cases as in the proof of Theorem 4.2. The cases 1 - 3 are analogous to that proof. We have only to consider case 4: ((f_{-P},x) and (f_{-P},z) feasible).

Path-disjoint network:

$$\begin{aligned} & b_P(f_{-P},x) - b_P(f_{-P},z) \\ = \ & c_P(f_{-P},x) - c_P(f_{-P},z) \\ = \ & c_P(x) - c_P(z) & (20) \\ = \ & c_P(x) - c_P(z) + \sum_{P_k \in \mathcal{P} \setminus \{P\}} c_{P_k}(f_{P_k}) - \sum_{P_k \in \mathcal{P} \setminus \{P\}} c_{P_k}(f_{P_k}) \\ = \ & \Pi(f_{-P},x) - \Pi(f_{-P},z) \ . & (21) \end{aligned}$$

(20) and (21) hold as we have a path-disjoint network.

Strictly increasing costs:
Assume

$$\begin{aligned}
& b_P(f_{-P}, x) - b_P(f_{-P}, z) = c_P(f_{-P}, x) - c_P(f_{-P}, z) > 0 \\
\Leftrightarrow \quad & x - z > 0 && (22) \\
\Leftrightarrow \quad & \sum_{P_k \in \mathcal{P}} c_{P_k}(f_{-P}, x) - \sum_{P_k \in \mathcal{P}} c_{P_k}(f_{-P}, z) > 0 && (23) \\
\Leftrightarrow \quad & \Pi(f_{-P}, x) - \Pi(f_{-P}, z) > 0 \ .
\end{aligned}$$

(22) is true as we have strictly increasing costs $c_e(f_e)$ and due to the same reason $c_P(f_{-P}, x) - c_P(f_{-P}, z) \geq 0 \ \forall \ P \in \mathcal{P}$ holds, which is used in (23).

□

In [21] the following definition is given: A flow f is non-dominated if there exists no flow f such that the following clause holds:

$$[\forall \ P \in \mathcal{P} : b_P(f) \geq b_P(f^*) \ \wedge \ \exists \ P \in \mathcal{P} : b_P(f) > b_P(f^*)] \, .$$

Lemma 4.4. *In path player games that are played on path-disjoint networks or in path player games with cost functions $c_e(f_e)$ that are strictly increasing, there is an equilibrium f^* which is non-dominated.*

Proof. Consider the equilibrium f^* which is given as a maximizer of the potential function (19). Theorem 1.11 in [8] proves that a flow maximizing the sum of the benefits over all player in a game is non-dominated. □

In general it is not necessarily true that a maximizer f of a potential function is also non-dominated. For instance, see [21], to find a path player game in which each equilibrium is dominated. As the maximizer f is an equilibrium, it has to be dominated in this case.

5 An Exact Potential for an Extended Benefit Function

We have already pointed out in Example 2.1 that the path player game as stated originally is not an exact potential game. The problem is the discontinuity of the benefit function at the point when the flows become infeasible. It is possible to handle this drawback by a slight extension of the standard benefit function, by carrying the cost function $c_P(f)$ into the infeasible region. With this approach, we are able to present an exact potential function for the path player game.
In a path player game, the *extended benefit function* is given by

$$b_P^{ext}(f) = \begin{cases} c_P(f) & if \quad \sum_{P_k \in \mathcal{P}} f_{P_k} \leq r \\ -M + c_P(f) & if \quad \sum_{P_k \in \mathcal{P}} f_{P_k} > r \end{cases} \ . \tag{24}$$

The extended benefit function describes in fact a realistic model of economic situations. A player gets punished for sending too much flow, but nevertheless, he receives the income created by the costs. For instance, consider a company that is producing more goods than it is allowed to by a pollution regulation and gets punished. Nevertheless, this company is selling all produced goods and thus obtains income from them.

Theorem 5.1. *The function*

$$\Pi(f) = \begin{cases} \sum_{e\in E} c_e(f_e) & \text{if} \quad \sum_{P\in\mathcal{P}} f_P \le r \\ -M + \sum_{e\in E} c_e(f_e) & \text{if} \quad \sum_{P\in\mathcal{P}} f_P > r \end{cases} \tag{25}$$

is an exact potential function for any path player game with extended benefit function.

Proof. Consider any player P and any two flows (f^x) and (f^z). We distinguish four cases. Proposition 4.1 is used in each case.
Case 1: (f^x) infeasible, (f^z) infeasible:

$$\begin{aligned} \Pi(f^x) - \Pi(f^z) &= -M + \sum_{e\in E} c_e(f_e^x) - \left(-M + \sum_{e\in E} c_e(f_e^z)\right) \\ &= -M + c_P(f^x) - (-M + c_P(f^z)) = b_P(f^x) - b_P(f^z) \end{aligned}$$

Case 2: (f^x) infeasible, (f^z) feasible:

$$\begin{aligned} \Pi(f^x) - \Pi(f^z) &= -M + \sum_{e\in E} c_e(f_e^x) - \sum_{e\in E} c_e(f_e^z) \\ &= -M + c_P(f^x) - c_P(f^z) = b_P(f^x) - b_P(f^z) \end{aligned}$$

Case 3: (f^x) feasible, (f^z) infeasible:

$$\begin{aligned} \Pi(f^x) - \Pi(f^z) &= \sum_{e\in E} c_e(f_e^x) - \left(M + \sum_{e\in E} c_e(f_e^z)\right) \\ &= c_P(f^x) - (M + c_P(f^z)) = b_P(f^x) - b_P(f^z) \end{aligned}$$

Case 4: (f^x) feasible, (f^z) feasible:

$$\begin{aligned} \Pi(f^x) - \Pi(f^z) &= \sum_{e\in E} c_e(f_e^x) - \sum_{e\in E} c_e(f_e^z) \\ &= c_P(f^x) - c_P(f^z) = b_P(f^x) - b_P(f^z) \end{aligned}$$

Resulting from these four cases we conclude that

$$b_P(f^x) - b_P(f^z) = \Pi(f^x) - \Pi(f^z) \ ,$$

and thus, $\Pi(f)$ is an exact potential function. □

6 Computation of Equilibria by Improvement Sequences

One approach to obtain an equilibrium, is to solve the optimization problem given in Theorem 3.11. The obvious drawback is that in general we will not find all equilibria by this computation. In this section, another approach is presented. Dependent on the initial flows, the algorithm is in principle able to deliver all equilibria in a path player game. A proper choice of the set of initial flows is in fact an open question and an analysis of the attraction region of the equilibria profiles is a topic of future research.
In Section 2 we have introduced sequences and the finite improvement property (FIP). A finite improvement sequence terminates with an equilibrium, which is a motivation to study this approach. Due to their infinite number of strategies, path player games doesn't satisfy FIP and we have to look for alternatives. We will investigate two different approaches in this section. The first one will use *best-reply improvement sequences* which are proposed in [12]. We will show in this section that in path player games, best-reply improvement sequences are in general not finite. Nevertheless, we will present classes of path player games, where best-reply sequences are finite and thus end with an equilibrium. The second approach will analyze ϵ-improvement sequences, which yield *approximate equilibria*.
First, we consider the finite best-reply property, which is a greedy approach: at each step the active player chooses a best reply as a new strategy, such that the improvement of the active player's benefit is maximized.
If in each step the active player shifts to a best reply strategy with respect to the strategies of his competitors, i.e. if $f^k_{P(k)} \in f^{max}_{P(k)}\left(f^{k-1}_{-P(k)}\right)$ does hold for all $k = 1, \ldots, N$, the sequence is called a *best-reply sequence*. A best-reply sequence is called a *best-reply improvement sequence* if a player will only get active if he can obtain a strict improvement, i.e. if $f^{k-1}_{P(k)} \notin f^{max}_{P(k)}\left(f^{k-1}_{-P(k)}\right)$ does hold for all $k = 1, \ldots, N$. (Recall that f^{max} was introduced in (2).) A game in which every best-reply improvement sequence is finite, satisfies the *finite best-reply property (FBRP)*. The finite improvement property implies the finite best-reply property but the reverse is not true. Unfortunately, also best-reply improvement sequences may be infinite in path player games, FBRP is not satisfied in general. This behavior is illustrated by the following example.

Example 6.1. *Consider a path player game with two players and a flow rate $r = 1$. Each of the two paths consists of two edges. One is owned exclusively by the player of this path, the other one is shared with the second player (see Figure 3).*
Let f_1, f_2 be the flows of player 1 and 2. The cost of the exclusively used edges are given by $c_{e_1}(f_1) = -f_1^2$ for player 1 and $c_{e_2}(f_2) = -f_2^2$ for player 2. The cost of the commonly used edge is $c_{e_3}(f) = -(f_1 + f_2 - 1)^2$. Thus, we get the cost of the paths as $c_1(f) = -f_1^2 - (f_1 + f_2 - 1)^2$ and $c_2(f) = -f_2^2 - (f_1 + f_2 - 1)^2$.
Given a fixed f_2, the first player will choose $f_1 = 1/2 - f_2/2$ as best reply, while player 2 will choose $f_2 = 1/2 - f_1/2$ for a given f_1. This best reply mapping has a fixed point at $f^* = (1/3, 1/3)$ which is also the unique equilibrium of the game. See Figure 4 for an illustration of the set of feasible solutions and the best reply strategies. The equilibrium

f^* can only be reached by the best-reply mapping if the starting point f^0 has either $f_1^0 = 1/3$ or $f_2^0 = 1/3$ or both. Any other start point will create an infinite best-reply sequence, see e.g. Figure 5.

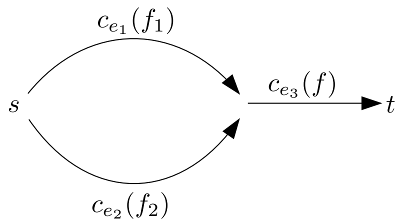

Figure 3: Game network

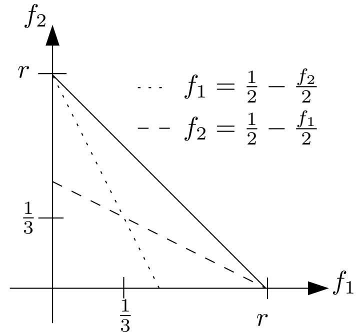

Figure 4: Best-reply strategies

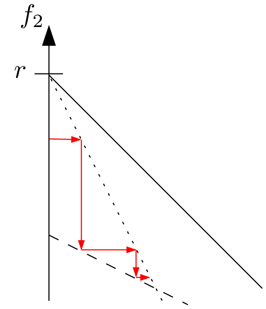

Figure 5: Best-reply improvement sequence

Nevertheless, we can present a class of path player games that satisfy FBRP.

Theorem 6.1. *Consider a path player game where for all players $P \in \mathcal{P}$ and for all $f_{-P} \in \mathbb{R}_+^{|\mathcal{P}|-1}$ the best reaction sets satisfy*

$$f_P^{max} \subseteq \{0, d_P\} \ . \tag{26}$$

This game satisfies FBRP.

Proof. Consider a path player game with n players $P = P_1, \ldots, P_n$ and a best-reply improvement sequence $\varphi = (f^0, f^1, \ldots, f^k, \ldots)$. Let $d^k = r - \sum_{P \in \mathcal{P}} f_P^k$ be the *free flow rate at step k.*
We introduce a fictitious player $P = P_{n+1}$ who is "sending" the free flow rate as the own strategy, i.e. $f_{P_{n+1}}^k = d^k$ holds for all $k \geq 0$.
For each player $P = P_1, \ldots, P_n$ and each step $k \geq 0$, we define the set $A_P^k \subseteq \{P_1, \ldots, P_{n+1}\}$ that is a subset of the set of players (including the fictitious one). In addition, for each $k > 0$, the sets $A_{P_1}^k, \ldots, A_{P_{n+1}}^k$ have to be a partition of the set $\{P_1, \ldots, P_{n+1}\}$, i.e.

$$\bigcup_{P = P_1, \ldots, P_{n+1}} A_P^k = \{P_1, \ldots, P_{n+1}\} \ , \tag{27}$$

$$A_P^k \cap A_{P'}^k = \emptyset \quad \forall\ P \neq P',\ P, P' \in \{P_1, \ldots, P_{n+1}\} \ . \tag{28}$$

has to hold for all $k > 0$.
In the following, we use these sets to show that in each step k, each initial flow f_P^0 is contributing to the flow of exactly one player.

Claim: For each $k > 0$ there is a partition of $A_P^k, P = P_1, \dots, P_{n+1}$ (satisfying (27) and (28)), such that

$$f_P^k = \sum_{P_j \in A_P^k} f_{P_j}^0$$

holds for all $P = P_1, \dots, P_{n+1}$.

We prove the claim by induction:

<u>$k = 0$:</u>
Set $A_P^0 = \{P\} \ \forall \ P = P_1, \dots, P_{n+1}$. It follows that

$$\sum_{P_j \in A_P^0} f_{P_j}^0 = \sum_{P_j \in \{P\}} f_{P_j}^0 = f_P^0 \ .$$

<u>$(k-1) \to k$:</u>
Assume the claim is true for step $k-1$. Consider the active player $P(k)$, which is changing his flow from $f_{P(k)}^{k-1}$ to $f_{P(k)}^k$. By (26), one of the two following cases has to be satisfied:

$$(i) \ \ f_{P(k)}^k = 0 \ ,$$

$$(ii) \ \ f_{P(k)}^k = f_{P(k)}^{k-1} + d^{k-1} = f_{P(k)}^{k-1} + f_{P_{n+1}}^{k-1} \ .$$

In the second case, we use that the fictitious player is representing the free flow rate. Automatically, his flow $f_{P_{n+1}}^k$ is also changed by the move of the active player. According to the cases (i) and (ii) we have:

$$(i) \ \ f_{P_{n+1}}^k = f_{P_{n+1}}^{k-1} + f_{P(k)}^{k-1} \ ,$$

$$(ii) \ \ f_{P_{n+1}}^k = 0 \ .$$

Furthermore, $f_P^k = f_P^{k-1}$ holds for all $P \notin \{P(k), P_{n+1}\}$.
For case (i), set $A_{P(k)}^k = \emptyset$, $A_{P_{n+1}}^k = A_{P_{n+1}}^{k-1} \cup A_{P(k)}^{k-1}$, and $A_P^k = A_P^{k-1} \ \forall \ P \notin \{P(k), P_{n+1}\}$. This is a partition, as (27) and (28) are satisfied. Furthermore, we have

$$\sum_{P_j \in A_{P(k)}^k} f_{P_j}^0 = \sum_{P_j \in \emptyset} f_{P_j}^0 = 0 \ ,$$

$$\sum_{P_j \in A_{P_{n+1}}^k} f_{P_j}^0 = \sum_{P_j \in A_{P_{n+1}}^{k-1}} f_{P_j}^0 + \sum_{P_j \in A_{P(k)}^{k-1}} f_{P_j}^0 = f_{P_{n+1}}^{k-1} + f_{P(k)}^{k-1} \ , \tag{29}$$

and for all $P \notin \{P(k), P_{n+1}\}$:

$$\sum_{P_j \in A_P^k} f_{P_j}^0 = \sum_{P_j \in A_P^{k-1}} f_{P_j}^0 = f_P^{k-1} = f_P^k \ .$$

Note that in (29) we use that $A_{P_{n+1}}^{k-1}$ and $A_{P(k)}^{k-1}$ are disjoint, since by induction hypothesis (28) is satisfied. For case (ii), set $A_{P(k)}^{k} = A_{P(k)}^{k-1} \cup A_{P_{n+1}}^{k-1}$, $A_{P_{n+1}}^{k} = \emptyset$, and $A_P^k = A_P^{k-1}\ \forall\ P \notin \{P(k), P_{n+1}\}$. This is a partition, as (27) and (28) are satisfied. Furthermore, we have

$$\sum_{P_j \in A_{P(k)}^k} f_{P_j}^0 = \sum_{P_j \in A_{P(k)}^{k-1}} f_{P_j}^0 + \sum_{P_j \in A_{P_{n+1}}^{k-1}} f_{P_j}^0 = f_{P(k)}^{k-1} + f_{P_{n+1}}^{k-1} \ , \tag{30}$$

$$\sum_{P_j \in A_{P_{n+1}}^k} f_{P_j}^0 = \sum_{P_j \in \emptyset} = 0 \ ,$$

and for all $P \notin \{P(k), P_{n+1}\}$:

$$\sum_{P_j \in A_P^k} f_{P_j}^0 = \sum_{P_j \in A_P^{k-1}} f_{P_j}^0 = f_P^{k-1} = f_P^k \ .$$

In (30) it is used that $A_{P_{n+1}}^{k-1}$ and $A_{P(k)}^{k-1}$ are disjoint, since by induction hypothesis (28) is satisfied. This last case finishes the proof of the claim.

There is a finite number of possibilities to partition $\{P_1, \ldots, P_{n+1}\}$ into sets A_P^k. Thus, the number of different flows f^k that can be obtained by a best-reply improvement sequence is finite. Furthermore, in a best-reply improvement sequence, no flow f is visited twice. This is true as the path player game is an ordinal/exact potential game and thus, for a best-reply improvement sequence, a strict improvement of the ordinal/exact potential function $\Pi(f)$ is required in each step. Since by this argumentation, a cycle is not existing in any best-reply improvement path, and on the other hand the number of different flows f^k is finite, each best-reply improvement path is finite. □

¿From the previous theorem the next corollary follows immediately, as the considered cost functions attain their maximum on the boundary of a compact interval.

Corollary 6.2. *The following classes of path player games satisfy FBRP:*

- *PPG with cost functions $c_e(f_e)$ that are linear in f_P*
- *PPG with cost functions $c_e(f_e)$ that are strictly increasing in f_P*
- *PPG with cost functions $c_e(f_e)$ that are strictly convex in f_e*

Previous results allow us to use best-reply improvement sequences to determine equilibria for the presented classes of path player games in a finite number of steps. For general path player games, Example 6.1 has illustrated that best-reply improvement sequences may be infinite. Therefore, we need to consider different ways for the determination of equilibria.

In the next approach we determine *approximate equilibria*, which are described in [14].

For $\epsilon > 0$, a sequence $\varphi = (f^0, \ldots, f^k, \ldots)$ is an *ϵ-improvement sequence*, if for all $k \geq 1$ it holds that $b_{P(k)}(f_k) > b_{P(k)}(f_{k-1}) + \epsilon$. A game Γ in which for all $\epsilon > 0$ every ϵ-improvement sequences is finite, satisfies *approximate finite improvement property (AFIP)*.

Theorem 6.3. *The path player game satisfies AFIP.*

Proof. Consider an ϵ-improvement sequence φ. First assume the initial flow f^0 of φ is infeasible. Then, either no player is able to improve his benefit, that means f^0 is an equilibrium and φ terminates with $f^N = f^0$ and $l(\varphi) = 0$.
Otherwise, there is at least one player which is able to improve the own benefit by creating a feasible flow within one step. Then f^1 of φ is feasible.
For an improvement sequence it holds that if f^k is feasible, the subsequent flow f^{k+1} is feasible, too. Thus, it is sufficient for the rest of the proof to consider the set of feasible flows $\mathbb{F}$ since ϵ-improvement sequences with $l(\varphi) > 0$ will either start in $\mathbb{F}$ or jump into $\mathbb{F}$ in the first step.
The benefit functions in path player games are bounded for $f \in \mathbb{F}$, as $b_P(f) = c_P(f)$ is continuous over the closed set $\mathbb{F}$. Thus, the exact restricted potential function $\Pi(f)$, which is existing according to Theorem 3.6, is bounded, as well. As an ϵ-improvement sequence increases the exact restricted potential function values by at least an $\epsilon > 0$ in each step, each ϵ-improvement sequence has to be finite. □

Note that AFIP is also satisfied for any exact potential game with bounded benefit functions, which is a result of [14]. A maximal finite ϵ-improvement sequence yields an *ϵ-equilibrium* f^ϵ as a terminal flow, i.e. a solution where $b_P(f^\epsilon_{-P}, f^\epsilon_P) \geq b_P(f^\epsilon_{-P}, f_P) - \epsilon$ holds for all $P \in \mathcal{P}$ and for all $f_P > 0$ and $\epsilon > 0$. Note that ϵ-equilibria may be infeasible. An infeasible ϵ-equilibrium can be only obtained as terminal flow of an ϵ-improvement sequence, with an infeasible equilibrium as initial flow (see [18] for a characterization of infeasible equilibria in path player games). In this case, the initial and the terminal flow coincide, the sequence has length one. We will neglect this degenerated case and concentrate on feasible ϵ-equilibria.
If we increase the precision for an ϵ-improvement sequence, by decreasing ϵ, we obtain a sequence of feasible ϵ-equilibria. By the following lemma, accumulation points of that sequence are equilibria, such that ϵ-improvement sequences can be used for the computation of equilibria with a given precision.

Lemma 6.4. *Consider a path player game and a sequence of feasible ϵ-equilibria $f^*(\epsilon)$ that is given for $\epsilon \to 0$. Any accumulation point f^* is a feasible equilibrium of the path player game.*

Proof. First, note that feasible ϵ-equilibria exist for each $\epsilon > 0$ as path player games satisfy AFIP. Furthermore, by feasibility of ϵ-equilibria, the sequence $\{f^*(\epsilon)\}_{\epsilon \to 0}$ is bounded. Thus, by the Theorem of Bolzano-Weierstrass, it has an accumulation point, i.e. we can find a subsequence $f^*(\bar{\epsilon}^k)$ that is converging to f^* for $k \to \infty$. (Furthermore, $\bar{\epsilon}^k \overset{k\to\infty}{\longrightarrow} 0$ holds for each subsequence of the original sequence.)
By the definition of ϵ-equilibria one gets for each $\epsilon > 0$ that

$$b_P(f^*_{-P}(\epsilon), f^*_P(\epsilon)) - b_P(f^*_{-P}(\epsilon), f_P) \geq \epsilon \quad \forall P \in \mathcal{P}, \forall f_P \geq 0 \ . \tag{31}$$

Now consider the limit of (31):

$$\lim_{\epsilon\to 0} b_P(f^*_{-P}(\epsilon), f^*_P(\epsilon)) - \lim_{\epsilon\to 0} b_P(f^*_{-P}(\epsilon), f_P) \geq \lim_{\epsilon\to 0} \epsilon \quad \forall P \in \mathcal{P}, \forall f_P \geq 0 \ .$$

As we assume to have continuous functions $c_e(f_e)$, the benefit $b_P(f)$ is continuous for any feasible f and we rewrite:

$$\begin{aligned} & b_P\left(\lim_{\epsilon\to 0} f^*_{-P}(\epsilon), \lim_{\epsilon\to 0} f^*_P(\epsilon)\right) - b_P\left(\lim_{\epsilon\to 0} f^*_{-P}(\epsilon), f_P\right) \geq 0 \quad \forall P \in \mathcal{P}, \forall f_P \geq 0 \\ \Leftrightarrow \quad & b_P\left(f^*_{-P}, f^*_P\right) - b_P\left(f^*_{-P}, f_P\right) \geq 0 \quad \forall P \in \mathcal{P}, \forall f_P \geq 0 \ , \end{aligned}$$

i.e. f^* is a feasible equilibrium. □

The following theorem is obtained directly from the previous lemma.

Theorem 6.5. *In path player games with continuous cost functions $c_e(f_e)$, equilibria exist.*

The above argument provides an alternative proof of the existence of pure strategy equilibria in path player games, motivated by the existence of potential functions for that kind of games. The first existence proof, which can be found in [18], uses a fixed point argument. Note that Monderer and Shapley have provided in [14] a sufficient condition for the existence of equilibria in infinite potential games. However, as path player games have non-continuous benefit functions and moreover, mutually dependent strategies sets of players, they do not satisfy those requirements. Hence, all the results in this sections are completely new.

7 Summary

In this paper, the existence of potential functions for path player games is proved. In particular, we found three different types of potential functions, which are used to develop methods for the computation of equilibria in path player games. First, the maximizers of potential functions are equilibria. We use this fact to generate an optimization problem where optimal solutions are equilibria. Second, improvement sequences that yield equilibria are analyzed. Finally, an alternative proof for the existence of pure-strategy equilibria is derived from the existence of potential functions.
In terms of future research in this field, it will be interesting to consider improvement sequences and the corresponding attraction regions of equilibria. That means to identify sets of starting points for which improvement sequences will end in one equilibrium. Furthermore, a generalization of path player games, the *games on polyhedra*, see [23], is currently considered under the aspect of potential functions. It turns out that potential functions can be found for special instances of such games. This is used to prove the existence of equilibria, which is not given in general for games on polyhedra. Moreover, the already mentioned line planning game, which is a potential game itself, is developed further and in particular tested with real-world data within the European project ARRIVAL [1].

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