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The Path Player Game: Introduction and Equilibria

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Abstract

We introduce the path player game, a noncooperative network game with a continuum of mutually dependent set of strategies. This game models network flows from the point of view of competing network operators. The players are represented by paths in the network. They have to decide how much flow shall be routed along their paths. The competitive nature of the game is due to the following two aspects: First, a capacity bound on the overall network flow links the decisions of the players. This network capacity can be considered as a feasibility constraint, which leads to mutually dependent strategy sets of the players. In this sense, the path player game is generalized Nash equilibrium game. Second, edges may be shared by several players which might have conflicting goals.
In this paper, we restrict ourselves to study the existence of equilibria in path player games, which is a non-trivial task due to non-continuity of payoff functions and the infinite, mutually dependent strategy sets. We prove the existence of equilibria in pure strategies by using a fixed point method. Moreover this approach takes into account the dependencies of the players' strategy sets such that we are even able to state the existence of an equilibrium that satisfies the capacity bound. Furthermore, we analyze different instances of path player games in more detail and present characterizations of equilibria for these cases.

1 Introduction

Various types of games on networks have been studied in recent years. For instance in *routing games* [24, 5], flow has to be transported from origin to destination nodes. In *load balancing games* [20, 6], load is to be assigned to resources. In *facility location games* and *service provider games* [29, 7], facilities have to be located and assigned to the demand points. *Network design games* [9, 2] and *coordination games* [14, 17, 4], describe the generation of networks. In [13, 15], properties of *social networks* are studied.

In this work we study a new type of routing game. Usually, in routing games the problem of sending flow in a network is considered from the point of view of the flow itself, assuming that the flow can choose a path from origin to destination. Another interesting aspect, which has not been considered yet, is the behavior of the path owners, when they are allowed to choose the amount of flow that will be sent along them. This new approach models systems where paths are owned by decision makers, like in public transportation, energy or information networks. The decision makers offer a certain bandwidth to be used by the flow, like a bandwidth of electricity, or a certain daily frequency of trains. Equilibria in this model describe a stable market situation among competing path owners. Thus, the existence and characterization of equilibria is an important research topic and will be investigated in this paper.

A rough description of path player game is given as follows: Consider a network G and a set of players $\mathcal{P}$, one for each path $P \in \mathcal{P}$ where the flow is defined as $f : \mathcal{P} \to \mathbb{R}^+$. Each player's strategy is to choose a nonnegative flow f_P. An upper bound to the flow is given by the flow rate r, which shall not be exceed by the summarized flow in the network. The flow rate is motivated by the limited capacity of the network resources and a violation is penalized in any case. In practice, this bound may arise from society regulations like a limitation of traffic for ecological or security reasons. Furthermore, each single player has a lower bound, the security limit ω_P. This aspect plays a central role in the application to the line planning problem, a question of transport optimization. Both bounds may be violated, but with consequences for the benefit (or payoff). An equilibrium in this game is given, when each player is playing a benefit-maximizing strategy.

More detailed, the benefit is given by three parts: First of all, cost functions $c_e(x)$ are assigned to the edges e that are dependent on the flow x sent on that edge. If the bounds r and ω_P are satisfied, the income of a player is derived by the sum of costs over the edges that belong to the path. The other two parts of the payoff come into operation if the bounds are violated. Then a constant benefit is payed. Violation of the flow rate r (which causes an infeasible flow) results in a negative benefit, a penalty. Violation of the security limit ω_P is not necessarily penalized.

Note that the benefit is not necessarily strictly increasing. So a player is not necessarily interested in routing as much flow as possible. Handling too much flow could mean increasing operating costs, for instance due to over-hours or additional maintenance. The competitive aspect of the game is given on the one hand by the flow rate that has to be satisfied by the network flow. Another interesting component of the game is that the paths may own edges shared with other paths, as the flow on an edge depends on the flow on the paths which are using the edge. This can have positive or negative effects for the players. It may happen that players are forced into situations where too much flow on a certain edge (created by competing paths) decreases their benefit, or the other way around, that the flow provided by other paths increases their income. Thus, the players sharing an edge may have different objectives regarding that edge, which leads to competitive situations.

In path player games we deal with a non-continuous benefit caused by the special treatment of flow violating bounds r and ω_P. If we consider just feasible flows, the restriction by the flow rate implies mutual dependencies of the players' strategy sets as the maximal amount of flow a player may choose is bounded by the strategies chosen by the other players. Games that take into account dependencies of strategy sets are called generalized Nash equilibrium (GNE) games, see [16, 10]. An often studied question in GNE games is the existence of equilibria, as it is in general not given. Path player games are instances of GNE games with a simple structure, that allows to state the existence of equilibria and moreover, provides several methods for computation of equilibria. In [22] the concept of path player games is extended to games on polyhedra. These games first allow a more general description of the dependencies among strategy sets and second are not necessarily based on a network structure anymore. See the reference for a study of computation of equilibria in these cases.

In path player games, the penalty for violating flow rate r is a simple but efficient way to avoid dependencies among strategy sets, i.e. fixed strategy sets are obtained. For further analysis, it is worthwhile to consider both models, the approach using infeasibility penalty and fixed strategy sets as well as the GNE path player game with dependent strategy sets. As the first model is allowing infeasible game situations and thus is not bound to dependent strategy sets, it is more general and easier to handle. On the other hand, without the infeasibility penalty, the second approach provides a simpler (continuous) payoff. This is of advantage for instance in the analysis of a potential function (see [23]), which turns out to exist for the path player game in both versions.

Moreover, it turns out that path player games may have multiple equilibria. This motivates analysis of the relation of equilibria and non-dominated solutions in the sense of Pareto: A flow is dominated by a second flow if in the second flow, none of the players has to accept less benefit while there is at least one player that receives a higher benefit. This question is considered in [27]. It is interesting that in path player games, in fact all relations between the set of non-dominated solutions and equilibria are possible. In particular, in the paper mentioned, an example similar to the Prisoner's Dilemma is presented, where each equilibrium is dominated and each non-dominated solution is non-stable. But also classes of games with nice behavior are described, where the set of equilibria and the set of non-dominated solution are equal, or at least, each non-dominated solution is an equilibrium.

As the path player game models situations in which several providers of a commodity are sharing a network, its applications can be found for instance in public transport, telecommunication or information networks. Currently, an application to the line planning problem is under consideration, which is the following: To create a public transportation network, like a railway or bus network, lines have to be installed. In particular, the lines are given by their stops, for instance a railway line may go from Hamburg to Basel with stops in Hannover, Frankfurt and Karlsruhe. Assigned to each line is the frequency, i.e. the number of times a line travels within a given time horizon, for instance twice a day. The line plan has to respect some constraints. First, there is a given customers demand that has to be transported from several origins to destinations and has to be satisfied by the frequencies. On the other hand, upper bounds on the edges are given to limit the frequencies on the edges. These bounds are usually given for security reasons, e.g. it is a rule that only one train is allowed to be on a block at a certain time. Summarizing, the question of line planning is: Which lines and what frequencies shall be installed such that the constraints are satisfied? In terms of objectives there are several approaches, for instance cost- or customer-oriented ones, see [25]. For the line planning game, we minimize the average delay of each line.

Based on the path player game, the line planning game is developed and is currently under consideration in the framework of the European project *ARRIVAL* [1]. In this concept, the lines are players that want to minimize their delay with respect to given capacity constraints and customers' demands. This model is a building block for the larger process of setting up traffic networks for real-world problems like it is a goal of *ARRIVAL*. It turns out that efficient methods for computing equilibria in line planning games can be developed. See [26] for first results and numerical studies using data from interregional trains in Germany. Figure 1 illustrates a line concept resulting from those studies.

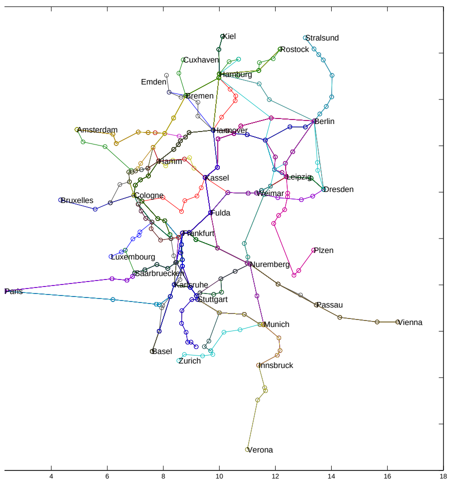

Figure 1: Line concept: Interregional trains in Germany

Concerning existing literature, we describe in the following three types of network games that are related to path player games: We have already mentioned routing games, where the network flow is analyzed from the point of view of the flow. A routing game is played like the path player game on a congested network, and the flow is assumed to consist of a finite or infinite number of players. These players act independently and selfishly. Each of them chooses a path from the source to the destination that minimizes the cost of traveling along that path. This model can be seen as a counterpart to the path player game, as it represents the point of view of the travelers, i.e. of the flow. On the contrary, in the path player game, we analyze the situation from the path owner's point of view. Here we want to maximize the income one gets from the flow using a path. Note that the cost functions in routing games are sometimes interpreted as latencies. Thus, an increasing latency shall prevent that too much flow is going over the edge, in some sense it acts as a substitute for the missing capacity constraint. Our interpretation is different, as we assume that the cost paid by the flow is meant as income for the path owners.

The strategies of the players in a path player game can be taken as offering bandwidth to the flow. In fact, our model is related to *bandwidth allocation games*, as described in [19, 18]. In bandwidth allocation games capacitated links are used by several players. The players send bids to a central manager and the manager answers with prices that are proportional to the bids. Moreover, he cares for satisfying the capacity constraints. Each user has its own utility function that determines his payoff depending on the price and the bid. For these type of games it is distinguished between *price taking users* and *price anticipating users*. The latter ones take into account the reasoning of the manager and adjust their bids, while the first ones just accept the price. Only the second approach represents a game. On the contrary to this model, our model considers no capacities on the edges, although the flow rate r is corresponding to a capacity in a single edge bandwidth allocation game. Also the "bids" in the path player game, that means the strategies, are not answered by a manager, but are directly accepted. So, the path player game is a simpler approach, which enables us to get further results. In the path player game we allow general continuous and nonnegative cost functions, while in bandwidth allocation strictly increasing, continously differentiable and concave functions (so called *elastic traffic*) are required. Furthermore, in bandwidth allocation the existence of equilibria can not be guaranteed. In the path player game we are able to prove the existence for continuous cost functions.

Another model describing the behavior of path owners is that of *path auctions*, see e.g. [8, 3]. Here each edge is owned by a player, and a central manager has the task to buy a shortest path from s to t from the edge owners. The edge owners know the price of their edge, but they are allowed to report a wrong price if they benefit from lying. The question is to develop a payment mechanism such that every edge owner is interested to tell the truth. Such a mechanism is called *truth telling*. This model is in a sense related to ours, assume our network consists of parallel edges from s to t, our path owners would be edge owners as well. Nevertheless, in the path player game we are in a previous step analyzing the game aspect and as a consequence we are able to obtain further results.

In this paper, we prove the existence of pure-strategy equilibria in path player games using a fixed point approach. As we have non-continuous payoffs and a continuum of strategies, we can not use the results existing in literature. Furthermore, the proof takes into account the dependencies among strategy sets such that even the existence of feasible equilibria should be shown. Afterwards, we discuss equilibria for several instances of path player games. In particular, if we assume strictly increasing cost functions on all edges, we obtain a necessary condition for equilibria. This result can be strengthened, if we assume in addition non-compensative security (NCS) property. If this property is given for a game, it assures that a player takes advantage of the security payment only if he is forced to. Consequently, NCS property is given if we have no security payment in the game. For games with strictly increasing costs and NCS property, we derive even a necessary and sufficient condition.

The paper is organized as follows. In Section 2 we introduce the game model. Afterwards, in Section 3 we define equilibria in this class of games and show that feasible equilibria in pure strategies exists. In Section 4 further properties of path player games necessary, later on, for describing equilibria, are discussed. In Section 5 we analyze the equilibria for strictly increasing cost functions and give necessary and necessary and sufficient conditions for a profile of flows to be in equilibrium. The paper ends with some suggestions for future work.

2 The Model

We consider a given network $G = (V, E)$ with vertices $v \in V$ and edges $e \in E$. A path P in G is given by a sequence of edges $e \in E$: $P = (e_1, ..., e_k)$. By $\mathcal{P}$ we denote the set of all paths P in G from the single source s to the single sink t, thus the set $\mathcal{P}$ is given by the structure of the network G. Each of the paths $P \in \mathcal{P}$ represents a player[1] in the path player game. Each player proposes an amount of flow f_P that he wants to be routed along his path. The complete flow is represented by a function $f : \mathcal{P} \to \mathbb{R}^+$, while the flow on path P is denoted by f_P. For each edge $e \in E$, the flow f_e along the edge can hence be determined by the sum of the flows on paths that contain e, i.e.

$$f_e = \sum_{P: e \in P} f_P \ .$$

We assume that the demand is high enough to ensure that the players can implement the flow they proposed. Note that this is a considerable difference to bidding games, like bandwidth allocation or path auction games.

Each edge e is associated with a cost function $c_e(\cdot)$, that depends on the flow on e. The cost function represents the income of the edge owners and we assume these

[1] In the course of this paper we will denote both, the path and the corresponding player with P, as both notations are handled equivalently.

functions to be continuous and nonnegative for nonnegative flows, i.e. $c_e(x) \geq 0$ for $x \geq 0$. If the edge belongs to more than one owner, we assume that each player receives the same income c_e. (Note that it is possible to generalize this model by allowing the owners to share the fee in an arbitrary way.)

To calculate the cost on a path P, we sum up the costs of the edges belonging to that path, i.e.,

$$c_P(f) = \sum_{e \in P} c_e(f_e).$$

These costs are, however, not directly the benefit of player P since there are two more issues to handle:

- We require that the sum of flows in the network is bounded by a given *flow rate* r. It can be interpreted as a network capacity. We call a flow f *feasible* for a flow rate r if $\sum_{P \in \mathcal{P}} f_P \leq r$ holds. If the flow rate is exceeded, the flow is called *infeasible* and all players receive a penalty of $-M$, with M being a large number.

  Note that a feasible flow need not necessarily use the complete flow rate. This makes sense in an economic context, where the resource providers only satisfy the complete demand if this maximizes their income, but not if it is too costly for instance due to overtime or additional maintenance of the resources.

- Furthermore, in the game a security system for the players is implemented: If the flow of a player P lies below the so called *security limit* $\omega_P \geq 0$, he will receive a fixed *security payment* $\kappa_P > -M$. In this case, the path P is called *underloaded*, while we call P *loaded*, if $f_P > \omega_P$. For positive κ_P, the security limit and payment serve as an insurance that guarantees a fixed income for each player. On the other hand, if $\kappa_P < 0$, the security payment is a penalty for underloaded paths. This penalty may represent for instance additional costs for maintaining an unused resource.

Summarizing, we obtain the *benefit function* in the path player game:

Definition 2.1. *The* benefit function *of player $P \in \mathcal{P}$ in a path player game for $f \geq \vec{0}_{|\mathcal{P}|}$ is given as:*

$$b_P(f) = \begin{cases} c_P(f) & if \quad \sum_{P \in \mathcal{P}} f_P \leq r \quad \wedge \quad f_P \geq \omega_P \\ \kappa_P & if \quad \sum_{P \in \mathcal{P}} f_P \leq r \quad \wedge \quad f_P < \omega_P \\ -M & if \quad \sum_{P \in \mathcal{P}} f_P > r \end{cases},$$

where $c_P(f) = \sum_{e \in P} c_e(f_e)$.

Some remarks about path player games should be added.

- There is a continuum of strategies as a player is allowed to choose any non-negative real number. The benefit (or payoff) a player obtains after fixing a strategy depends on the strategies of all players.

- The path player game is noncooperative and thus it is possible that the flow created by the decisions of the players is not feasible. For instance if the benefit is a nondecreasing function, each player will try to get as much flow as possible such that the sum of all proposed flows may exceed the flow rate.

There are two approaches for dealing with the latter issue. First, path player games can be considered as a generalized Nash equilibrium game ([16, 10]). That means, only feasible flows with $\sum_{P \in \mathcal{P}} f_P \leq r$ are considered leading to mutual dependent strategy sets of the players. This approach is studied in detail in [22], where a more general setting, namely *games on polyhedra* are considered. Path player games are instances of games on polyhedra. Furthermore, in [23] generalized path player games are studied and potential functions for such classes of games are proven to exist.

In this paper we follow a second approach to handle infeasible flows. Namely, infeasible flows are allowed, but punished with a negative payoff in the benefit function. This approach has one drawback: It turns out that also an infeasible flow may be an equilibrium situation. Nevertheless, we will prove in Theorem 3.4 that feasible equilibria do exist so that we can draw our attention to feasible equilibria.

3 Equilibria for General Benefit Functions

In this section we analyze equilibria in path player games for general benefit functions while, later in Section 5, we derive additional results for strictly increasing cost functions. The definition of equilibria in path player games follows the definition of a *Nash equilibrium* (see e.g. [21]): A flow f^* is an *equilibrium* in a given path player game if and only if for all players $P \in \mathcal{P}$ and for all $f_P \geq 0$ it holds that

$$b_P(f^*_{-P}, f^*_P) \geq b_P(f^*_{-P}, f_P) \quad . \tag{1}$$

We will call the equilibrium *feasible* if f^* is a feasible flow, *infeasible* otherwise. An equilibrium is a game situation where none of the players is able to obtain a better outcome by changing his strategy. Such a situation characterizes a stable state of the system.

In order to find equilibria in the path player game we have to look at the benefit of a single player who changes his own strategy, while the strategies of the competitors remain fixed. We define $f_{-P} \in \mathbb{R}_+^{|\mathcal{P}|-1}$ by deleting the component belonging to path P, such that we can fix the strategies f_{-P} of the competitors and just consider the influence of f_P. The *one-dimensional cost function* assigned to a player $P \in \mathcal{P}$ for a given flow f_{-P} is denoted by

$$\tilde{c}_P(f_P) = c_P(f_{-P}, f_P) = \sum_{e \in P} c_e \left(f_P + \sum_{P_k \in \mathcal{P} \setminus \{P\} : e \in P_k} f_{P_k} \right) \quad .$$

Note that the term $\sum_{P_k: e \in P_k \wedge P_k \neq P} f_{P_k}$ is constant. It is clear that if $c_e(f_e)$ are convex (concave) functions, then $\tilde{c}_P(f_P)$ is also a convex (concave) function. We need two other notations:

- The *one-dimensional benefit* for a player P and a flow $f_P \geq 0$ with respect to the given flow f_{-P} is denoted by

$$\tilde{b}_P(f_P) = b_P(f_{-P}, f_P) \ .$$

- The *decision limit* of player P for a given flow f_{-P} is

$$d_P = r - \sum_{P_k \in \mathcal{P} \setminus \{P\}} f_{P_k} \ .$$

  The set $[0, d_P]$ is called the *decision interval* of player P, it contains all feasible strategies for P.

From the definition of the decision limit we obtain the following corollary. It says that if there is one player sending as much flow as possible (without violating the decision limit), then this is true for all players.

Corollary 3.1. *Any flow f satisfies: If there is a player P_k with $f_{P_k} = d_{P_k}$ then all players $P \in \mathcal{P}$ satisfy $f_P = d_P$.*

Using the one-dimensional cost function $\tilde{c}_P(f_P)$ defined before and the decision limit d_P, we are able to describe the one-dimensional benefit function in more detail.

$$\tilde{b}_P(f_P) = \begin{cases} \tilde{c}_P(f_P) & if \quad f_P \leq d_P \ \wedge \ f_P \geq \omega_P \\ \kappa_P & if \quad f_P \leq d_P \ \wedge \ f_P < \omega_P \\ -M & if \quad f_P > d_P \end{cases} \ .$$

Figure 2 illustrates an example of a one-dimensional benefit function. The function $\tilde{b}_P(f_P)$ is characterized by three parts: The two constant regions generated by the security payment κ_P, the infeasibility penalty $-M$, and the middle part, created by the cost function $\tilde{c}_P$. As the players want to maximize their benefit, we define the *best reaction set* for a player P with respect to a given flow f_{-P} as

$$f_P^{max} = \{f_P \geq 0 : f_P \text{ maximizes } \tilde{b}_P(f_P)\} \ .$$

These sets are useful for a first characterization of equilibria. First, we need the following result.

Lemma 3.2. *Consider a path player game with cost functions $c_e(f_e)$ being continuous for all edges $e \in E$. Then, the sets f_P^{max} are nonempty for all $P \in \mathcal{P}$.*

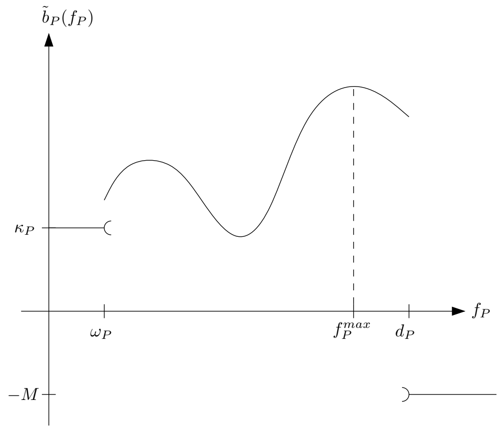

Figure 2: one-dimensional benefit function

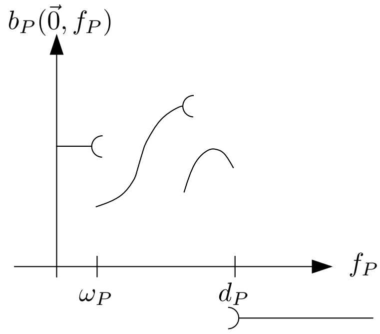

Figure 3: f_P^{max} is empty

Proof. Consider the intervals $I_1 = [0, \omega_P)$, $I_2 = [\omega_P, d_P]$ and $I_3 = (d_P, \infty)$. Since $\tilde{b}_P(f_P)$ is constant on I_1 and I_3, maxima exist for these intervals. The existence of a maximum on I_2 is confirmed by Weierstrass extreme value theorem since $\tilde{b}_P(f_P)$ is continuous on I_2 and I_2 is compact. The maximum of these three single maxima hence is the overall maximum. □

Figure 3 shows that in the case of non-continuous cost functions a benefit maximizing flow need not exits. We will use the following reformulation of equilibria in terms of the best reaction sets.

Corollary 3.3. *In a path player game a flow f^* is an equilibrium if and only if for all $P \in \mathcal{P}$ with respect to f^*_{-P} it is satisfied that*

$$f^*_P \in f_P^{max} \quad .$$

For infinite games with continuous benefits it is known that there exists an equilibrium in mixed strategies if the strategy spaces are nonempty and compact.

Even more, if we assume continuous and quasi-concave benefit functions, there exists a pure-strategy equilibrium (see [11]). In our game we cannot assume continuous benefit functions. In addition, we have to deal with variable strategy sets if we just consider feasible flows. Therefore, it is not evident that in path player games feasible equilibria do always exist. We will in the following prove even more: the existence of feasible equilibria in *pure strategies*.

Theorem 3.4 (Existence of feasible equilibria)**.** *In a path player game with continuous cost functions $c_e(f_e)$ for all edges $e \in E$, there exists at least a feasible equilibrium in pure strategies $\hat{f}$ such that $\hat{f}_P \in f_P^{max} \; \forall \; P \in \mathcal{P}$.*

Proof. Consider the set of feasible flows $\mathbb{F} = \left\{f : f_P \geq 0 \; \forall \; P \in \mathcal{P} \; \wedge \; \sum_{P\in\mathcal{P}} f_P \leq r\right\}$. The set $\mathbb{F}$ is closed, bounded and convex. Furthermore consider the single-value function, $T : \mathbb{F} \to \mathbb{R}^{|\mathcal{P}|}$ defined as $T(f) = f'$ whose components $f'_P = t(f_P)$ are given by

$$f'_P = f_P + \begin{cases} \min\left\{f_P^m - f_P; \frac{f_P^m - f_P}{\sum_{P_k\in\mathcal{P}:f_{P_k}<f_{P_k}^m}(f_{P_k}^m - f_{P_k})} \cdot d\right\} & if \quad f_P < f_P^m \\ f_P^m - f_P & if \quad f_P \geq f_P^m \end{cases} , \tag{2}$$

where $f_P^m = \min\{f_P^{max}\}$ is chosen as the smallest flow that is benefit maximizing[2] and $d = r - \sum_{P\in\mathcal{P}} f_P$ is the flow left that can be distributed among the players maintaining feasibility. Note that it holds for all $P \in \mathcal{P}$ that $d = d_P - f_P \geq 0$. Note furthermore, that by Lemma 3.2 f_P^m exists and that by definition of f_P^{max} it holds that

$$0 \leq f_P^m \leq d_P = r - \sum_{P_k\in\mathcal{P}\setminus\{P\}} f_{P_k} \quad . \tag{3}$$

A visualization and interpretation of the function T can be found right after this proof. In the following we prove that T is a continuous function of $\mathbb{F}$ into itself. Then, by Brouwer's fixed point theorem a fixed point $f = T(f)$ exists in $\mathbb{F}$. Finally, we will show that each fixed point in $\mathbb{F}$ is representing an equilibrium in pure strategies, so that we will be able to guarantee the existence of a feasible equilibrium in pure strategies.

Part a) ($T : \mathbb{F} \to \mathbb{F}$)
First note that $f'_P \geq 0 \; \forall \; P \in \mathcal{P}$. Denote the sets $\mathcal{P}_1 = \{P \in \mathcal{P} : f_P < f_P^m\}$ and $\mathcal{P}_2 = \{P \in \mathcal{P} : f_P \geq f_P^m\}$.

$$\begin{aligned} \sum_{P\in\mathcal{P}} f'_P \quad &= \quad \sum_{P\in\mathcal{P}_1} \left(f_P + \min\left\{ f_P^m - f_P; \frac{f_P^m - f_P}{\sum_{P_k\in\mathcal{P}:f_{P_k}<f_{P_k}^m}(f_{P_k}^m - f_{P_k})} \cdot d \right\}\right) \\ &\quad + \sum_{P\in\mathcal{P}_2} (f_P + f_P^m - f_P) \end{aligned}$$

[2] Note that for the proof it is not important which $f_P \in f_P^{max}$ is chosen for f_P^m as long as it is well-defined.

$$
\begin{aligned}
&= \sum_{P\in\mathcal{P}} f_P + \sum_{P\in\mathcal{P}_1} \min\left\{ f_P^m - f_P; \frac{f_P^m - f_P}{\sum_{P_k\in\mathcal{P}:f_{P_k}<f_{P_k}^m}(f_{P_k}^m - f_{P_k})}\cdot d \right\} \\
&\quad + \overbrace{\sum_{P\in\mathcal{P}_2}(f_P^m - f_P)}^{\leq 0} \\
&\leq \sum_{P\in\mathcal{P}} f_P + \sum_{P\in\mathcal{P}_1} \frac{f_P^m - f_P}{\sum_{P_k\in\mathcal{P}:f_{P_k}<f_{P_k}^m}(f_{P_k}^m - f_{P_k})}\cdot d \\
&= \sum_{P\in\mathcal{P}} f_P + d = \sum_{P\in\mathcal{P}} f_P + r - \sum_{P\in\mathcal{P}} f_P \\
&= r\ .
\end{aligned}
$$

Therefore, $f' \in \mathbb{F}$ since $f'_P \geq 0\ \forall\ P \in \mathcal{P}$ and $\sum_{P\in\mathcal{P}} f'_P \leq r$.

Part b) ($T(f)$ is continuous)
We distinguish the following exhaustive cases:

i) $f_P > f_P^m$:
$f'_P = f_P^m\ \forall\ f_P > f_P^m$, i.e. $t(f_P)$ is continuous

ii) $f_P = f_P^m + 0$:
$f'_P = f_P^m$ for $f_P = f_P^m + 0$, i.e. $t(f_P)$ is continuous to the right $f_P = f_P^m + 0$

iii) $f_P < f_P^m$:
Consider $g(f) = f_P^m - f_P$ and $h(f) = \frac{f_P^m - f_P}{\sum_{P_k\in\mathcal{P}:f_{P_k}<f_{P_k}^m}(f_{P_k}^m - f_{P_k})}\cdot d$. The functions $g(f)$ and $h(f)$ are continuous and so the minimum of both functions is continuous too. It follows that $t(f_P)$ with $f'_P = f_P + \min\{g(f); h(f)\}$ is continuous.

iv) $f_P = f_P^m - 0$:
Consider the following marginal value of the mapping that we take for each flow f where $f_P \to f_P^m - 0$:

$$
\lim_{f:f_P\to f_P^m-0}\left(\overbrace{f_P}^{\to f_P^m} + \min\left\{ \overbrace{f_P^m - f_P}^{\to 0}; \overbrace{\frac{f_P^m - f_P}{\sum_{P_k\in\mathcal{P}:f_{P_k}<f_{P_k}^m}(f_{P_k}^m - f_{P_k})}}^{\geq 0}\cdot \overbrace{d}^{\geq 0} \right\} \right) = f_P^m\ .
$$

Thus, $t(f_P)$ is continuous to the left at $f_P = f_P^m - 0$.

Hence, T is continuous.

Part c) ($\hat{f} = T(\hat{f}) \Rightarrow \hat{f}$ is a pure strategy equilibrium)
Since T maps $\mathbb{F}$ into itself and we have proved that a fixed point $\hat{f} = T(\hat{f})$ exists then $\hat{f} \in \mathbb{F}$. Therefore $\hat{f}$ is an equilibrium in pure strategies.

Moreover, we can explicitly describe the form of such an equilibrium. Indeed, as $\hat{f} = T(\hat{f})$ then $\hat{f}'_P = \hat{f}_P$ for each path $P \in \mathcal{P}$ which in turns implies that the bracket in (2), that we will denote by K_P, equals zero. Hence $K_P = 0$ for all $P \in \mathcal{P}$.
First note that $\hat{f}_P < \hat{f}_P^m$ can not occur since from $K_P = 0$ and $\hat{f}_P^m - \hat{f}_P > 0$ it follows that $d = 0$. Then, from (3) we get

$$0 = d = r - \sum_{P \in \mathcal{P}} \hat{f}_P \geq \hat{f}_P^m - \hat{f}_P \ .$$

This implies that $\hat{f}_P \geq \hat{f}_P^m$, which means by (2) and as $K_P = 0$ that $\hat{f}_P = \hat{f}_P^m \in f_P^{max}$. In conclusion, the equilibrium satisfies

$$\hat{f}_P \in f_P^{max} \ \forall \ P \in \mathcal{P} \ . \qquad \square$$

Figures 4 and 5 illustrate the mapping T. The mapping can be interpreted as a simple auction where the players bid the flow they want to route over their paths. In particular, each player asks to receive the flow f_P^m. Then, each player receives a flow f'_P which depends on all bids and on the amount of flow that can be distributed without exceeding the flow rate r. If the current flow of a player P is greater than or equal to f_P^m, then he is given exactly $f'_P = f_P^m$, as reducing flow will not violate the flow rate. If $f_P < f_P^m$ holds, i.e. P asks for a larger flow, we have to distinguish two cases. The first case is illustrated in Figure 4. Here, $\sum_{P_k \in \mathcal{P}: f_{P_k} < f_{P_k}^m} (f_{P_k}^m - f_{P_k}) > d$ holds, i.e. the players want to increase their flow, but ask for more flow than available. In this case the flow rate would be violated if each player would receive his bid. Thus, each player receives a fraction of d proportional to his bid and smaller as his bid. In the second case, the sum of the players' bids is not exceeding r: $\sum_{P_k \in \mathcal{P}: f_{P_k} < f_{P_k}^m} (f_{P_k}^m - f_{P_k}) \leq d$. Each player will receive exactly his bid, which is illustrated in Figure 5.

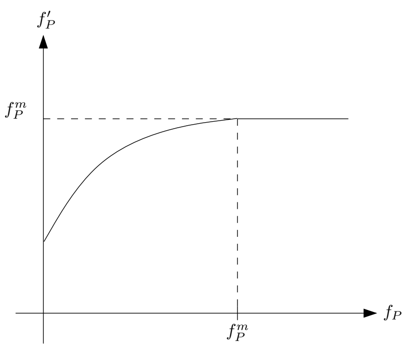

Figure 4: Players receive flow smaller than bid

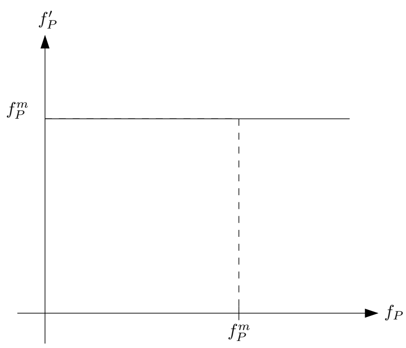

Figure 5: Players receive exact bids

Note hat in a path player game infeasible equilibria may occur. They are fully characterized in the next lemma.

Lemma 3.5. *In a path player game a flow f is an infeasible equilibrium if and only if for all paths P in $\mathcal{P}$ the following is satisfied:*

$$\sum_{P\in\mathcal{P}} f_P \geq r + \max_{P\in\mathcal{P}} f_P \ .$$

Proof. ($\sum_{P\in\mathcal{P}} f_P \geq r + \max_{P\in\mathcal{P}} f_P \ \Rightarrow\ f$ infeasible equilibrium)
Consider a flow f such that $\sum_{P\in\mathcal{P}} f_P \geq r + \max_{P\in\mathcal{P}} f_P$ holds. This flow is infeasible as $\max_{P\in\mathcal{P}} f_P > 0$ holds. In addition for all paths P in $\mathcal{P}$ the following is true:

$$d_P = r - \sum_{P_k\in\mathcal{P}\setminus\{P\}} f_{P_k} = r - \left(\sum_{P_k\in\mathcal{P}} f_{P_k} - f_P\right) \leq r - \overbrace{\left(\sum_{P_k\in\mathcal{P}} f_{P_k} - \max_{P\in\mathcal{P}} f_P\right)}^{\geq r} \leq 0$$

$$\begin{aligned} &\Rightarrow\ \tilde{b}_P(f_P) = -M \ \forall\ f_P \\ &\Rightarrow\ f_P^{max} = [0,\infty) \end{aligned}$$

Therefore, we conclude that $f_P \in f_P^{max}\ \forall\ P \in \mathcal{P}$ and thus, f is an equilibrium.

(f infeasible equilibrium $\Rightarrow\ \sum_{P\in\mathcal{P}} f_P \geq r + \max_{P\in\mathcal{P}} f_P$)
Consider a flow f such that $\sum_{P\in\mathcal{P}} f_P > r$ and $f_P \in f_P^{max}\ \forall\ P \in \mathcal{P}$, i.e. f is an infeasible equilibrium. Assume that the claim is not true, i.e. $\sum_{P\in\mathcal{P}} f_P < r + \max_{P\in\mathcal{P}} f_P$. Let $\bar{P}$ be such that $\max_{P\in\mathcal{P}} f_P = f_{\bar{P}}$. Then,

$$d_{\bar{P}} = r - \sum_{P\in\mathcal{P}\setminus\{\bar{P}\}} f_P = r - \left(\sum_{P\in\mathcal{P}} f_P - f_{\bar{P}}\right) = r - \overbrace{\left(\sum_{P\in\mathcal{P}} f_P - \max_{P\in\mathcal{P}} f_P\right)}^{<r}$$

$$\begin{aligned} &\Rightarrow\ d_{\bar{P}} > 0 \\ &\Rightarrow\ \exists\ f'_{\bar{P}} : \tilde{b}_{\bar{P}}(f'_{\bar{P}}) > -M \\ &\Rightarrow\ f_{\bar{P}} \notin f_{\bar{P}}^{max}\ , \end{aligned}$$

which contradicts the assumption and thus the claim follows. □

It is a consequence that infinitely many infeasible equilibria exist in path player games.

4 Properties of Path Player Games

In this section we describe properties of path player games that will be needed for the characterization of equilibria.

Path-disjoint Network A set of paths $\bar{\mathcal{P}}$ is called *disjoint* if for all pairs $P_1, P_2 \in \mathcal{P}$ with $P_1 \neq P_2$ it holds that $P_1 \cap P_2 = \emptyset$.
We call a network *path-disjoint* if the set $\mathcal{P}$ of all paths from s to t is disjoint. In a path disjoint network, $c_P(f)$ only depends on f_P and is independent from f_{-P}. In the literature, cost functions c_P with $c_P(f) = c_P(\,\cdot\,, f_P)$ are also known as *separable functions* (e.g. see [12]).

Trivial Games We will call a game with flow rate r and security limits ω_P *trivial*, if $\sum_{P\in\mathcal{P}} \omega_P > r$ holds, and *nontrivial* otherwise. In trivial games, it is possible that the entire flow rate r is used, even if all players route $f_P < \omega_P$ for all $P \in \mathcal{P}$, which cannot happen in nontrivial games.

Lemma 4.1. *Let f be a feasible flow in a nontrivial path player game. Then there exists at least a $P \in \mathcal{P}$ such that $d_P \geq \omega_P$.*

Proof. Consider a nontrivial path player game, i.e. $\sum_{P\in\mathcal{P}} \omega_P \leq r$ and a given flow f. It holds for all $P \in \mathcal{P}$ that

$$\begin{aligned}
& d_P && = r - \sum_{P_k\in\mathcal{P}\setminus\{P\}} f_{P_k} = r - \sum_{P_k\in\mathcal{P}} f_{P_k} + f_P \\
\Rightarrow\quad & \textstyle\sum_{P\in\mathcal{P}} d_P && = |\mathcal{P}| \cdot r - |\mathcal{P}| \cdot \sum_{P\in\mathcal{P}} f_P + \sum_{P\in\mathcal{P}} f_P \\
& && = |\mathcal{P}| \cdot r - (|\mathcal{P}| - 1) \cdot \sum_{P\in\mathcal{P}} f_P \\
& && = r + (|\mathcal{P}| - 1) \cdot \overbrace{\left(r - \sum_{P\in\mathcal{P}} f_P\right)}^{\geq 0} \\
\Rightarrow\quad & && \sum_{P\in\mathcal{P}} d_P \geq r \geq \sum_{P\in\mathcal{P}} \omega_P \\
\Rightarrow\quad & && \exists\, P \in \mathcal{P} : d_P \geq \omega_P
\end{aligned}$$

□

Non-Compensative Security Property A path player game is called a game with *non-compensative security (NCS) property* if for all paths $P \in \mathcal{P}$ and for all flows f_{-P} with $d_P \geq \omega_P$ there exists at least a path P with flow $f_p \geq \omega_P$ satisfying $\tilde{b}_P(f_P) > \kappa_P$.

In games with NCS property, no player P will choose the security payment κ_P when a flow $f_P \geq \omega_P$ is possible. If a player has the possibility to earn benefit by receiving income by his "productivity", i.e. by getting income from the cost function c_P, he has no reason to take advantage of the security limit. The security payment shall only be used if the player has no other choice due to the strategies of his competitors, i.e. if $d_P < \omega_P$. The NCS property is an interesting attribute of games as it will enable the characterization of equilibria for strictly increasing costs (see

Section 5). Note that as we assume nonnegative costs, a game where $\kappa_P < 0$ holds for all P in $\mathcal{P}$ has NCS property. In all other cases, it is not so easy to recognize if a game has NCS property. However, in some cases, the NCS property of a game follows from the following property of the benefit function. A benefit function $b_P(f)$ with $\omega_P < r$ has the *non-compensative security (NCS) property* if

$$\kappa_P < c_P(0, \ldots, 0, \omega_P, 0, \ldots, 0) =: c_P(\vec{0}_{|\mathcal{P}|-1}, \omega_P) \tag{4}$$

holds. If κ_P is sufficiently small, a player on an underloaded path gets a benefit which is lower than the income he would get if he was able to route a flow of value ω_P over that path, while no other player routes anything. The idea is that no player should have an incentive to choose his path to be underloaded if he is able to route a flow $f_P \geq \omega_P$.
To illustrate benefit functions with NCS property, let us consider a benefit function $b_P(f)$, where all players apart from P are routing a zero-flow, i.e. $b_P(f) = b_P(\vec{0}_{|\mathcal{P}|-1}, f_P)$. A function $b_P(f)$ as shown in Figure 6 does not have NCS property

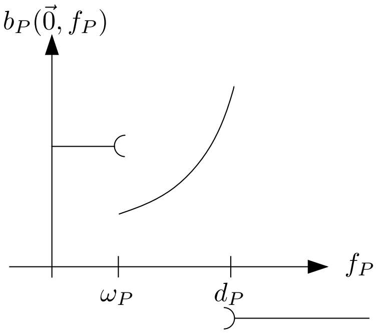

Figure 6: No NCS property

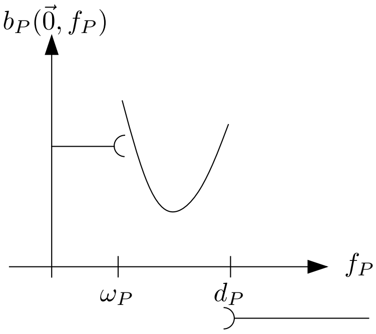

Figure 7: NCS property

as the player P will choose the security payment instead of the income obtained by routing ω_P. In general, that does not mean, that the player always prefers the benefit κ_P. It may happen (like in this illustration) that there is a flow $f_P > \omega_P$ with $b_P(\vec{0}_{|\mathcal{P}|-1}, f_P) > \kappa_P$. However, for a benefit function without NCS property, we can not guarantee that there will be a flow f_P, that provides a higher benefit than κ_P. On the contrary, a benefit function as the one shown in Figure 7 allows the player to increase his benefit when routing more than ω_P.
Let us now consider the relation between games with NCS property and benefit functions with NCS property. Unfortunately, a game that possesses benefit functions with NCS property is not necessarily a game with NCS property. Consider a path P with $d_P \geq \omega_P$, whose benefit functions possess NCS property. It does not necessarily hold that P is in any case able to obtain a benefit greater than κ_P. In general networks, player may share edges. It is possible that on an edge e with decreasing benefit some of the players sharing e have incentive to raise the flow f_e even

if edge e induces a loss, if they can compensate that loss by gains on other edges. Consequently, $b_P(f_P) \leq \kappa_P \; \forall \; f_P \geq \omega_P$ could hold, i.e. the game would possess no NCS property. We call this effect of influencing the benefit of the competitors *edge sharing effect*.

For instance see Figure 8, where P_1 would accept a decreasing income from edge e,

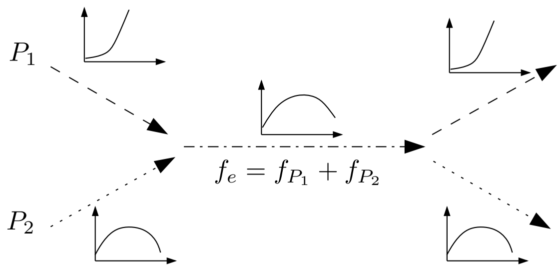

Figure 8: Edge sharing effect

as this loss is compensated by the remaining edges. At the same time, P_2 does not want to increase f_e too much, as at a certain point his benefit $b_P(f)$ will decrease. Nevertheless, P_2 can not avoid that P_1 increases the flow, i.e. he is forced into a situation where sending flow can create loss. Note that in this situation, a positive ω_P and κ_P fulfill the purpose of being an insurance, as it helps the player P_2 to escape the harmful behavior of the competitors.

As the edge sharing effect may destroy the NCS property of games, we investigate additional assumptions which prevent the edge sharing effect, obtaining situations where benefit functions with NCS property induce games with NCS property.

Lemma 4.2. *Let us consider a path player game with cost functions $c_e(f_e)$ that are monotonically increasing for all edges $e \in E$ and benefit functions $\tilde{b}_P(f_P)$ that possess NCS property for all paths $P \in \mathcal{P}$. Such a game is a game with NCS property.*

Proof. Consider a path $P \in \mathcal{P}$ and a flow f_{-P} with $d_P \geq \omega_P$. By the definition of the benefit function, for all $f_P \in [\omega_P, d_P]$ we get that

$$\begin{aligned}
\tilde{b}_P\left(f_P\right) = \tilde{c}_P(f_P) &= \sum_{e\in P} c_e\left(f_P + \sum_{P_k \in \mathcal{P}\setminus\{P\}: e\in P_k} f_{P_k}\right) && (5)\\
&\geq \sum_{e\in P} c_e\left(\omega_P + \sum_{P_k \in \mathcal{P}\setminus\{P\}: e\in P_k} 0\right) && (6)\\
&= c_P\left(\vec{0}_{|\mathcal{P}|-1}, \omega_P\right) > \kappa_P \;. && (7)
\end{aligned}$$

Condition (5) holds due to the definition of $c_P(f)$, while (6) holds because of the monotonically increasing cost functions $c_e(f_e)$. (7) is true as $b_P(f)$ possesses NCS

property.
We conclude that $\tilde{b}_P(f_P) > \kappa_P$ for all $P \in \mathcal{P}$ and for all feasible f with $f_P \geq \omega_P$ and thus the game has NCS property. $\square$

Lemma 4.3. *Let us consider a path player game on a path-disjoint network G. Furthermore, let the benefit functions $b_P(f)$ possess NCS property for all paths $P \in \mathcal{P}$. Such a game is a game with NCS property.*

Proof. Consider a path $P \in \mathcal{P}$ and a flow f_{-P} with $d_P \geq \omega_P$ and set $f_P = \omega_P$. As the resulting flow f is feasible, it holds that

$$\begin{aligned} \tilde{b}_P(f_P) &= \tilde{c}_P(f_P) \\ &= c_P(\vec{0}_{|\mathcal{P}|-1}, f_P) > \kappa_P \ . \end{aligned} \tag{8}$$

Note that (8) holds since G is path-disjoint and $b_P(f)$ has NCS property. Hence, the lemma follows. $\square$

The following lemma does not assume benefit functions with NCS property but a similar condition for at least one edge in each path, to obtain a game with NCS property.

Lemma 4.4. *A path player game where each path P satisfies that*

$$\bar{E}_P = \{e : e \in P \ \wedge \ e \notin P_k, \forall \ P_k \in \mathcal{P} \setminus \{P\}\} \neq \emptyset \ \forall \ P \in \mathcal{P} \ ,$$

possesses the NCS property if

$$\sum_{e \in \bar{E}_P} c_e(\omega_P) > \kappa_P \ \forall \ P \in \mathcal{P} \ .$$

Proof. For a flow f consider all paths P with $d_P \geq \omega_P$ and the corresponding flows f_{-P}. Set $f_P = \omega_P$, therefore the resulting flow is feasible. Then, we obtain that

$$\begin{aligned} \tilde{b}_P(f_P) &= \tilde{c}_P(\omega_P) \\ &= \overbrace{\sum_{e \in \bar{E}_P} c_e(\omega_P)}^{>\kappa_P} + \sum_{e \in P \setminus \{\bar{E}_P\}} \overbrace{c_e(f_e)}^{\geq 0} > \kappa_P \ , \end{aligned}$$

and thus the lemma follows. $\square$

5 Equilibria for Strictly Increasing Cost Functions

In this section we present characterizations of equilibria under the assumption of strictly increasing cost functions. We will obtain a necessary condition for equilibria and even a necessary and sufficient condition if the game has in addition NCS property or if we consider a game with no security limit. There are further results for other types of costs functions, which we will omit here. In short, for differentiable costs a necessary condition can be found. For differentiable and concave cost functions the necessary condition will become also sufficient in a game with no security limit. Finally, for convex costs we are able to determine a dominating strategy set. See [28] for details.
The next proposition will be useful for the proofs in this section.

Proposition 5.1. *Consider a path player game with strictly increasing cost functions $c_e(f_e)$. Then the one-dimensional benefit functions $\tilde{b}_P(f_P)$ are also strictly increasing for $f_P \in [\omega_P, d_P]$.*

The proof of this proposition is based on the fact that $\tilde{c}_P(f_P)$ is a sum of strictly increasing functions $c_e(f_P + \sum_{P_k \in \mathcal{P}\setminus\{P\}} f_{P_k})$ and so it is again strictly increasing.
To obtain some of the results we require the non-security-property, i.e. we set $\omega_P = 0 \; \forall \; P \in \mathcal{P}$. The benefit functions and the one-dimensional benefit functions take the following simplified form that will appear every time we require the non-security-limit property:

$$b_P(f) = \begin{cases} c_P(f) & if \quad \sum_{P \in \mathcal{P}} f_P \leq r \\ -M & if \quad \sum_{P \in \mathcal{P}} f_P > r \end{cases} , \tag{9}$$

$$\tilde{b}_P(f_P) = \begin{cases} \tilde{c}_P(f_P) & if \quad f_P \leq d_P \\ -M & if \quad f_P > d_P \end{cases} . \tag{10}$$

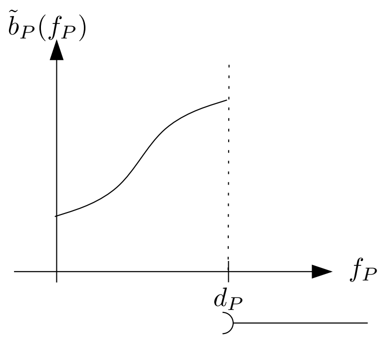

Figure 9: Strictly increasing cost function and no security limit

We obtain a necessary and sufficient condition for equilibria in path player games with strictly increasing costs.

Lemma 5.2. *In a path player game with strictly increasing cost functions $c_e(f_e)$ on all edges $e \in E$ and security limit $\omega_P = 0$ for all $P \in \mathcal{P}$, a flow f is a feasible equilibrium if and only if*

$$\sum_{P \in \mathcal{P}} f_P = r \quad .$$

Proof. (f *equilibrium* $\Rightarrow \sum_{P\in\mathcal{P}} f_P = r$)
Let f be a feasible equilibrium and assume that $\sum_{P\in\mathcal{P}} f_P < r$.

$$\begin{aligned} &\Rightarrow \quad f_P < d_P \;\forall\; P \in \mathcal{P} \\ &\Rightarrow \quad \tilde{c}_P(f_P) < \tilde{c}_P(f_P + \epsilon) \;\forall\; \epsilon \in (0, d_P],\; \forall\; P \in \mathcal{P} \\ &\Rightarrow \quad f_P \notin f_P^{max} \quad , \end{aligned}$$

i.e. f is not an equilibrium and hence,

$$\sum_{P \in \mathcal{P}} f_P = r \quad .$$

Note that (11) follows as $\tilde{c}_P(f_P)$ is strictly increasing for all P in $\mathcal{P}$.

($\sum_{P\in\mathcal{P}} f_P = r \Rightarrow f$ *equilibrium*)
As $\sum_{P\in\mathcal{P}} f_P = r$ holds, it implies that $f_P = d_P$ for all $P \in \mathcal{P}$. Furthermore, for all $e \in E$, $\tilde{c}_e(f_e)$ is a strictly increasing function, thus $\tilde{b}_P(f_P)$ is strictly increasing over $[0, d_P]$ (see Prop.5.1) and $f_P \in f_P^{max} \;\forall\; P \in \mathcal{P}$. Hence, by Corollary 3.3, f is an equilibrium. □

Now we need to investigate situations, where we can not assume $\omega_P = 0$ for all paths $P \in \mathcal{P}$. If we consider a game with strictly increasing cost functions $c_e(f_e)$ together with general security limit $\omega_P \geq 0$, the statement of Lemma 5.2 still holds provided that we assume, in addition, to have a nontrivial game with NCS property (see Section 4).

Lemma 5.3. *Consider a game with strictly increasing cost functions $c_e(f_e)$ on all edges $e \in E$. Assume, that the game is nontrivial and it satisfies NCS property. Then a flow f is a feasible equilibrium if and only if $\sum_{P\in\mathcal{P}} f_P = r$.*

Proof. (f *feasible equilibrium* $\Rightarrow \sum_{P\in\mathcal{P}} f_P = r$)
Consider a feasible equilibrium f and assume that $\sum_{P\in\mathcal{P}} f_P < r$, i.e. $f_P < d_P$ for all $P \in \mathcal{P}$. Due to non-triviality we can find a path $\bar{P}$ such that $d_{\bar{P}} \geq \omega_{\bar{P}}$ (see Lemma 4.1). We distinguish two cases:

Case 1: $f_{\bar{P}} \geq \omega_{\bar{P}} \Rightarrow \tilde{b}_{\bar{P}}(d_{\bar{P}}) > \tilde{b}_{\bar{P}}(f_{\bar{P}})$ (due to Proposition 5.1), which contradicts f being a feasible equilibrium.

Case 2: $f_{\bar{P}} < \omega_{\bar{P}} \Rightarrow \exists\; \hat{f}_{\bar{P}} \geq \omega_{\bar{P}}$ such that $\tilde{b}_{\bar{P}}(\hat{f}_{\bar{P}}) > \kappa_{\bar{P}} = \tilde{b}_{\bar{P}}(f_{\bar{P}})$ (due to NCS property), which contradicts f being a feasible equilibrium.

The above implies that $\sum_{P \in \mathcal{P}} f_P = r$.

($\sum_{P \in \mathcal{P}} f_P = r \;\Rightarrow\; f$ *feasible equilibrium*)
Consider a flow with $\sum_{P \in \mathcal{P}} f_P = r$, i.e. $f_P = d_P$ for all $P \in \mathcal{P}$. We analyze the two cases:

Case 1: $f_P \geq \omega_P$: As there exists at least one $\hat{f}_P \geq \omega_P$ such that $\tilde{b}_P(\hat{f}_P) > \kappa_P$ (due to NCS property), and as $\tilde{b}_P(f_P)$ is strictly increasing over $[\omega_P, d_P]$ (see Prop.5.1), and in particular, $\tilde{b}_P(f_P) \geq \tilde{b}_P(\hat{f}_P) > \kappa_P$ it holds that $f_P^{max} = \{d_P\}$.

Case 2: $f_P < \omega_P$: As $\tilde{b}_P(f_P)$ is constant over $[0, \omega_P)$ and $d_P < \omega_P$, it holds that $d_P \in f_P^{max}$.

Hence, we conclude that f is a feasible equilibrium as $f_P \in f_P^{max}\ \forall\ P \in \mathcal{P}$. □

Unfortunately the converse of Lemma 5.3 does not hold: A game that satisfies the property

"A flow f is a feasible equilibrium if and only if $\sum_{P \in \mathcal{P}} f_P = r$" (#)

does not have to be neither nontrivial nor satisfy the NCS property. For an illustration we present the following examples:

Example 5.1. (#) $\not\Rightarrow$ *NCS property.*
Consider a game on a network with two paths, as illustrated in Figure 10. A flow rate $r = 1$ has to be routed from s to t. On both paths the costs are $c_P(x) = x$, but the security limits and security payments differ: $\omega_1 = \kappa_1 = 1$ and $\omega_2 = \kappa_2 = 0$.
In this game a flow f with $\sum_{P \in \mathcal{P}} f_P < r$ can not be an equilibrium as $f_2^{max} = \{d_2\}$ for all f_{-2}, i.e. player 2 would in any case use up the remaining flow rate. On the other hand, each flow f with $\sum_{P \in \mathcal{P}} f_P = r$ is an equilibrium flow. If $\sum_{P \in \mathcal{P}} f_P = r$ holds, player 2 can not find any better strategy as he will always try to get as much flow as possible, while player 1 is also not able to improve his payoff as his benefit function is anyway constant over $[0, 1]$. That means, this game fulfills condition (#). Nevertheless, the game has not NCS property. There is no $f_1 \geq \omega_1$ with $\tilde{b}_1(f_1) > \kappa_1$ and so path 1 is destroying the NCS property of the game.

Example 5.2. (#) $\not\Rightarrow$ *non-triviality.*
Consider the game illustrated in Figure 11. The graph consists of two paths, and we choose $\omega_1 = 2$ and $\omega_2 = 0$. The remaining components of the game, as cost functions and security payments may be chosen arbitrarily, but it is important that the cost functions are strictly increasing.
With a similar argument as in Example 5.1, it is possible to show that this game fulfills condition (#). Nevertheless, the game is trivial, as $\sum_{P \in \mathcal{P}} \omega_P > r$.

If we consider a game with strictly increasing cost functions and general security limit, but we can not ensure NCS property or the non-triviality of the game, we are still able to give a necessary condition for a profile of flows to be an equilibrium.

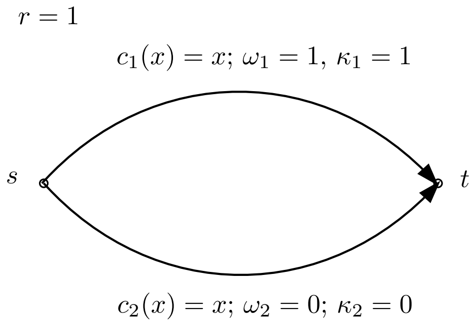

Figure 10: Game graph of example 5.1

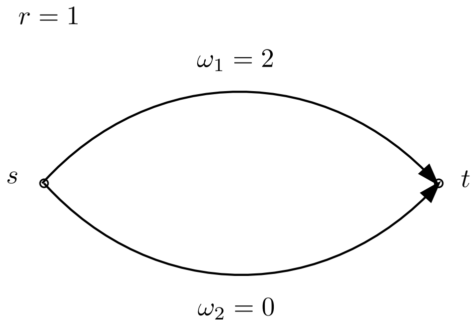

Figure 11: Game graph of example 5.2

Lemma 5.4. *If a flow f in a path player game with strictly increasing cost functions $c_e(f_e)$ on all edges $e \in E$ is a feasible equilibrium then at least one of the following two cases holds:*

(*i*) $\sum_{P \in \mathcal{P}} f_P = r$

(*ii*) $f_P < \omega_P \; \forall \; P \in \mathcal{P}$.

Proof.
Let f be a feasible equilibrium and assume that (i) is not true, i.e. $\sum_{P \in \mathcal{P}} f_P < r$ then $f_P < d_P \; \forall \; P \in \mathcal{P}$.
Assume case (ii) is also not true, i.e. $\exists \; \bar{P}$ with $f_{\bar{P}} \geq \omega_{\bar{P}}$. Then $\tilde{b}_{\bar{P}}(f'_{\bar{P}}) > \tilde{b}_{\bar{P}}(f_{\bar{P}}) \; \forall \; f'_{\bar{P}} \in (f_{\bar{P}}, d_{\bar{P}}]$, as according to Proposition 5.1, $\tilde{b}_P(f_P)$ is strictly increasing over this domain. It follows that $f_{\bar{P}} \notin f_{\bar{P}}^{max}$. This contradicts f being an equilibrium, hence $f_P < \omega_P$ for all $P \in \mathcal{P}$. $\square$

To illustrate Lemma 5.4 we present two examples of feasible equilibria, where (i) and (ii) do not hold.

Example 5.3. *Consider a path player game with two vertices s and t, which are connected by two edges, i.e. $\mathcal{P} = \{1, 2\}$. A flow rate $r = 1$ has to be routed between the two vertices. We set the security limits $\omega_1 = \omega_2 = 0.5$, the security payment $\kappa_1 = \kappa_2 = 1$ and the cost functions $c_P(x) = x, P \in \{1, 2\}$. The flow $f = (0.2, 0.2)$ with $b_1(f) = b_2(f) = 1$ is an equilibrium for which (ii) holds and (i) is not satisfied.*

Example 5.4. *Consider the following path player game: There are two vertices s and t which are connected by two edges, i.e. $\mathcal{P} = \{1, 2\}$. This game is a game on a path-disjoint network. A flow rate $r = 2$ has to be routed from s to t. Furthermore, the paths possess security limits $\omega_P = 1$, security payments $\kappa_P = 1$ and cost functions $c_P(f_P) = 1 + f_P$.*
The flow $f = (0.5, 1.5)$ with $\tilde{b}_1(0.5) = 1$ and $\tilde{b}_2(1.5) = 2.5$ destroys property (ii) while (i) does hold. This flow is an equilibrium as none of the players is able to improve the current payoff.

The following lemma provides a statement about the converse of Lemma 5.4.

Lemma 5.5. *Consider a path player game with strictly increasing cost functions $c_e(f_e)$. Let f be a flow with the following properties:*

(i) $\sum_{P\in\mathcal{P}} f_P = r$

(ii) $f_P < \omega_P \;\forall\; P \in \mathcal{P}$

Then, f is a feasible equilibrium.

Proof.
For all players P in $\mathcal{P}$ and for all $\epsilon > 0$, we have:

$$\begin{aligned} \tilde{b}_P(f_P+\epsilon) &= -M < \tilde{b}_P(f_P) \\ \tilde{b}_P(f_P-\epsilon) &= \kappa_P = \tilde{b}_P(f_P), \text{ if } \epsilon \le f_P \;. \end{aligned}$$

It follows that for all $P \in \mathcal{P}$ and for all $\bar{f}_P > 0$

$$\tilde{b}_P(f_P) \ge \tilde{b}_P(\bar{f}_P)$$

holds, and thus f is a feasible equilibrium. $\square$

Example 5.5 ($(i) \wedge \neg(ii) \not\Rightarrow$ f is feasible equilibrium)**.** *Let us consider a path player game with strictly increasing cost functions $c_e(f_e)$. Furthermore, consider a feasible flow f such that $\sum_{P\in\mathcal{P}} f_P = r$ holds and that there exists $\bar{P} \in \mathcal{P}$ with $f_{\bar{P}} \ge \omega_{\bar{P}}$. It is possible to construct a game such that $\tilde{b}_{\bar{P}}(f_{\bar{P}}) < \kappa_{\bar{P}}$ holds (see Figure 12) and thus, the flow f is not an equilibrium:*
Set $r = 1$, $\omega_1 = \omega_2 = 0.25$ and the security payment $\kappa_1 = \kappa_2 = 2$. For cost functions $c_P(x) = x$ with $P = \{1, 2\}$ the flow $f = (0.5, 0.5)$ fulfills (i) but not (ii). This flow with $b_1(f) = b_2(f) = 0.5$ is not an equilibrium as $f_1^{max} = f_2^{max} = [0, 0.25)$.

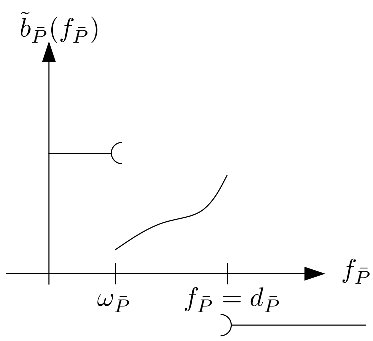

Figure 12: $\tilde{b}_{\bar{P}}(f_{\bar{P}}) < \kappa_{\bar{P}}$

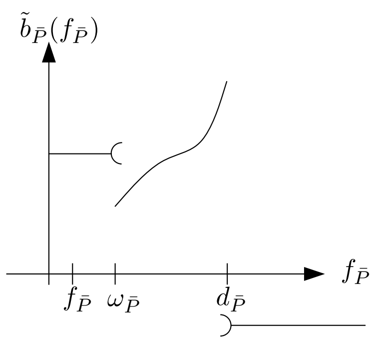

Figure 13: $\tilde{b}_{\bar{P}}(f_{\bar{P}}) < \tilde{b}_{\bar{P}}(d_{\bar{P}})$

Example 5.6 ($\neg(i) \ \wedge \ (ii) \ \not\Rightarrow \ f$ is feasible equilibrium)**.** *Let us consider a path player game with strictly increasing cost functions* $c_e(f_e)$*. Furthermore, consider a feasible flow* f *such that* $\sum_{P \in \mathcal{P}} f_P < r$ *and* $f_{\bar{P}} < \omega_{\bar{P}}$ *holds for all* $P \in \mathcal{P}$*. Thus, it holds for all* P *that* $f_P < d_P$ *and it is possible to construct a game such that*

$$\exists \ \bar{P} : \tilde{b}_{\bar{P}}(f_{\bar{P}}) = \kappa_{\bar{P}} < \tilde{b}_{\bar{P}}(d_{\bar{P}})$$

holds, (see Figure 13) i.e. f *is not an equilibrium:*
Set $r = 1$*,* $\omega_1 = \omega_2 = 0.5$ *and* $\kappa_1 = \kappa_2 = 0.1$*. For cost functions* $c_P(x) = x, P = \{1, 2\}$ *a flow* $f = (0.45, 0.45)$ *with* $b_1(f) = b_2(f) = 0.1$ *is no equilibrium as* $f_1^{max} = f_2^{max} = 0.55$*.*

We have seen, that a feasible flow with property (ii) need not be an equilibrium. This doesn't change if we assume to have a trivial game or a game without NCS property. The following example illustrates the assertion.

Example 5.7. *Consider a game with two vertices* s *and* t *and two edges connecting the vertices, i.e.* $\mathcal{P} = \{1, 2\}$ *on a path-disjoint network. A flow rate* $r = 5$ *has to be routed from* s *to* t*. Both paths possess a security limit* $\omega_P = 3$*, i.e. the game is trivial. Furthermore, the security payment is* $\kappa_P = 1$ *for* $P \in \{1, 2\}$ *and the cost functions are* $c_1(f_1) = f_1$ *and* $c_2(f_2) = f_2/10$*. This game possesses no NCS property as there is no* $f_2 > \omega_2$ *such that* $\tilde{b}_2(f_2) > \kappa_2$*.*
Consider the feasible flow $f = \vec{0}_{|\mathcal{P}|}$*. The flow* f *fulfills* (ii) *and* $d_P = r$ *for all* P*. Nevertheless, since* $\tilde{b}_1(d_1) = 3 > \tilde{b}_1(0) = \kappa_1 = 1$ *this flow is not an equilibrium.*

6 Conclusion

In this paper, we presented results for equilibria in a new network game, the path player game. We proved the existence of feasible equilibria in pure strategies. Furthermore, we presented a necessary condition for equilibria if the cost functions are strictly increasing. If the game satisfies in addition the NCS property, we obtained even a necessary and sufficient condition.
For future research, it will be a promising extension to consider not just paths but complete subgraphs as players. This reflects the fact that in real-world situations network providers usually own a subnetwork. Furthermore, some applications require integer solutions. Thus, the extension of the path player game to an integer version is of interest and has been already implemented for the line planning game in [28]. The same holds true for the extension to multiple source-sink pairs. The first results obtained for the line planning game (see [26, 28]) are promising and motivate further research in this field.
As in path player games we often have multiple equilibria, repeated or stochastic versions of the game could be considered to refine the set of equilibria. Finally, it is open work to analyze the situation as an optimization problem, that means to look for minimal cost flows in the network, and to compare them with the equilibria of the game.

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