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Abstract

We consider the problem of locating a circle with respect to existing facilities on the plane, such that the sum of weighted distances between the circle and the facilities is minimized. The problem properties are analyzed, and we give solution procedures.

1 Introduction

Drezner, Steiner and Wesolowsky [3] considered the problem of locating a circle on the plane with respect to existing facilities and suggested it as a model for the out of roundness problem. These authors primarily treated a minimax model, locating the circle so as to minimize the maximum distance between the circle and the facilities.

Here we consider the corresponding weighted minisum model: locate a circle so as to minimize the sum of weighted distances between the circle and the facilities. Applications include the out of roundness problem and the problem of locating circular facilities, e.g. a circular irrigation pipe in a field, circular conveyor belts, or ring roads. In the former case, the minisum solution may be used to evaluate the amount of rework required for an out-of-round part. Circular facilities such as ring roads are also of practical interest; see Pearce [8] and Suzuki [12]. The related problem of locating a circle on a sphere is examined in Brimberg et al. [1], and applications in diverse areas, including medical/biological and search-and-rescue, are noted. These problems may be transformed to our model by projecting the given points on various planes, and locating a circle relative to the projected points. A discrete formulation of our problem is also studied in Labbé et al. [5]. We investigated the minimax version of the model in [2].

The next section of the paper introduces the notation we will be using. Section 3 discusses some important properties of the mathematical model for the general case where the radius of the circle is variable. These properties are used in Section 4 to develop a solution approach. Section 5 examines the fixed radius case, which interestingly appears to be more difficult to solve. The last section provides a brief conclusion with suggestions for possible future research.

2 Notation

We use the following notation.

Let $\{A_1, \ldots, A_n\}$ be a given set of existing facilities, i.e. the number of existing facilities is n, $J = \{1, 2, \ldots, n\}$ and facility j is located at $A_j = (a_j, b_j)$ with associated positive weight w_j, for $j = 1, \ldots, n$. The existing facilities are also called the fixed points.

The circle to be located, C, is determined by its center, $X = (x, y)$, and its radius, r. We use the shortcut $C = (X, r)$.

The Euclidean distance between the center and facility j is denoted by

$$d_j(X) := d(X, A_j), \text{ for } j = 1, \ldots, n.$$

The shortest Euclidean distance between the circle C and a facility j is denoted as $d_j(C)$ and is given as $r - d(X, A_j)$, if the facility is inside the circle, and as $d(X, A_j) - r$, if it is outside. (If the facility is on the circle, the distance is 0, and both expressions apply). Summarizing, in general we have

$$d_j(C) := d(C, A_j) = |d(X, A_j) - r| \text{ for } j = 1, \ldots, n.$$

For a given circle $C = (X, r)$, it is convenient to define the index sets of facilities outside, on, and inside the circle:

$$\begin{aligned} J_+(C) &= \{j : d(X, A_j) > r\}, \\ J_0(C) &= \{j : d(X, A_j) = r\}, \\ J_-(C) &= \{j : d(X, A_j) < r\}. \end{aligned}$$

Note that the sets depend on the center X and on the radius r of the circle C. If it is clear which circle C is meant, we may simply write J_+, J_0, J_-. We say that a circle $C = (X, r)$ intersects a point A if $d(X, A) = r$.

The problem we consider is denoted by (P) and given as

$$\min f(C) = f(X,r) = \sum_{j=1}^{n} w_j d_j(C) = \sum_{j=1}^{n} w_j |d(X,A_j) - r|.$$

This problem may either have a finite solution (which is a circle with a finite radius) or it may have a solution with $r \to \infty$. In the latter case the resulting optimum is a straight line.

In the following, we consider first the general problem (P) and then the special case (Pr), where the radius of the circle is fixed in advance.

3 Finding a circle with variable radius

If $n \leq 3$, any circle that intersects all the existing facilities is optimal, and the objective function has an optimal value of 0. (For the special case, where $n = 3$ and the existing facilities are collinear, the optimum is not achieved, but approaches the straight line through them as $r \to \infty$.) Therefore, in the remainder of the discussion of problem (P), we will assume that $n \geq 4$.

The next result shows that a point ($r = 0$) is always inferior to a circle.

Lemma 1 *The optimal solution of problem (P) must have a positive radius.*

Denote any 'circle' degenerated to a point X_0 by $C_0 = (X_0, 0)$. Its objective function is $f(X_0, 0) = \sum_{j=1}^{n} w_j d(X_0, A_j) > 0$, since $n > 3$. Now consider a circle C_1 with positive radius intersecting X_0. For each j, we have $d_j(C_1) \leq d(X_0, A_j) = d_j(C_0)$. The inequality must be strict for at least one j, if the existing facilities are non-collinear. In this case,

$$\begin{aligned} f(C_1) &= \sum_{j=1}^{n} w_j d_j(C_1) \\ &< \sum_{j=1}^{n} w_j d_j(C_0) = f(C_0). \end{aligned}$$

On the other hand, if the A_j are collinear the straight line through them ($r \to \infty$) gives the best solution with objective function value zero. In both cases, C_0 cannot be optimal. $\diamond$

From this result, it follows that a point facility ($r = 0$) can never be an optimal solution of the circle location problem (P). However, the other extreme, a straight line (the limit of a circle with $r \to \infty$) may solve problem (P), as shown by the following example.

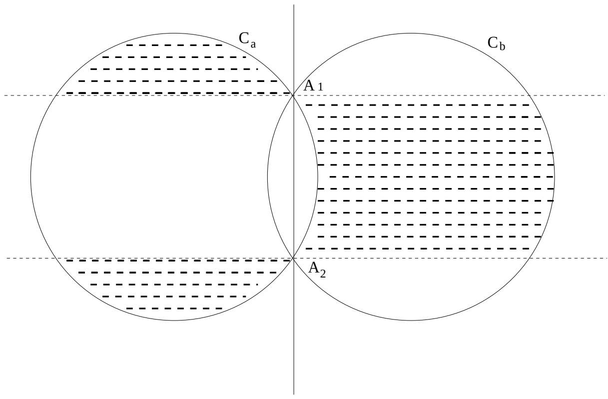

Figure 1: Illustration of the set $\mathcal{C}_a^- = \mathcal{C}_b^+$ for the region enclosed by the two circles C_a and C_b, and in which existing facilities can occur within the proof of lemma 2.

Let $n = 4$, $A_1 = (0,0)$, $A_2 = (1,10)$, $A_3 = (1,0)$, $A_4 = (1,-10)$, and let $w_1 = 1$, $w_2 = w_3 = w_4 = 100$. The limiting optimal solution of (P) for this instance is given by the vertical line through A_2, A_3, A_4. Under general conditions, however, the optimal solution must have a finite radius, as shown by the next result.

Lemma 2 *Suppose that $n \geq 5$ and that no triple of the existing facilities is collinear. Then no optimal solution of problem (P) is a straight line, i.e., each optimal solution has a finite radius.*

Proof:
If the optimal solution of (P) occurs in the limiting sense, $r \to \infty$, then this solution is a straight line which also solves the linear facility minisum location problem. It is well known (see [4, 11]) that each optimal solution of the Euclidean line location problem intersects at least two of the existing facilities, and hence, by the assumption in the lemma, exactly two of them. Therefore, consider a straight line l through any pair of existing facilities; without loss of generality we may assume that this pair is A_1, A_2. Denote by J_1 and J_2 the set of existing facilities on either side of the straight line l, i.e. $\{1, 2, \ldots, n\} = \{1, 2\} \cup J_1 \cup J_2$. Construct two circles $C_a = (X_a, r)$ and $C_b = (X_b, r)$ of the same radius r, both with center points on the bisector B_{A_1,A_2} between A_1 and A_2, on opposite sides of l, and such that both circles intersect A_1 and A_2 (i.e. $J_0(C_a) = J_0(C_b) = \{A_1, A_2\}$). If r is large enough, we also obtain $J_+(C_a) = J_1$ and $J_-(C_a) = J_2$ for C_a and analogously $J_-(C_b) = J_1$ and $J_+(C_b) = J_2$ for C_b. Consider an existing facility $A_j \notin \{A_1, A_2\}$. Let l^{per} be the line through A_j perpendicular to l, and denote by Z_a, Z_l, Z_b the intersection points between l^{per} with C_a,l, and C_b, respectively (which exist since all A_j are contained in one of the circles C_a or C_b).

Due to the symmetry we have $\delta_j = |Z_a - Z_l| = |Z_b - Z_l|$. We now want to relate the distance from a point A_j to the line l with the distances from A_j to Z_a and Z_b. To this end, we denote

$$\begin{array}{ccc} \mathcal{C}_a^- = \{X : d(X, Z_a) \leq d(X, l)\} & , & \mathcal{C}_a^+ = \{X : d(X, Z_a) > d(X, l)\} \\ \mathcal{C}_b^- = \{X : d(X, Z_b) < d(X, l)\} & , & \mathcal{C}_b^+ = \{X : d(X, Z_b) \geq d(X, l)\}; \end{array}$$

see Figure 1 for an illustration. Note that due to the construction $\mathcal{C}_a^+ = \mathcal{C}_b^-$,$\mathcal{C}_a^- = \mathcal{C}_b^+$. Further defining

$$\epsilon_j = \begin{cases} \delta_j & \text{if } j \in \mathcal{C}_a^+ \\ -\delta_j & \text{if } j \in \mathcal{C}_b^+ \end{cases}$$

we obtain $d(A_j, C_a) \leq d(A_j, l) + \epsilon_j$ and $d(A_j, C_b) \leq d(A_j, l) - \epsilon_j$ for all $j = 1, 2, \ldots, n$. Since $n \geq 5$ both inequalities hold strictly for at least one index.
If we further assume l to be optimal, we hence get

$$\begin{aligned} \sum_{j=1}^{n} d(A_j, l) &\leq \sum_{j=1}^{n} d(A_j, C_a) < \sum_{j=1}^{n} d(A_j, l) + \sum_{j=1}^{n} \epsilon_j, \text{ and} \\ \sum_{j=1}^{n} d(A_j, l) &\leq \sum_{j=1}^{n} d(A_j, C_b) < \sum_{j=1}^{n} d(A_j, l) - \sum_{j=1}^{n} \epsilon_j, \end{aligned}$$

which cannot be satisfied at the same time; a contradiction and hence l cannot be optimal. $\diamond$

Note that the strict result of Lemma 2 also holds in most cases for $n = 4$ existing facilities. There is only one exception, namely, if A_3 and A_4 are both on the bisector B_{A_1,A_2} of A_1 and A_2 on opposite sides of l. In this constellation the line l through A_1 and A_2 has the same objective value as the two circles C_a and C_b in the proof.

For line location it is well known that there always exists an optimal line intersecting two of the given existing facilities. We now discuss the question, if such a property also holds for the location of a circle. The first result that we mention has already been shown in [3]. Since it is an important building block for the subsequent theorem, we state it with proof.

Lemma 3 *There exists an optimal circle for problem (P) which intersects at least one existing facility location.*

Proof:
From Lemma 1 we know that $r > 0$. Furthermore, if the optimal solution is a straight line ($r \to \infty$) we know from results for Euclidean line location (see [13]) that there is an optimal line which even intersects two of the A_j. Therefore we

only need to consider the case where the optimal solution has a finite radius, i.e. $0 < r < \infty$.

Take any circle (X, r). Fixing X but leaving r as variable turns out to be a one-dimensional location problem

$$\min f(r) = \sum_{j=1}^{n} w_j |d(A_j, X) - r|$$

for which it is well known that an optimal solution $r^* = d(A_{j^*}, X)$ exists; the resulting circle (X, r^*) intersects A_{j^*}. $\diamond$

The above result shows that the optimal radius satisfies the *median property*, yielding the following corollary. It says that the sum of weights inside an optimal circle and the sum of weights outside the circle cannot differ too much. The result will be useful later.

Corollary 1 *Let $C = (X, r)$ be an optimal solution of problem (P) with corresponding sets J_+, J_-, J_0. Then we have that*

$$\begin{aligned} \sum_{j \in J_- \cup J_0} w_j &\geq \sum_{j \in J_+} w_j, \quad \textit{and} \\ \sum_{j \in J_+ \cup J_0} w_j &\geq \sum_{j \in J_-} w_j, \quad \textit{or, equivalently} \\ \left| \sum_{j \in J_-} w_j - \sum_{j \in J_+} w_j \right| &\leq \sum_{j \in J_0} w_j. \end{aligned}$$

Lemma 3 shows that there exists an optimal solution intersecting one of the existing facilities. We first show that in general, there need not exist an optimal circle intersecting three of the existing facilities. The first example has been given in [11] thanks to [9]. A similar example is illustrated next.

Consider the following set of $n = 6$ existing facilities $A_1 = (0, 6)$, $A_2 = (-5, 0)$, $A_3 = (-4, 0)$, $A_4 = (4, 0)$, $A_5 = (5, 0)$, $A_6 = (0, -6)$ with equal weights $w_j = 1$ (see Figure 2). In this example, the circle $C^* = C((0, 0), 5)$ with center $(0, 0)$ and radius $r = 5$ leads to

$$f(C^*) = 4$$

which may be shown to be better than all circles passing through three of the existing facilities.

The example above and Lemma 3 now pose the question, if there always exists an optimal circle intersecting two existing facilities. This has already been mentioned as an open question in [11]. The next theorem gives a positive answer. It should be noted that this theorem extends a well known result for linear facilities ($r \to \infty$).

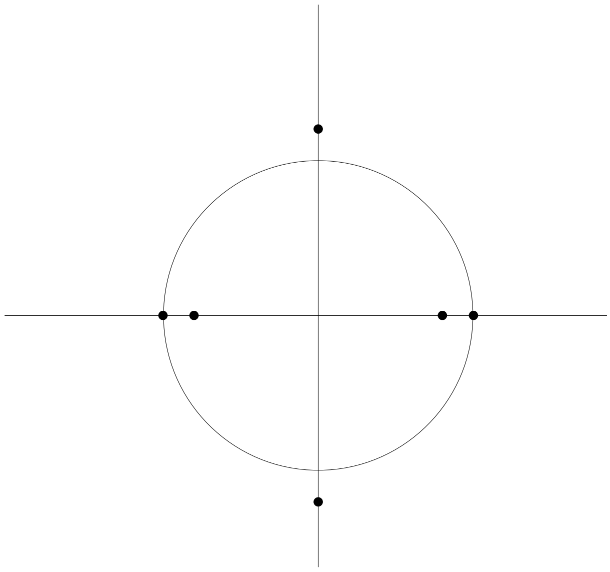

Figure 2: An example where no optimal circle intersects more than two existing facilities.

Theorem 1 *All optimal circles for problem (P) intersect at least two existing facility locations.*

Proof:
As in the preceding proof of Lemma 3 we only need to consider the case that $0 < r < \infty$: As before we know that $r > 0$ due to Lemma 1. If the optimal solution is a straight line ($r \to \infty$) we know from [4] that *all* optimal lines intersect at least two of the A_j.

Take an optimal circle (X, r'). From Lemma 3 we know that there exists a circle (X, r), $0 < r < \infty$ which has at least the same objective as (X, r') and intersects at least one of the A_j, say A_s. We consider two cases:

(i) X does not coincide with an existing facility, i.e. $X \neq A_t$ for all $t \in \{1, \ldots, n\}$:

Assuming that C intersects exactly A_s we perturb the center X of the circle, but adapting the radius such that the perturbed circle still intersects A_s. In a (sufficiently small) neighborhood about $X = (x, y)$ the objective function is hence differentiable and can be rewritten as

$$g(X) = \sum_{j \in J_-} w_j(d_s(X) - d_j(X)) + \sum_{j \in J_+} w_j(d_j(X) - d_s(X)). \qquad (1)$$

Using that

$$\begin{array}{rclrcl} \frac{\partial d_j}{\partial x} & = & -\cos \Theta_j, & \frac{\partial d_j}{\partial y} & = & -\sin \Theta_j, \\ \frac{\partial^2 d_j}{\partial x^2} & = & \frac{(\sin \Theta_j)^2}{d_j(X)}, & \frac{\partial^2 d_j}{\partial y^2} & = & \frac{(\cos \Theta_j)^2}{d_j(X)}, \end{array}$$

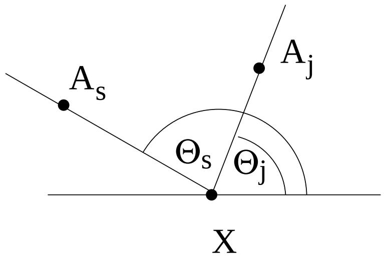

Figure 3: The definition of the angles in the formulas of the derivatives.

where $\cos\Theta_j = \frac{a_j - x}{d_j(X)}$, $\sin\Theta_j = \frac{b_j - y}{d_j(X)}$ (see Figure 3),
we obtain the following second derivatives of g:

$$\frac{\partial^2 g}{\partial x^2} = \sum_{j\in J_-} w_j \left(\frac{(\sin\Theta_s)^2}{d_s(X)} - \frac{(\sin\Theta_j)^2}{d_j(X)} \right) + \sum_{j\in J_+} w_j \left(\frac{(\sin\Theta_j)^2}{d_j(X)} - \frac{(\sin\Theta_s)^2}{d_s(X)} \right),$$

$$\frac{\partial^2 g}{\partial y^2} = \sum_{j\in J_-} w_j \left(\frac{(\cos\Theta_s)^2}{d_s(X)} - \frac{(\cos\Theta_j)^2}{d_j(X)} \right) + \sum_{j\in J_+} w_j \left(\frac{(\cos\Theta_j)^2}{d_j(X)} - \frac{(\cos\Theta_s)^2}{d_s(X)} \right).$$

It follows that

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \sum_{j\in J_-} w_j \left(\frac{1}{d_s(X)} - \frac{1}{d_j(X)} \right) + \sum_{j\in J_+} w_j \left(\frac{1}{d_j(X)} - \frac{1}{d_s(X)} \right) < 0,$$

since $d_s(X) > d_j(X)$ for all $j \in J_-$ and $d_s(X) < d_j(X)$ for all $j \in J_+$. Hence we conclude that at least one of these second-order derivatives is negative, such that $g(X)$ cannot be a local minimum. Hence there exists a circle C^* which is strictly better than the circle (X, r) (and hence also than the original circle (X, r')).

(ii) X coincides with an existing facility, i.e. $X = A_t$ for some $t \in \{1, \dots, n\} \setminus \{s\}$:

In this case, the objective function (1) as treated in Case (i) is not differentiable. Hence we separate the term for $j = t$ and obtain

$$g(X) = w_t(d_s(X) - d_t(X)) \quad (2)$$
$$+ \sum_{j\in J_- \setminus\{t\}} w_j(d_s(X) - d_j(X)) + \sum_{j\in J_+} w_j(d_j(X) - d_s(X)). \quad (3)$$

The second part of the objective (3) describes a reduced circle location problem without A_t. From the first case we know that $X = A_t$ cannot be optimal for the reduced problem. Moreover, for $X = A_t$ the first part (2) of our objective obtains a local maximum, since

$$\begin{aligned} & d_s(X) - d_t(X) \leq d_s(A_t) - d_t(A_t) \\ \iff & d(A_s, X) \leq d(A_s, A_t) + d(A_t, X), \end{aligned}$$

and the latter holds for all X due to the triangle inequality. Together, $X = A_t$ cannot be an optimal solution.

In summary, we have shown that all optimal circles must intersect at least two of the existing facilities. $\diamond$

4 Solution approaches for the variable radius case

The objective function $f(X, r)$ is observed above to contain two sums: one with positive weighted Euclidean distances for the existing facilities outside the circle ($j \in J_+$), and one with negative weighted Euclidean distances for those within the circle ($j \in J_-$). Furthermore, the sets J_+ and J_- depend on the center X and on the radius r of the circle. It follows that $f(X, r)$ has a complex shape that is non-convex in general. This makes the problem difficult to solve relative to its counterpart, the location of a single point facility in the plane. Due to the non-convexity, a local search will only guarantee a local optimum. Such procedures are further complicated by non-differentiability of $f(X, r)$ whenever J_+ or J_- changes (i.e. one or more existing facilities are added or removed from J_0) or when X coincides with an existing facility. In the following we will use the discretization approach of the previous section to design an algorithm for the circle location problem.

First of all, by Lemma 2 we do not have to consider the limiting case $r \to \infty$, if no triple of the given facilities is collinear. In case that the points are not in general position, we have to check all lines passing through at least two of the existing points to find the best possible line. Checking all lines requires $O(n^3)$ time, but more sophisticated approaches which solve the Euclidean line location problem in $O(n^2)$ are available, see [6, 4].

As pointed out above, dealing with circles with finite radius is more complicated. Based on Theorem 1, however, we know that the center point X of each optimal solution lies on a bisector B_{st} of a pair of existing facilities A_s and A_t, such that we may reduce the search for the optimal solution to a series of one-dimensional searches along all bisectors. We analyze the situation if we move X along the bisector B_{st}.

Since the Euclidean distance is invariant under axis rotation and translation, the reference axes may be reoriented such that the bisector B_{st} becomes the x-axis and the origin is at the mid-point of the line segment $[A_s, A_t]$. We hence may assume that

$$A_s = (0, b_s),\ A_t = (0, -b_s),$$

while the coordinates of all other existing facilities are given as $A_j = (a_j, b_j)$. Since the circle (X, r) we are looking for is required to intersect A_s and A_t we obtain

$$X = (x, 0) \text{ and } r = d_s(X) = \sqrt{x^2 + b_s^2}.$$

The objective function hence is only dependent on $x \in \mathbb{R}$ and takes the form

$$g(x) = \sum_{j \in J_-} w_j \left(\sqrt{x^2 + b_s^2} - \sqrt{(x - a_j)^2 + b_j^2} \right) + \sum_{j \in J_+} w_j \left(\sqrt{(x - a_j)^2 + b_j^2} - \sqrt{x^2 + b_s^2} \right). \quad (4)$$

Beginning at the origin and moving the center X of the circle to the left (or to the right) along the bisector, we see that the circle radius increases, and points will leave J_- and enter J_+ or vice versa. Each point A_j, non-collinear with A_s and A_t has a unique *intersection point* $X_j = (x_j, 0)$ where the three bisectors B_{st}, B_{sj} and B_{tj} intersect, and where this transition occurs. For the circle with center X_j it holds that

$$\{A_j, A_s, A_t\} \subseteq J_0,$$

i.e., the circle intersects A_j at $x = x_j$, while A_j is outside the circle for all x on one side of x_j and inside the circle for all x on the opposite side of x_j.

Thus, there are $\mathrm{O}(n)$ intersection points X_j on the bisector B_{st} that can be ordered from left to right in $O(n \log n)$ time. Note that the sets J_-, J_+, J_0 only can change at intersection points, i.e. in the interior of each interval I between adjacent intersection points they are independent of the specific point $X \in I$. We now may eliminate segments $I \subseteq B_{st}$ between adjacent intersection points that do not comply with Corollary 1. More precisely, whenever for any X in I we have that

$$\left| \sum_{j \in J_-} w_j - \sum_{j \in J_+} w_j \right| > w_s + w_t,$$

we can eliminate the segment $I \subseteq B_{st}$ which contains x.

As an example, we took the instance of a circle location problem depicted in Figure 2. In Figure 4 we graphed the bisector for each pair of existing facilities. The relevant part of the bisectors (i.e. the sections that might contain an optimal solution and hence have to be analyzed) are denoted as $\mathcal{B}$. They are indicated in bold in Figure 4 for the sample problem in Figure 2. Note that the Median-Voronoi diagram is a strict subset of $\mathcal{B}$.

If the numerical search along each of the remaining $\mathrm{O}(n)$ eligible segments of B_{st} is bounded by $\mathrm{O}(Kn)$, the optimal solution on B_{st} is obtained to a desired accuracy

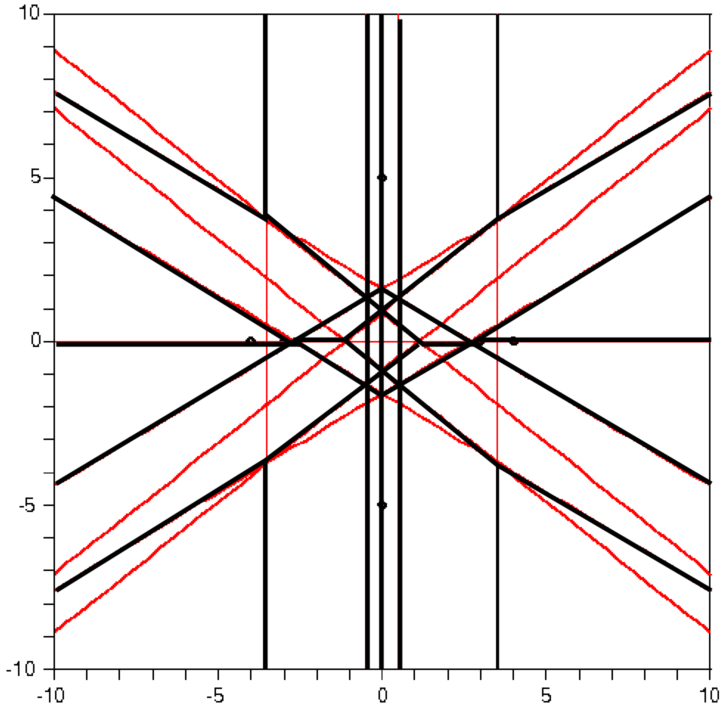

Figure 4: The bisectors and their relevant parts $\mathcal{B}$ (shown in bold) for the example depicted in Figure 2.

in $O(Kn^2)$ time. As there are $O(n^2)$ bisectors to examine, the complexity of the solution procedure is bounded by $O(Kn^4)$, making the algorithm suitable for smaller problem instances.

For larger problem instances we derive another result, justifying that in the case of many existing facilities the optimal circle is very likely to contain three of them. Being more specific, let us consider the limiting case (P_lim) defined below:

1. The number of existing facilities $n \to \infty$.
2. The existing facilities are given as (r, Θ), where $r \in \mathbb{R}$ and $\Theta \in [\theta_1, \theta_2]$ are two independent random variables measured from optimal center X^* obtained in the limit $n \to \infty$, and the random variable r is bounded.
3. The points $A_j = Y$ with positive probability in the distribution do not form the contour of a circle.
4. The distribution of the angles Θ_j does not differ inside and outside the optimal circle $C^* = (X^*, r^*)$. Formally, we require

$$E(\sin^2 \Theta_j(X^*)|j \in J_+(C^*)) = E(\sin^2 \Theta_j(X^*)|j \in J_-(C^*)),$$

   where $E(x|M)$ denotes the expectation of the random variable x given the event M.
5. All weights w_j are equal, and may be set to $1/n$.

In the following we will see that the asymptotic behavior of this problem reveals a useful property. First, we analyze the distribution of the existing facilities A_j. From Conditions 2 and 3, we obtain

$$E(\frac{1}{d_j(X^*)}|j \in J_+(C^*)) * c = E(\frac{1}{d_j(X^*)}|j \in J_-(C^*))$$

where the constant $c > 1$.

As an example, consider the case where the A_j are uniformly distributed on a disc of unit radius (the density functions of r and Θ are, respectively, $2r$ for $0 \le r \le 1$ and $\frac{1}{2\pi}$ for $0 \le \Theta \le 2\pi$). With $X^* = (0, 0)$, the center of the disc, calculations show that $r^* = 1/\sqrt{2}, E(\frac{1}{d_j(X^*)}|j \in J_+(C^*)) = 2(2-\sqrt{2}), E(\frac{1}{d_j(X^*)}|j \in J_-(C^*)) = 2\sqrt{2}$; hence $c = \sqrt{2}/(2-\sqrt{2}) > 1$.

Lemma 4 *In the limiting case (P_lim) the optimal circle intersects at least three of the existing facilities.*

Proof:
According to Theorem 1 we know that any optimal solution is located on one of the bisectors B_{st} between two existing facilities A_s and A_t. We hence consider the objective function g along such a bisector B_{st} as given in (4). Our goal is to show that g is concave between any pair of adjacent intersection points X_j (defined on page 10) in the vicinity of X^*. To this end, we calculate the second derivative of g by looking at the derivatives of the terms appearing in (4).

$$\begin{aligned} d_j(x) &= \sqrt{(a_j - x)^2 + b_j^2}, \\ d_j'(x) &= -\frac{(a_j - x)}{d_j(x)} = -\cos\Theta_j, \\ d_j''(x) &= \frac{b_j^2}{(d_j(x))^3} = \frac{\sin^2\Theta_j}{d_j(x)}. \end{aligned}$$

We obtain

$$g'(x) = \cos\Theta_s \left(\sum_{j\in J_+} w_j - \sum_{j\in J_-} w_j \right) + \sum_{j\in J_-} w_j \cos\Theta_j - \sum_{j\in J_+} w_j \cos\Theta_j \quad (5)$$

$$\begin{aligned} g''(x) &= \sum_{j\in J_-} w_j \left(\frac{\sin^2\Theta_s}{d_s} - \frac{\sin^2\Theta_j}{d_j} \right) + \sum_{j\in J_+} w_j \left(\frac{\sin^2\Theta_j}{d_j} - \frac{\sin^2\Theta_s}{d_s} \right) \\ &= \left(\sum_{j\in J_-} w_j - \sum_{j\in J_+} w_j \right) \frac{\sin^2\Theta_s}{d_s} - \sum_{j\in J_-} w_j \frac{\sin^2\Theta_j}{d_j} + \sum_{j\in J_+} w_j \frac{\sin^2\Theta_j}{d_j}, \quad (6) \end{aligned}$$

where in all expressions d_j, d_s and Θ_j, Θ_s depend on the variable X (i.e. on the center of the circle we are looking for). As remarked before the sets J_- and J_+ do not change between any pair of adjacent intersection points X_j.

Now let us fix some point X as center of a circle and assume that the median circle with center X only intersects A_s and A_t, i.e. $|J_0| = 2$. Recall that $w_j = 1/n$ for all $j = 1, \ldots, n$, hence we know from Corollary 1 that $\left|\sum_{j\in J_-} w_j - \sum_{j\in J_+} w_j\right| \le 2/n$. Since $\sin^2(\Theta_s)$ is bounded and the median radius $d_s(x)$ approaches a finite value as $n \to \infty$, the first term of the second derivative $\to 0$. We now compare the second and the third term using a stochastic approximation which is valid if $|J_-|$ and $|J_+|$ are large enough:

$$\begin{aligned} \sum_{j\in J_-} \frac{\sin^2\Theta_j}{d_j} &\approx |J_-| E\left(\frac{\sin^2\Theta_j}{d_j} | j \in J_- \right) \\ &= |J_-| E(\sin^2\Theta_j | j \in J_-) E(\frac{1}{d_j} | j \in J_-) \quad (\Theta_j, d_j \text{ are independent}) \\ &= \frac{|J_-|}{|J_+|} |J_+| \cdot \frac{E(\sin^2\Theta_j | j \in J_-)}{E(\sin^2\Theta_j | j \in J_+)} \cdot E(\sin^2\Theta_j | j \in J_+) \end{aligned}$$

$$
\begin{aligned}
& \cdot \frac{E(\frac{1}{d_j}|j \in J_-)}{E(\frac{1}{d_j}|j \in J_+)} \cdot E(\frac{1}{d_j}|j \in J_+) \\
\approx\ & \frac{|J_-|}{|J_+|} \cdot |J_+| \cdot E(\sin^2 \Theta_j | j \in J_+) \cdot c \cdot E(\frac{1}{d_j}|j \in J_+) \\
\approx\ & c \sum_{j \in J_+} \frac{\sin^2 \Theta_j}{d_j}, \quad \text{since } \frac{|J_-|}{|J_+|} \approx 1.
\end{aligned}
$$

Summarizing,

$$
-\sum_{j \in J_-} \frac{\sin^2 \Theta_j}{d_j} + \sum_{j \in J_+} \frac{\sin^2 \Theta_j}{d_j} \approx (1-c) \sum_{j \in J_+} \frac{\sin^2 \Theta_j}{d_j} < 0,
$$

and hence, $g(x)$ is strictly concave between the pair of adjacent intersection points X_j. Consequently, the optimal center X^* in the limiting sense must coincide with an X_j. This means that the optimal circle intersects at least three of the existing facilities. $\diamond$

This result hence allows the following (heuristic) approach for examples with larger n: Determine for each triple of existing facilities A_s, A_t, A_j the circle C_{stj} intersecting all three of them and take the best of these circles. The center point of the circle C_{stj} is the intersection point of two of the three bisectors B_{st}, B_{sj}, B_{tj}. In contrast to the first approach presented we do not require any numerical search along the relevant segments of the bisectors.

5 Finding a circle with fixed radius

When the radius of the circle is given, the only decision variable is the center of the circle, and we consider the problem (Pr)

$$
\min f_r(X) = \sum_{j=1}^{n} w_j |d(X, A_j) - r|.
$$

We first remark that an optimal circle with fixed radius need not intersect any of the existing facilities, as the following example shows: Consider $n = 6$ existing facilities given by $A_1 = (1.1 \cos 60^o, 1.1 \sin 60^o)$, $A_2 = (1.1 \cos 60^o, -1.1 \sin 60^o)$, $A_3 = (-1.1, 0)$, $A_4 = (0.9 \cos 60^o, 0.9 \sin 60^o)$, $A_5 = (0.9 \cos 60^o, -0.9 \sin 60^o)$, and $A_6 = (-0.9, 0)$ (see Figure 5 for an illustration). Assume that the weights $w_1 = w_2 = w_3 = 100$ and $w_4 = w_5 = w_6 = 1$. The radius r should be fixed to 1. Then the (unique) best circle with radius 1 is the circle $((0,0), 1)$ with center $X^* = (0,0)$, which does not intersect any of the existing facilities.

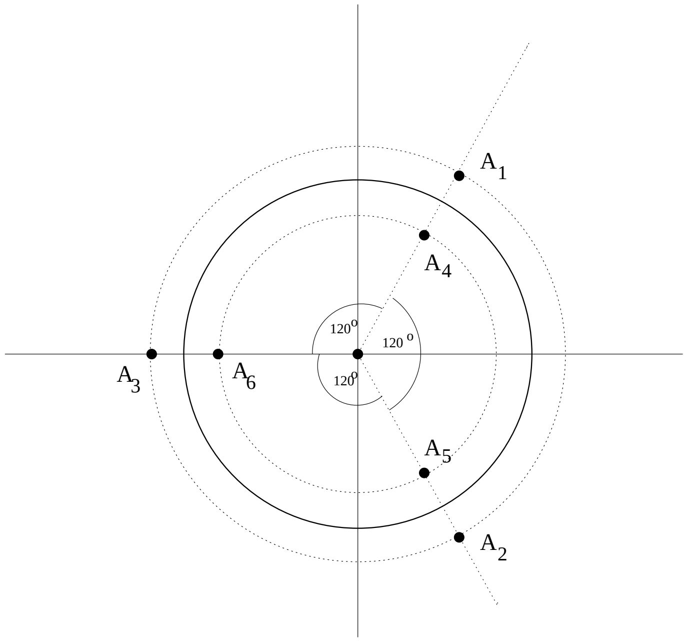

Figure 5: An example where no optimal circle with fixed radius intersects an existing facility.

The example illustrates that we need not look for results similar to Lemma 3 or Theorem 1. However, we will in the following show some other interesting properties. The first relates problem (Pr) to the well-known Weber problem, (e.g., see [7]), where the goal is to find a point $X \in \mathbb{R}^2$ minimizing the sum of distances to the existing facilities, i.e.,

$$\text{minimize} \quad \sum_{j=1}^{n} w_j d(X, A_j).$$

The solution to this problem turns out to be the solution to our problem, when the given radius is sufficiently small.

Lemma 5 *Let X^* be an optimal solution to the Weber problem, and assume that $d(X^*, A_j) \geq r$ for $j = 1, \ldots, n$. Then X^* is an optimal solution to problem (Pr) with given radius r.*

Proof:
Let X be an arbitrary point in the plane. We have

$$f_r(X^*) \quad = \quad \sum_{j=1}^{n} w_j |d(X^*, A_j) - r|$$

$$\begin{aligned} &= \sum_{j=1}^{n} w_j d(X^*, A_j) - \sum_{j=1}^{n} w_j r \\ &\leq \sum_{j=1}^{n} w_j d(X, A_j) - \sum_{j=1}^{n} w_j r \\ &\leq \sum_{j=1}^{n} w_j |d(X, A_j) - r| = f_r(X). \end{aligned}$$

Note also that if the existing facilities are not collinear, the Weber objective function is strictly convex (e.g., see [7]), and therefore, the first inequality in the preceding relation is satisfied in a strict sense, for all $X \neq X^*$. $\diamond$

The previous lemma deals with the case of an empty optimal circle, i.e., if $J_- = \emptyset$. The next result, however, presents cases in which the circle either intersects or contains at least one existing facility.

Lemma 6 *Let E^* denote the set of optimal solutions of the Weber problem, and let*

$$\underline{d} = \max_{X \in E^*} \{ \min_{j=1,\ldots,n} d(X, A_j) \}.$$

If $r \geq \underline{d}$ then $|J_0 \cup J_-| \geq 1$ in any optimal solution of (Pr).

Proof:
Let $X^* \in E^*$ denote an optimal solution of the Weber problem with

$$\min_{j=1,\ldots,n} d(X^*, A_j) = \underline{d}.$$

Consider any point X (with corresponding circle (X, r)) such that J_0 and J_- are both empty. Then $X \notin E^*$ since $r \geq \underline{d}$. Now move X along the line segment $[X, X^*]$ towards X^*, until the circle first touches an existing facility, and let $Y \in [X, X^*]$ denote the point where this occurs. Due to the convexity of the Weber objective function, $X^* - X$ is a descent direction and hence

$$\begin{aligned} f_r(Y) &= \sum_{j=1}^{n} w_j |d_j(Y) - r| = \sum_{j=1}^{n} w_j d_j(Y) - \sum_{j=1}^{n} w_j r \\ &< \sum_{j=1}^{n} w_j d_j(X) - \sum_{j=1}^{n} w_j r = f_r(X). \end{aligned}$$

Consequently, X cannot be an optimal solution of (Pr). $\diamond$

Combining Lemma 5 with Lemma 6 we see that if $r \leq \underline{d}$, a solution of (Pr) is readily obtained from the set E^*, and $J_- = \emptyset$ in this case. On the other hand, if $r > \underline{d}$, the (closed) disc formed by an optimal circle must contain at least one of the fixed points (i.e. $J_0 \cup J_- \neq \emptyset$). The next observation considers the opposite case of a circle containing all existing facilities in its interior.

Lemma 7 *Let (X, r) be a circle such that $d(X, A_j) < r$ for $j = 1, \ldots, n$. Then X is not an optimal solution to problem (Pr) with given radius r.*

Proof:
For X we obtain

$$\begin{aligned} f_r(X) &= \sum_{j=1}^{n} w_j |d(X, A_j) - r| \\ &= -\sum_{j=1}^{n} w_j d(X, A_j) + r \sum_{j=1}^{n} w_j, \end{aligned}$$

where the first part of this function is the negative of the classical Weber function and strictly concave in X if the A_j are noncollinear, and strictly concave in at least one direction otherwise, while the second part of the objective is constant. Consequently, no X belonging to

$$\{X : d(X, A_j) < r, j = 1, \ldots, n\} = \bigcap_{j=1,\ldots,n} \text{int}(A_j, r),$$

can be optimal, where $\text{int}(A_j, r)$ denotes the set of points contained in the interior of the circle $C = (A_j, r)$. Hence, in the optimal case, $J_+ \cup J_0 \neq \emptyset$. $\diamond$

If a circle does not intersect any existing facility, the sum of the outside weights must be larger than the sum of the inside weights for the circle to be optimal, as shown in the following result.

Lemma 8 *If an optimal solution to problem (Pr) has $J_0 = \emptyset$, then*

$$\sum_{j \in J_+} w_j > \sum_{j \in J_-} w_j.$$

Proof:
In a similar fashion as with variable radius, it may be shown that $X = A_j$ cannot be an optimal solution for any j. Hence we need only consider solutions X that do not coincide with an existing facility, for which the objective function is infinitely differentiable. Furthermore, since $J_0 = \emptyset$, if follows from Lemma 7 that $J_+ \neq \emptyset$ at the optimal solution being considered.
The objective function may be written as

$$f_r(X) = \sum_{j \in J_+} w_j (d(X, A_j) - r) + \sum_{j \in J_-} w_j (r - d(X, A_j)).$$

After differentiation we obtain

$$\frac{\partial^2 f_r(X)}{\partial x^2} + \frac{\partial^2 f_r(X)}{\partial y^2} = \sum_{j \in J_+} \frac{w_j}{d_j(X)} - \sum_{j \in J_-} \frac{w_j}{d_j(X)}.$$

Consider an optimal solution, X^*, and for contradiction assume that

$$\sum_{j \in J_+} w_j \leq \sum_{j \in J_-} w_j.$$

Then we have

$$\begin{aligned} \frac{\partial^2 f_r(X^*)}{\partial x^2} + \frac{\partial^2 f_r(X^*)}{\partial y^2} &= \sum_{j \in J_+} \frac{w_j}{d_j(X^*)} - \sum_{j \in J_-} \frac{w_j}{d_j(X^*)} \\ &< \sum_{j \in J_+} \frac{w_j}{r} - \sum_{j \in J_-} \frac{w_j}{r} \leq 0. \end{aligned}$$

Therefore at least one of the second-order partial derivatives would have to be negative, contradicting the fact that the objective function achieves a local minimum at X^*. $\diamond$

The stated properties may be embedded in a general branch-and-bound procedure such as the big square small square (BSSS) method [10] to simplify the search for an optimal solution of problem (Pr). The steps are outlined below.

Step 1.
Solve the associated Weber problem ($r = 0$) to obtain the median point X_m. If Lemma 5 is satisfied, stop; X_m is an optimal solution of problem (Pr); otherwise use Lemma 6 to set the smallest rectangle that must contain an optimal solution.

Step 2.
Use a general branch-and-bound procedure where the original rectangle in step 1 is divided systematically into progressively smaller cells as needed, until an optimal solution is determined to a desired accuracy. A lower bound on the objective function for any cell G may be calculated as follows:

$$LB = \sum_{j=1}^{n} w_j \max\{0, \min\{r - \overline{d_j}, \underline{d_j} - r\}\},$$

where $\underline{d_j} = \min_{X \in G} d_j(X)$, $\overline{d_j} = \max_{X \in G} d_j(X)$, $j = 1, \ldots, n$, are easily determined from the coordinates of the four corner points of G. An upper bound is readily obtained by calculating the objective function at the centroid X_c of G; $UB = f_r(X_c)$. Lemmas 7 and 8 may be incorporated as additional fathoming rules.

6 Conclusion

We have considered the problem of locating a circle on the plane so as to minimize the sum of weighted distances between some given facilities and the circle. The main result is that any optimal circle intersects at least two facilities. This has allowed us to develop a solution procedure with complexity $O(n^4)$, where n is the number of facilities. In many cases the optimal circle will intersect three facilities, so a heuristic procedure is to consider the circles based on all triplets, and pick the best one.

We also considered the special case, where the radius of the circle was given. For this situation we investigated several properties, allowing us to solve the problem quite efficiently in many cases.

Plans for future research include: using other norms, such as rectangular, general ℓ_p and block norms; considering the multi-circle problem with potential applications in clustering and data mining; and programming and testing the algorithms.

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