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The Big Cube Small Cube Solution Method for Multidimensional Facility Location Problems

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Abstract

In this paper we propose a general solution method for (non-differentiable) facility location problems with more than two variables as an extension of the big square small square technique (BSSS). We develop a general framework based on lower bounds and discarding tests for every location problem. We demonstrate our approach on three problems: the Fermat-Weber problem with positive and negative weights, the median circle problem, and the p-median problem. For each of these problems we show how to calculate lower bounds and discarding tests. Computational experiences are given which show that the proposed solution method is fast and exact.

Keywords: approximation algorithms, facility location problem, p-median problem, Fermat-Weber problem, continuous location, global optimization, non-differentiable optimization.

1 Introduction

In this paper, we deal with some continuous, non-convex, and even non-differentiable location problems with up to six variables. Our solution method is a generalization of the big square small square technique to the multidimensional case.

Hansen et al. (1985) suggested the **big square small square** (BSSS) technique for some location problems on the plane with two variables. Plastria (1992) generalized this method to the **generalized big square small square** (GBSSS) technique. Using triangles instead of squares, Drezner and Suzuki (2004) proposed the **big triangle small triangle** (BTST) method, which was further generalized in Drezner (2007). Note that all these techniques are branch-and-bound solution methods for problems on the plane with two variables. They require lower bounds for each square or triangle.

The interval branch-and-bound algorithm is a more general optimization technique, which can be applied to problems in every dimension using interval analysis, see Hansen (1979), Hansen (1980), or Hansen (1992) for a survey on global optimization using interval analysis. Here, lower bounds are obtained by inclusion functions and the main task is to find efficient discarding tests. Apart from Hansen (1992), some discarding tests are suggested in Tóth et al. (2007), Fernández et al. (2007), and Fernández et al. (2006). Note that for both, lower bounds using inclusion functions and most of the discarding tests, a differentiable objective function and information about the gradient are required.

Since in most location problems, the objective functions are not differentiable, we will develop specific lower bounds and moreover, present some special discarding tests for the problems we consider. We will demonstrate our algorithm on three problems: the Fermat-Weber problem with positive and negative weights, the median circle problem, and the p-median problem.

The **Fermat-Weber problem** (see, e.g. Drezner et al. (2001) for a survey) is to find a location for a new facility which minimizes the weighted sum of Euclidean distances to a set of existing facilities. In the case that all weights are positive, these problems are convex and can be efficiently solved by a fixed point iteration, see Weiszfeld (1937) and Brimberg (1995). But if some facilities are disadvantageous, we have to deal with some negative weights. Nickel and Dudenhöffer (1997) calculated a finite dominating set (FDS) for the case that all distance functions are polyhedral gauges. Moreover, since the objective functions for these problems are a difference of two convex functions, Tuy et al. (1995) suggest to approximate the solution with the global D.C. optimization method. Conversely, Drezner and Suzuki (2004) used the BTST method and could show that their technique is much more efficient for Fermat-Weber problems on the plane than the D.C. method. Our goal is to solve the *three-dimensional* Fermat-Weber problem with positive and negative weights.

Considering irrigation pipes or ring roads, the **median circle problem** is to locate a circle on the plane which minimizes the sum of weighted distances between the circle and a set of n existing facilities. This problem was first formulated and analyzed in Drezner et al. (2002). Further investigations can be found in Schöbel (1999) and Brimberg et al. (2008).

The latter suggest a solution approach with a run time of $O(n^4)$. Using our solution method, we will show that on average the run time is almost proportional to the number of existing facilities.

The p**-median problem** has $2p$ variables: the problem is to locate p new facilities on the plane which minimize the weighted sum of distances to a set of existing facilities. Berman and Drezner (2007) solved the 2-median problem on a network, while Lau et al. (2001) used two dimensional meshes. The continuous p-median problem was revisited by Chen et al. (1998), solving the problem with the D.C. method. We will apply our solution method to the 2- and 3-median problem.

The remainder of our paper is organized as follows. In the next section we will extend the BSSS method to more than two variables. Section 3 will present notations and formulations for our three facility location problems. Before applying the solution method, we have to calculate some lower bounds for each of the problems, see Section 4. In the subsequent Section 5, we will discuss division rules and discarding tests. Using the presented lower bounds and discarding tests, Section 6 shows some of our computational results. Finally, we give a brief discussion.

2 The Big Cube Small Cube Method

This section describes the proposed solution method and can be applied to minimize an arbitrary function $f : \mathbb{R}^m \to \mathbb{R}$. Consider a feasible area C, which is an **orthotope** or shortly **cube** with sides parallel to the axes, say

$$C \;=\; [\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_m, \overline{x}_m] \;\subset\; \mathbb{R}^m.$$

In order to solve this problem we suggest the following algorithm.

Big-Cube-Small-Cube-Algorithm (BCSC)

Input A function $f : \mathbb{R}^m \to \mathbb{R}$ and a feasible cube $C \subset \mathbb{R}^m$.

Initialization (Phase 1)

1. Create a list $\mathcal{C}$ of cubes. Initialize $\mathcal{C} = \{C\}$, i.e. the list contains only the cube C.
2. The value of the objective function is evaluated at the center
$$c(C) := \left(\frac{1}{2}(\underline{x}_1, \overline{x}_1), \;\ldots\;, \;\frac{1}{2}(\underline{x}_m, \overline{x}_m)\right)$$
of C. The resulting value $f(c(C))$ is taken as **upper bound** UB and the center itself as incumbent $(x_1, \ldots, x_m)_{opt}$.
3. Calculate a **lower bound** $LB(C)$ for C.
(How to calculate good lower bounds is described in Section 4.)

Branch-and-Bound Phase (Phase 2)

1. **Selection rule**: Find the lowest lower bound LB_{min} for all cubes C' in $\mathcal{C}$.
2. **Termination rule**: If $LB_{min} + \varepsilon \cdot |LB_{min}| \geq UB$ stop: Return the optimal solution $(x_1, \ldots, x_m)_{opt}$ and UB as its value of the objective function.
3. **Division rule**: Select a cube $C' \in \mathcal{C}$ with $LB(C') = LB_{min}$ and split it into some smaller cubes.

   (How to use division rules to split the selected cube is shown in Section 5.)
4. Evaluate the value of the objective function at the center of each smaller cube. If at least one of these values is $\leq UB$, update UB to the lowest value of all smaller cubes and update $(x_1, \ldots, x_m)_{opt}$ to the center of the associated cube.
5. Calculate a lower bound for all smaller cubes (see Section 4).
6. Add the smaller cubes to $\mathcal{C}$ and delete the original cube from the list.
7. **Discarding test**: If $LB(C') + \varepsilon \cdot |LB(C')| > UB$ for a cube $C' \in \mathcal{C}$ discard it from $\mathcal{C}$. If UB has not changed it is sufficient to check only the smaller cubes of Step 6.
8. Apply some further discarding tests for the smaller cubes (see Section 5).
9. Return to Step 1 of Phase 2.

As shown in Drezner and Suzuki (2004) for the case $m = 2$ this algorithm returns an optimal solution $(x_1, \ldots, x_m)_{opt} \in C$ within a relative accuracy of $\varepsilon > 0$ if it terminates. In the following result we present a sufficient condition for the termination of our algorithm.

Lemma 1. *Suppose a function $f : \mathbb{R}^m \to \mathbb{R}$ and a feasible cube C. For all cubes*

$$C' \;=\; [\underline{x}'_1, \overline{x}'_1] \times \ldots \times [\underline{x}'_m, \overline{x}'_m] \;\subset\; C$$

denote by $c(C')$ the center of C' and define $\delta(C')$ as the length of the maximal width component of C', i.e.

$$\delta(C') \;=\; \max\{\overline{x}'_1 - \underline{x}'_1, \ldots, \overline{x}'_m - \underline{x}'_m\}.$$

Furthermore, assume $|LB(C')| \geq L$ for a fixed $L > 0$. Then the algorithm terminates after a finite number of steps for every $\varepsilon > 0$ if there exists a fixed constant $D > 0$ such that

$$f(c(C')) - LB(C') \;\leq\; D \cdot \delta(C') \tag{1}$$

for all cubes $C' \subset C$.

Proof. Suppose that Equation (1) is satisfied for all cubes $C' \subset C$. Then we have

$$UB - LB(C') \;\leq\; f(c(C')) - LB(C') \;\leq\; D \cdot \delta(C').$$

If all cubes in the list are small enough to satisfy

$$D \cdot \delta(C') \ < \ \varepsilon \cdot L$$

we have

$$UB - LB(C') \ < \ \varepsilon \cdot L \ \leq \ \varepsilon \cdot |LB(C')|$$

and therefore the termination condition of Step 2 is satisfied for all cubes C' if $\delta(C')$ is small enough. $\square$

Note that the assumptions given in Lemma 1 depend strongly on the objective function f and the bounding operation. For example, if f is Lipschitzian and strict positive, the calculation of L and D is very simple.

Moreover, we remark that ε is a relative accuracy. For problems with zero as minimal objective value, the termination condition of Step 2 may never be satisfied since we can not guarantee the existence of a constant $L > 0$ with $|LB(C')| \geq L$ for all cubes $C' \subset C$. In this case we suggest to use

$$LB_{min} + \varepsilon \cdot |LB_{min}| + \varepsilon_{abs} \ \geq \ UB,$$

i.e. to add a small absolute error $\varepsilon_{abs} > 0$ in the termination criteria. Along the lines of the proof of Lemma 1 we obtain the following results.

Lemma 2. *Consider the termination criteria*

$$LB_{min} + \varepsilon \cdot |LB_{min}| + \varepsilon_{abs} \ \geq \ UB.$$

If there exists a fixed constant $D > 0$ such that

$$f(c(C')) - LB(C') \ \leq \ D \cdot \delta(C')$$

for all cubes $C' \subset C$ the algorithm terminates after a finite number of steps for every $\varepsilon > 0$.

Furthermore, using the termination criteria

$$LB_{min} + \varepsilon_{abs} \ \geq \ UB$$

and assuming again that there exists a fixed constant $D > 0$ such that

$$f(c(C')) - LB(C') \ \leq \ D \cdot \delta(C')$$

for all cubes $C' \subset C$, the algorithm terminates after a finite number of steps and returns a solution with absolute error smaller than ε_{abs}.

3 Formulation of the location problems investigated

In this section we briefly introduce the three location problems for which we apply the BCSC algorithm. To this end we first need the following notation.

Let $\{A_1, \ldots, A_n\}$ be a given set of n existing facilities. In $\mathbb{R}^2$ we denote $A_k = (a_k, b_k)$, in $\mathbb{R}^3$ let $A_k = (a_k, b_k, c_k)$. The Euclidean distance between X and A_K is denoted as

$$d_k(X) \ := \ d(X, A_k).$$

Note that d_k is a convex function in X which is differentiable if $X \neq A_k$.

For each of the existing facilities we have furthermore given a weight $w_k \in \mathbb{R}$, $k = 1, \ldots, n$ representing the importance of the existing facility. In many location problems the weights are assumed to be positive.

The three-dimensional Fermat-Weber Problem with positive and negative weights

Let $A_k = (a_k, b_k, c_k)$, $k = 1, \ldots, n$ be a given set of existing facilities in three dimensions. In this problem we allow the weights w_k to be positive or negative. The Fermat-Weber problem with positive and negative weights is to minimize

$$f(X) \ = \ f(x, y, z) \ = \ \sum_{k=1}^{n} w_k \cdot d_k(X). \tag{2}$$

This function can be expressed as a difference between two convex functions. In general it is neither convex nor concave.

The Median Circle Problem

With positive weights $w_k > 0$, the problem is to locate a circle so as to minimize the sum of weighted distances between the circle and the facilities $A_k = (a_k, b_k)$ on the plane. Describing a circle by its center $X = (x, y)$ and its radius $r \geq 0$, we have to minimize the function

$$f(X, r) \ = \ f(x, y, r) \ = \ \sum_{k=1}^{n} w_k \cdot |d_k(X) - r|. \tag{3}$$

This is a problem on the plane, nevertheless involving three variables. Its objective function is neither convex nor concave and, furthermore, not differentiable at all solutions (X, r) with $d_k(X) = r$ and $k \in \{1, \ldots, n\}$. This is important, since any optimal solution of the median circle problem satisfies $d_k(X) = r$ for at least two $k \in \{1, \ldots, n\}$ according to Brimberg et al. (2008), and hence the optimal solution is always attained at a non-differentiable point.

The p-Median Problem

The p-median problem is to find p new locations

$$X_1 \;=\; (x_1, y_1),\; \ldots,\; X_p \;=\; (x_p, y_p)$$

on the plane. With positive weights $w_k > 0$ we have to minimize

$$f(X_1, \ldots, X_p) \;=\; f(x_1, y_1, \ldots, x_p, y_p) \;=\; \sum_{k=1}^{n} w_k \cdot \min\{d_k(X_1), \ldots, d_k(X_p)\}. \tag{4}$$

This problem deals with $2p$ variables and is in general $\mathcal{N}P$ hard, see Megiddo and Supowit (1984). We will solve the p-median problem for fixed $p = 2$ and $p = 3$.

4 Calculating Lower Bounds

For each cube $C \in \mathcal{C}$, we are interested in a lower bound on the value of the objective function inside that cube. One possible approach is to use interval analysis, see Hansen (1992). Here we develop specific lower bounds for the three problems we consider.

We are using the same concept for all three problems: We bound and approximate every summand of the objective function by a concave function from below. Since the sum of concave functions is concave again, we end up with a concave optimization problem over a cube whose optimal solution is a lower bound on the optimal solution of our original problem. The optimal solution of the concave approximation can be calculated easily since it is obtained at one of the vertices of the cube C. We now detail this approach for each of our three location problems.

The Fermat-Weber Problem

Consider the cube

$$C \;=\; [x_{min},\, x_{max}] \times [y_{min},\, y_{max}] \times [z_{min},\, z_{max}]$$

and one summand of the objective function, say

$$g_k(x, y, z) \;:=\; w_k \cdot d_k(x, y, z).$$

We are looking for a concave function $m_k(x, y, z)$ with $m_k(x, y, z) \;\leq\; g_k(x, y, z)$ for all $(x, y, z) \in C$.

For $w_k < 0$ the function $g_k(x, y, z)$ itself is concave and we assign $m_k(x, y, z) := g_k(x, y, z)$.

Now consider $k \in \{1, \ldots, n\}$ with $w_k > 0$. In this case $g_k(x, y, z)$ is convex. It can be bounded from below by its subgradient function, e.g. at the point

$$U \;=\; (u, v, w) \;=\; \left(\frac{1}{2}(x_{min} + x_{max}),\; \frac{1}{2}(y_{min} + y_{max}),\; \frac{1}{2}(z_{min} + z_{max})\right).$$

For $U \neq A_k$ the subgradient function is given by the tangent plane of f at U, i.e.

$$\begin{aligned} h_k(x,y,z) &:= g_k(U) + \frac{\partial g_k}{\partial x}(U)\cdot(x-u) + \frac{\partial g_k}{\partial y}(U)\cdot(y-v) + \frac{\partial g_k}{\partial z}(U)\cdot(z-w) \\ &= w_k \cdot \left(d_k(U) + \frac{(u-a_k)}{d_k(U)}\cdot(x-u) + \frac{(v-b_k)}{d_k(U)}\cdot(y-v) + \frac{(w-c_k)}{d_k(U)}\cdot(z-w) \right) \end{aligned}$$

and we assign $m_k(x,y,z) := h_k(x,y,z)$. On the other hand, if $U = A_k$ we choose the subgradient $m_k(x,y,z) = 0$. Summing up the functions $m_k(x,y,z)$ for $k = 1,\ldots,n$ at the vertices of C we obtain the following lower bound.

Theorem 3. *Let C be a cube and denote by $V_1,\ldots,V_8$ the eight vertices of C. Define*

$$m_k(x,y,z) = \begin{cases} g_k(x,y,z) & \text{if} \quad w_k < 0, \\ h_k(x,y,z) & \text{if} \quad w_k > 0 \text{ and } A_k \neq U, \\ 0 & \text{else} \end{cases} .$$

Then

$$LB_1(C) = \min\left\{ \sum_{k=1}^{n} m_k(V_1),\ \sum_{k=1}^{n} m_k(V_2),\ \ldots,\ \sum_{k=1}^{n} m_k(V_8) \right\}$$

is a lower bound for the Fermat-Weber problem on C.

Proof. We have to show that $f(x,y,z) \geq LB_1(C)$ for all $x \in C$. To this end, first note that $m_k(x,y,z) \geq g_k(x,y,z)$ for all $x \in C$. This is obvious if $w_k < 0$ since we defined $m_k(x,y,z) = g_k(x,y,z)$ in this case. On the other hand, if $w_k > 0$ we know that $g_k(x,y,z)$ is convex and hence always bounded from below by its subgradient.

Hence for all $(x,y,z) \in C$ we obtain

$$f(x,y,z) = \sum_{k=1}^{n} w_k \cdot d_k(x,y,z) = \sum_{k=1}^{n} g_k(x,y,z) \geq \sum_{k=1}^{n} m_k(x,y,z) =: \ell(x,y,z).$$

Note that each of the $m_k(x,y,z)$ is a concave function and hence $\ell(x,y,z)$ is a concave function itself. In order to find the minimum of a concave function over a polyhedron it is sufficient to investigate the extreme points of the polyhedron which are in our case the eight vertices of the cube C. This yields

$$\min_{(x,y,z)\in C} \ell(x,y,z) = \min\{\ell(V_1),\ldots,\ell(V_8)\} = LB_1(C)$$

and therefore finishes the proof. □

The Median Circle Problem

Consider the cube

$$C = [x_{min}, x_{max}] \times [y_{min}, y_{max}] \times [r_{min}, r_{max}]$$

and again one summand of the objective function, say

$$g_k(x,y,r) \ := \ w_k \cdot |d_k(x,y) - r| \ = \ w_k \cdot \max\{d_k(x,y) - r,\ r - d_k(x,y)\}.$$

As in the approach for the Fermat-Weber problem we are looking for concave functions $m_k(x,y,r)$ for $k = 1, \ldots, n$ with $m_k(x,y,r) \leq g_k(x,y,r)$ for all $(x,y,r) \in C$. We then can proceed analogously and minimize the sum $\ell(x,y,r) = \sum_{k=1}^{n} m_k(x,y,r)$ of these concave functions by evaluating it at the vertices of C.

To obtain the required concave functions m_k we consider the following three cases.

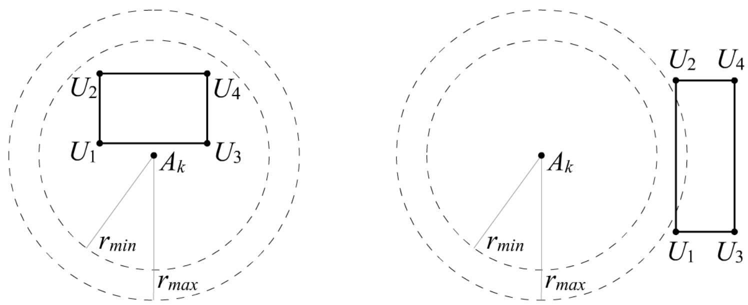

Figure 1: Examples for the two cases.

1. The four points

$$\begin{aligned} U_1 \ &:= \ (x_{min},\, y_{min}), \qquad & U_2 \ &:= \ (x_{min},\, y_{max}), \\ U_3 \ &:= \ (x_{max},\, y_{min}), \qquad & U_4 \ &:= \ (x_{max},\, y_{max}) \end{aligned}$$

are all located inside the circle with center A_k and radius r_{min}, see the left part of Figure 1 for an illustration. In this case we obtain that

$$g_k(x,y,r) \ = \ w_k \cdot (r - d_k(x,y))$$

is concave for all $(x,y,r) \in C$ and hence assign $m_k(x,y,r) \ := \ g_k(x,y,r)$.

2. The four points $U_1, \ldots, U_4$ are all located outside the circle with center A_k and radius r_{max}, see the right part of Figure 1. Here, we obtain

$$g_k(x,y,r) \ \geq \ w_k \cdot (d_k(x,y) - r)$$

for all $(x,y,r) \in C$. We hence define the convex function

$$q_k(x,y,r) \ := \ w_k \cdot (d_k(x,y) - r)$$

and bound it from below by its subgradient, e.g. at the point

$$T \ := \ (U,s) \ = \ (u,v,s) \ = \ \left(\frac{1}{2}(x_{min} + x_{max}),\ \frac{1}{2}(y_{min} + y_{max}),\ \frac{1}{2}(r_{min} + r_{max})\right).$$

For $U \neq A_k$ the subgradient of q in T is given by the tangent plant at T, i.e. by

$$\begin{aligned} n_k(x,y,r) &:= q_k(T) + \frac{\partial q_k}{\partial x}(T)\cdot(x-u) + \frac{\partial q_k}{\partial y}(T)\cdot(y-v) + \frac{\partial q_k}{\partial r}(T)\cdot(r-s) \\ &= q_k(T) + w_k \cdot \left(\frac{(u-a_k)}{d_k(U)}\cdot(x-u) + \frac{(v-b_k)}{d_k(U)}\cdot(y-v) - (r-s) \right) \\ &= w_k \cdot \left(d_k(U) + \frac{(u-a_k)}{d_k(U)}\cdot(x-u) + \frac{(v-b_k)}{d_k(U)}\cdot(y-v) - r \right) \end{aligned}$$

and we have that

$$g_k(x,y,r) \;\geq\; q_k(x,y,r) \;\geq\; n_k(x,y,r)$$

for all $(x,y,r) \in C$.

3. In all other cases there exists a $(x,y,r) \in C$ with $g_k(x,y,r) = 0$ and we define $m_k(x,y,r) := 0$ as (trivial) lower bound.

Proceeding analogously to Theorem 3 we have shown the following result.

Theorem 4. *Let C be a cube and denote by $V_1,\ldots,V_8$ the eight vertices of C. Define*

$$m_k(x,y,r) = \begin{cases} g_k(x,y,r) & \text{if} \quad d_k(U_j) \leq r_{min} \text{ for } j=1,\ldots,4, \\ n_k(x,y,r) & \text{if} \quad d_k(U_j) > r_{max} \text{ for } j=1,\ldots,4 \text{ and } A_k \neq U, \\ 0 & \text{else} \end{cases} .$$

Then

$$LB_2(C) = \min\left\{ \sum_{k=1}^{n} m_k(V_1),\; \sum_{k=1}^{n} m_k(V_2),\ldots,\; \sum_{k=1}^{n} m_k(V_8) \right\}$$

is a lower bound for the median circle problem.

The p-Median Problem

Consider the cube

$$C = [x^1_{min},\, x^1_{max}] \times [y^1_{min},\, y^1_{max}] \times \ldots \times [x^p_{min},\, x^p_{max}] \times [y^p_{min},\, y^p_{max}] \subset \mathbb{R}^{2n}$$

and one summand of the objective function, say

$$g_k(x_1,y_1,\ldots,x_p,y_p) := w_k \cdot \min\{d_k(x_1,y_1),\,\ldots\,,d_k(x_p,y_p)\}.$$

In the same way as before, we will bound g_k by a concave function m_k respecting

$$m_k(x_1,y_1,\ldots,x_p,y_p) \leq g_k(x_1,y_1,\ldots,x_p,y_p) \qquad \text{for all} \qquad (x_1,y_1,\ldots,x_p,y_p) \in C.$$

First, we define for $m = 1,\ldots,p$

$$P_m := (p_m, q_m) = \left(\frac{1}{2}(x^m_{min} + x^m_{max}),\; \frac{1}{2}(y^m_{min} + y^m_{max}) \right)$$

and we will use the supporting hyperplanes at these points:

$$\begin{aligned}
d_k(x_m, y_m) &\geq h_{k,m}(x_m, y_m) \\
&:= d_k(P_m) + \frac{\partial d_k}{\partial x_m}(P_m) \cdot (x_m - p_m) + \frac{\partial d_k}{\partial y_m}(P_m) \cdot (y_m - q_m) \\
&= d_k(P_m) + \frac{(p_m - a_k)}{d_k(P_m)} \cdot (x_m - p_m) + \frac{(q_m - b_k)}{d_k(P_m)} \cdot (y_m - q_m).
\end{aligned}$$

Next, we define the functions

$$m_k(x_1, y_1, \ldots, x_p, y_p) \;:=\; w_k \cdot \min\{h_{k,1}(x_1, y_1), \;\ldots\;, h_{k,p}(x_p, y_p)\}$$

if $d_k(P_m) \neq 0$ for all $k \in \{1, \ldots, n\}$, $m \in \{1, \ldots, p\}$, and $m_k(x_1, y_1, \ldots, x_p, y_p) := 0$ else. Note that m_k is concave since $h_{k,m}$ is linear and the minimum of linear functions is concave. Furthermore, we obtain

$$m_k(x_1, y_1, \ldots, x_p, y_p) \;\leq\; g_k(x_1, y_1, \ldots, x_p, y_p)$$

for all $(x_1, y_1, \ldots, x_p, y_p) \in C$. Summarizing our discussion we obtain the following result.

Theorem 5. *Let $C \subset \mathbb{R}^{2p}$ be a cube and denote by $V_1, \ldots, V_4^p$ the $2^{2p} = 4^p$ vertices of C. Then*

$$LB_3(C) \;=\; \min\left\{\sum_{k=1}^{n} m_k(V_1),\; \sum_{k=1}^{n} m_k(V_2),\; \ldots,\; \sum_{k=1}^{n} m_k(V_{4^p})\right\}$$

is a lower bound for the p-median problem.

5 Division Rule and Further Discarding Tests

There are several division rules and discarding tests for the interval branch-and-bound algorithm, see, e.g., Tóth et al. (2007), Fernández et al. (2007), and Fernández et al. (2006). Most of these rules and tests require twice differentiable objective functions. Since our functions are not even differentiable, we have to develop other approaches. We first describe our division rules and then suggest additional discarding tests for some of the location problems under consideration.

Division Rule

We will apply the following division rules.

For cubes in three dimensions (in the three-dimensional Fermat-Weber problem and in the circle location problem), the selected cube will be split into eight congruent smaller cubes as depicted in Figure 2.

Cuboids in higher dimensions (as in the p-median problem for $p \geq 2$) will be bisected in two smaller cuboids analogously to Tóth et al. (2007) perpendicular to the direction of the maximum width component.

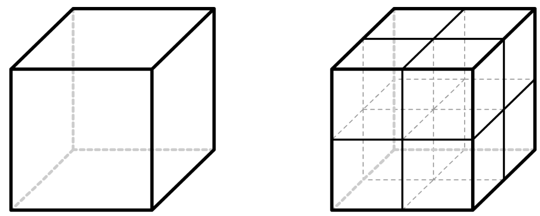

Figure 2: The split of a cube into eight small cubes.

Order Test for the p-median problem

If we are looking for p new locations as a vector in $\mathbb{R}^{2p}$ there are theoretically many symmetric solutions due to all possible permutations of the p new facilities. Our first discarding test aims to exclude such equivalent sets of new locations. This is done as follows.

We require

$$x_1 \ \leq \ x_2 \ \leq \ \ldots \ \leq \ x_p$$

for the optimal solution $(x_1, y_1, \ldots, x_p, y_p)_{opt}$. We then can delete all cubes

$$C \ = \ [x^1_{min}, \, x^1_{max}] \times [y^1_{min}, \, y^1_{max}] \times \ldots \times [x^p_{min}, \, x^p_{max}] \times [y^p_{min}, \, y^p_{max}]$$

with $x^m_{min} > x^s_{max}$ for some $m \in \{1, \ldots, p-1\}$ and $s \in \{m+1, p\}$. This reduces the number of cubes to be considered significantly.

Circle Feasibility Test for the median circle problem

The main result in Brimberg et al. (2008) is that all optimal solutions for the median circle problem intersect at least two existing facility locations. We use this result for a discarding test for the median circle problem as follows.

Consider an arbitrary cube C and check for every existing facility A_k if there is a circle $(X, r)_k \in C$ which intersects A_k. A cube C can be discarded from $\mathcal{C}$ if there is only one existing facility A_k with a circle $(X, r)_k \in C$ intersecting A_k.

We use this discarding test only for median circle problems with $n < 100$, since for problems with $n \geq 100$ only very few cubes could be excluded such that the test was numerically not efficient in these cases.

Weiszfeld Test for the p-median problem

Especially for the p-median problem, we found one more helpful discarding test.

Consider the cube

$$C \ = \ [x^1_{min}, \, x^1_{max}] \times [y^1_{min}, \, y^1_{max}] \times \ldots \times [x^p_{min}, \, x^p_{max}] \times [y^p_{min}, \, y^p_{max}]$$

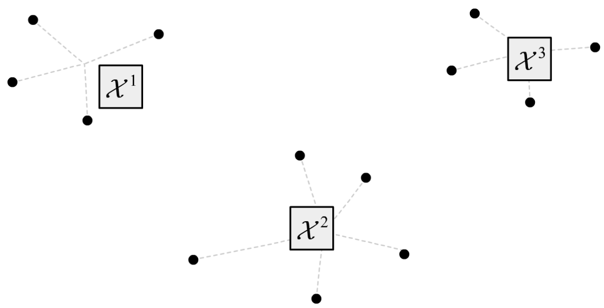

Figure 3: Example for the Weiszfeld test with $C = \mathcal{X}^1 \times \mathcal{X}^2 \times \mathcal{X}^3$.

and an arbitrary existing facility, say A_k. If there exists an $m \in \{1, \ldots, p\}$ such that the maximal distance between A_k and any X in

$$\mathcal{X}^m \;:=\; [x^m_{min},\, x^m_{max}] \,\times\, [y^m_{min},\, y^m_{max}]$$

is smaller than the minimal distance between A_k and any X in

$$\mathcal{X}^s \;=\; [x^s_{min},\, x^s_{max}] \,\times\, [y^s_{min},\, y^s_{max}]$$

for all $s \in \{1, \ldots, p\} \setminus \{m\}$, then A_k is associated with the square $\mathcal{X}^m$, see Figure 3.

If every existing facility A_k is associated with one square $\mathcal{X}^m$, we can split the p-median problem into p subproblems. Each subproblem is a 1-median problem, which can be solved up to every accuracy of $\delta > 0$ with the Weiszfeld algorithm, see Weiszfeld (1937). The p solutions (x_1, y_1) to (x_p, y_p) of these subproblems create together the point $(x_1, y_1, \ldots, x_p, y_p) \in \mathbb{R}^{2p}$ and we obtain

$$f(x_1, y_1, \ldots, x_p, y_p) \;\le\; f(x'_1, y'_1, \ldots, x'_p, y'_p) \,+\, p \cdot \delta \qquad \text{for all} \qquad (x'_1, y'_1, \ldots, x'_p, y'_p) \in C.$$

6 Computational Results

Our programs were coded in JAVA, compiled by JAVA 2 SDK 1.4, using double precision arithmetic. All tests were run on a 1.3 GHz computer with 512 MB of memory.

We generated $10 \le n \le 10,000$ existing facilities randomly in $[0,1]^3$ for the Fermat-Weber problem and in $[0,1]^2$ for the circle location and for the p-median problem. The weights were generated randomly in $[-1,1]$ for the Fermat-Weber problem and in $[0,1]$ for the other two problems.

Ten problems were run for different values of n for every problem. For accuracy, we applied $\varepsilon = 10^{-10}$ throughout all problems.

The Fermat-Weber Problem

The Fermat-Weber problem was solved in $C = [0,1]^3$ and for every value of n we generated $n/2$ positive and $n/2$ negative weights. These problems are very hard to solve and need the highest run times, see also Tuy et al. (1995). Our results are reported in Table 1. Therein, $\mathcal{C}_{max}$ is the maximum number of cubes in our list $\mathcal{C}$ throughout the branch-and-bound phase of the algorithm.

	Run time (sec.)			Iterations			$\mathcal{C}_{max}$		
n	Min	Max	Ave.	Min	Max	Ave.	Min	Max	Ave.
10	0.00	0.06	0.02	69	540	271.6	46	558	226.2
20	0.00	0.08	0.02	84	539	273.1	24	641	198.9
50	0.02	0.14	0.06	114	838	451.1	50	945	380.0
100	0.02	0.31	0.10	102	1,419	426.2	60	1,119	314.0
200	0.06	0.59	0.25	168	1,443	578.0	76	1,228	400.6
500	0.22	2.34	0.83	186	2,437	830.3	74	1,995	666.0
$1,000$	0.27	3.91	1.82	137	1,986	925.7	92	2,298	906.9
$2,000$	0.94	16.41	6.67	257	4,414	1,802.2	82	1,613	793.1
$5,000$	6.83	36.20	23.94	737	3,807	2,530.9	330	3,292	2,041.8
$10,000$	16.09	108.44	66.72	747	4,893	3,036.7	341	4,523	2,833.5

Table 1: Results for the Fermat-Weber problem for 10 runs using LB_1 and random weights.

As can be seen, our algorithm is very efficient. All problems with $n = 10,000$ were solved in less than two minutes with less than $5,000$ iterations and all problems with $n = 1,000$ were solved in less than four seconds with less than $2,000$ iterations.

For a computational comparison, we used the results from Drezner (2007). They solved the Fermat-Weber problem only in two dimensions, they used the same accuracy, and ran their programs on a computer with 2.8 GHz. For $n = 10,000$ existing facilities, they needed on average 54.05 seconds and for $n = 1,000$ problems 0.60 seconds. Note that these results are similar to our ones in three dimensions. Our standard deviation is quite high, but it is comparable to the standard deviation using BTST (see Phase 3 of the algorithm in Drezner and Suzuki (2004)).

For a further computational comparison, we also implemented the BTST method for the Fermat-Weber problem in two dimensions. Our experiments show that the BTST method is less efficient than BCSC for $n \geq 1,000$. The reason is that the BTST technique has to handle up to $2n - 5$ triangles in Phase 1, see Corollary 1 in Drezner and Suzuki (2004). In our BCSC method we do not have to calculate lower bounds for such a large number of cubes.

The Median Circle Problem

The optimal solution for the median circle problem may be a circle with a big radius or even a straight line, see Brimberg et al. (2008). In order to avoid these cases, we decided to solve the median circle problem in $C = [-1, 2] \times [-1, 2] \times [0, 3]$. Our results are shown in Table 2.

	Run time (sec.)			Iterations			$\mathcal{C}_{max}$		
n	Min	Max	Ave.	Min	Max	Ave.	Min	Max	Ave.
10	0.03	0.20	0.08	499	3,683	1,328.3	118	488	256.6
20	0.05	0.23	0.11	543	2,515	1,177.0	112	605	289.2
50	0.11	0.27	0.19	610	1,498	1,026.2	133	475	236.1
100	0.16	0.67	0.29	665	2,671	1,214.0	135	419	253.6
200	0.27	1.03	0.55	633	2,201	1,207.2	148	333	206.3
500	0.83	3.59	1.42	788	3,093	1,273.5	149	509	217.5
1,000	1.67	3.55	2.39	793	1,586	1,087.4	161	300	194.6
2,000	4.08	8.52	5.19	954	1,905	1,187.6	196	385	232.7
5,000	10.05	19.03	13.76	935	1,688	1,251.0	200	258	214.2
10,000	26.28	39.80	31.66	1,112	1,653	1,328.3	170	291	205.6

Table 2: Results for the median circle problem for 10 runs using LB_2 and random weights.

All problems could be solved surprisingly efficient. Our ten test problems with $n = 10,000$ existing facilities were solved in less than 40 seconds of computer time. Since the number of iterations is with an average of $1,200$ almost constant, the run time seems to be linear.

To the best of our knowledge this is the first numerical study of circle location problems such that a comparison to other numerical results is not possible.

The p-Median Problem

Using LB_3 and the proposed discarding tests, we solved the p-median problem in $C = [0, 1]^{2p}$ for $p = 2$ and $p = 3$. For the summarized results see Tables 3 and 4.

As the results show, the algorithm performed very well also for the p-median problem. Problems with $p = 2$ and $n = 10,000$ could be solved on average in 164.95 seconds and problems with $n = 1,000$ were solved in 10.40 seconds. Even the six dimensional 3-median problem for $n = 1,000$ could be solved on average in 958.91 seconds and problems with $n = 100$ on average in 116.43 seconds.

The p-median problem was also solved in Chen et al. (1998) by a D.C. outer approximation, but using a slower machine (SUN SPARC 2 with 28.5 mips processor, about 100 MHz) than ours. For $p = 2$ they used $\varepsilon = 10^{-5}$ and their results are on average 184.8 seconds for $n = 1,000$ and $2,717.8$ seconds for $n = 10,000$, which is more than 16 times of our run time. Problems for $p = 3$, $\varepsilon = 10^{-3}$ and $n = 30$ were solved in $1,079$ seconds, while problems with

	Run time (sec.)			Iterations			$\mathcal{C}_{max}$		
n	Min	Max	Ave.	Min	Max	Ave.	Min	Max	Ave.
10	0.02	0.13	0.05	149	1,511	522.7	45	235	99.1
20	0.05	0.20	0.12	564	1,909	1,163.4	105	277	204.9
50	0.14	0.52	0.31	796	2,798	1,726.3	130	506	303.0
100	0.41	0.97	0.70	1,374	3,214	2,289.5	239	452	369.8
200	1.03	2.30	1.66	1,779	4,120	2,917.6	212	703	455.5
500	3.22	6.30	4.88	2,331	5,287	3,445.7	343	725	494.5
1,000	8.52	13.91	10.40	3,469	5,643	4,209.2	430	789	575.7
2,000	18.44	30.36	22.73	3,792	6,018	4,586.7	479	851	612.9
5,000	54.36	82.53	67.25	4,391	6,690	5,458.6	548	895	716.7
10,000	131.95	210.59	164.95	5,289	8,443	6,512.6	600	1,041	818.7

Table 3: Results for the 2-median problem for 10 runs using LB_3 and random weights.

	Run time (sec.)			Iterations			$\mathcal{C}_{max}$		
n	Min	Max	Ave.	Min	Max	Ave.	Min	Max	Ave.
10	0.11	16.31	3.87	1,142	14,521	8,166.7	258	2,702	1,384.9
20	1.00	29.94	12.27	4,338	23,827	12,591.3	725	5,423	2,376.7
50	21.81	245.44	83.43	16,946	64,345	34,847.0	2,401	10,107	5,487.5
100	67.48	183.91	116.43	29,489	60,850	42,007.1	3,774	7,903	6,341.3
200	192.38	434.80	258.07	55,406	93,425	67,453.2	7,593	14,436	9,930.6
500	285.49	819.02	525.88	81,758	202,336	138,618.3	12,541	30,726	20,005.2
1,000	623.76	1552.94	958.91	95,318	212,948	140,197.8	13,415	30,029	19,635.0

Table 4: Results for the 3-median problem for 10 runs using LB_3 and random weights.

$n > 30$ ran out of memory. Note that these results were obtained although we used a higher accuracy ($\varepsilon = 10^{-10}$).

7 Discussion

We proposed a general solution method for multidimensional facility location problems which is a generalization of the big square small square technique (BSSS). To develop the approach we adapted the approaches for creating specific lower bounds from the BSSS method and used ideas for specific discarding tests similar to those in interval analysis methods. In contrast to the BSSS method, our approach can handle more than two variables. Its efficiency is still based on the calculation of good lower bounds for every cube. We remark that our method is applicable to non-differentiable objective functions which is often not the case in the interval analysis technique.

We implemented our method and demonstrated it on three example location problems for which we derived specific lower bounds and discarding tests. All problems could be solved efficiently even with large problem instances with up to $10,000$ existing facilities. The method has also been tested for other location problems, e.g. for circle location problems with Manhattan distance and for center circle problems, see Scholz (2007).

Future research in this area includes a comparison of different bounding techniques, the development of further discarding tests, and the application of the method to further location problems. Currently we are adapting the technique to solve multicriteria location problems. Also its application to ScheLoc problems (see Hamacher and Hennes (2007) or Elvikis et al. (2008)) is under research.

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