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# Direct and Inverse Sobolev Error Estimates for Scattered Data Interpolation via Spherical Basis Functions *†

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**Abstract**

The purpose of this paper is to get error estimates for spherical basis function (SBF) interpolation and approximation for target functions in Sobolev spaces less smooth than the SBFs, and to show that the rates achieved are, in a sense, best possible. Both of these results are new, and the inverse estimates are the first of their kind. In addition, we establish a Bernstein-type theorem, where the smallest separation between data sites plays the role of a Nyquist frequency.

* *Key words:* scattered data, spherical basis functions, radial basis functions, error estimates, inverse theorem, Bernstein theorem.

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## 1 Introduction

Fitting a surface to scattered data arising from sampling an unknown function defined on an underlying manifold comes up frequently in applied problems. When the underlying manifold is a sphere – or, more generally, the $n$-sphere $\mathbb{S}^n$ –, there are applications to geodesy, meteorology, astrophysics, geophysics, and other areas. Several review articles [5, 12], a book [6], and a recent volume [**21** (2004)] of the journal *Advances in Computational Mathematics* have been devoted to the topic itself or its applications.

Currently, there are two main approaches to solving such problems. For the case $\mathbb{S}^2$, one can use spherical triangles and employ a local polynomial approximation. This approach is described in a review article by Fasshauer and Schumaker [5], and recently Neamtu and Schumaker [20] have derived error estimates for it.

Another approach, and the topic of this paper, is to use *spherical basis functions* (SBFs), which we precisely define in Section 3. These functions go back to work of Schoenberg [26] and have properties similar to the familiar radial basis functions (RBFs). Like the RBFs, SBFs provide the means to interpolate scattered data, only on $\mathbb{S}^n$ rather than $\mathbb{R}^n$.

Early interpolation error estimates for SBFs mirrored the RBF results so that error estimates were obtained which applied only to functions lying in an underlying reproducing kernel Hilbert space, called the *native space*, determined by the interpolating class of SBFs. Thus the method of interpolating by SBFs was hampered with the same problem that afflicted interpolation by RBFs; namely, the error estimates applied to an unsatisfactorily small class of functions. A major improvement was presented in [17], where the first error estimates that applied to functions outside of the native space were given. Even there, the results did not apply to the standard Sobolev spaces. Indeed, it was only recently that Hubbert and Morton [10], who employed techniques similar to those used by Duchon [4] for $\mathbb{R}^n$, obtained such results for target functions in the native space of an SBF.

The SBFs we discuss here, like the ones in [10], have native spaces equivalent to Sobolev spaces. The thin-plate splines restricted to the sphere and Wendland's compactly supported RBFs [28] restricted to the sphere give rise to such SBFs.

The purpose of this paper is not only to get error estimates for SBF interpolation and approximation for target functions in Sobolev spaces less smooth than the SBFs, but also to show that the rates achieved are, in a sense, best possible. Both of these results are new, and the inverse estimates are the first of their kind for SBFs. For RBFs in $\mathbb{R}^n$ and in a native space setting, Schaback and Wendland [25] also have obtained inverse results.

Proving the inverse estimates requires two results of interest their own right. One is a technical tool involving centers (data sites). Namely, we show the existence of *nested* sequences of centers on $\mathbb{S}^n$ that have the property that they maintain uniformity, which here means the ratio of the mesh norm to the separation distance (or, radius, actually) remains bounded above as points are added. The second is establishing a Bernstein-type theorem, where the smallest separation between data sites plays the role of a Nyquist frequency.

The method of establishing these results is novel as well. We introduce a method that relates or "lifts" various norms and inequalities involving a given SBF on $\mathbb{S}^n$ to corresponding ones for a related RBF on $\mathbb{R}^{n+1}$, allowing one to apply results obtained previously for $\mathbb{R}^{n+1}$ to $\mathbb{S}^n$. Thus, this paper makes explicit how RBF error estimates on $\mathbb{R}^{n+1}$ give rise to SBF error estimates on $\mathbb{S}^n$. Previously, results on the two different manifolds, although parallel by nature, were obtained independently.

The paper is organized as follows. Section 2 contains a discussion of the geometry of sets of centers, including various terms we use throughout the sequel. It also includes the result about nested sequences mentioned above. Next, Section 3 reviews spherical harmonics, SBFs and their native spaces, and various Sobolev spaces. We describe the lifting method and obtain results using it in Section 4. In Section 5, we use the concept of a norming set to derive Sobolev-type error estimates for SBF interpolation when the target function $f$ is not smooth enough to be in the native space of the SBF. In the concluding section, Section 6, we establish the Bernstein inequality in inverse estimates mentioned above.

## 2 Geometry of Sets of Centers

We will let $X = \{x_j\}_{j=1}^N \subset \mathbb{S}^n$ be a set of $N$ distinct points on the sphere, and we will call $X$ a set of centers. There are three useful quantities we will associate with $X$: the *separation radius*, $q_X$, the *mesh norm*, $h_X$, and the *mesh ratio*, $\rho_X$. If $d(x,y)$ is the geodesic distance between two points $x$ and $y$ in $\mathbb{S}^n$, then these quantities are defined by

$$q_X := \frac{1}{2}\min_{j\neq k} d(x_j, x_k), \quad h_X := \max_{x\in\mathbb{S}^n}\min_j d(x, x_j), \text{ and } \rho_X := h_X/q_X.$$

For $\rho \geq 1$, we will say that a family $\mathcal{F} = \mathcal{F}_\rho(\mathbb{S}^n)$ comprising sets $X$ of centers is *$\rho$-uniform* if every $X \in \mathcal{F}$ satisfies $\rho_X \leq \rho$. Unless confusion would arise, we will not indicate either $\rho$ or $\mathbb{S}^n$, and just use $\mathcal{F}$ to designate a family. The specific $\rho$ or sphere $\mathbb{S}^n$ will be clear from the context.

On the circle $\mathbb{S}^1$, a set of $N$ equiangular points has $q = h = \pi/N$, and $\rho = 1$. For $n > 1$, it is clear that $\rho = 1$ cannot be achieved, and so the corresponding family $\mathcal{F}$ is empty. Which $\rho$ and $n$ have nonempty $\mathcal{F}$ is directly related to the extent one can uniformly distribute points on spheres and other manifolds [3, 8, 9, 22, 24]. For instance, Habicht and Van der Waerden [8] studied the best packing of $N$ non-overlapping hexagons on $\mathbb{S}^2$. A careful inspection of their proof shows that the $X$ they constructed has a mesh ratio $\rho_X \le 2/\sqrt{3} + CN^{-1/6}$, where $C$ is a constant independent of $N$.

On the other hand, when $\rho \ge 2$ there is a simple proof that $\mathcal{F}$ is nonempty no matter what $n$ is. Look at the set $X = \{\pm e_1, \pm e_2, \ldots, \pm e_n\} \subset \mathbb{S}^n$, where $e_j$ is the point on the unit sphere with all 0's except for a 1 in the $j^{th}$ position. $X$ has separation radius $q_X = \pi/4$, because the vectors involved are all orthogonal. A point on $\mathbb{S}^n$ at maximum distance from $X$ is $y = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$; here, $h_X = \cos^{-1}(\frac{1}{\sqrt{n}})$. Thus, $\rho_X = \frac{4}{\pi}\cos^{-1}(\frac{1}{\sqrt{n}}) < 2$, and so $\mathcal{F}$ is nonempty as long as $\rho \ge 2$. The proposition below shows that $\mathcal{F}$ also contains arbitrarily large, *nested* sets of centers that have additional properties that we will need in Section 6.

**Proposition 2.1.** *Let $\rho \ge 2$ and let $\mathcal{F}$ be a $\rho$-uniform family of sets of centers $X$. Then, there exists a sequence of sets $X_k \in \mathcal{F}$, $k = 0, 1, \ldots$, such that the sequence is nested, $X_k \subset X_{k+1}$, and such that at each step the mesh norms satisfy $\frac{1}{4}h_{X_k} < h_{X_{k+1}} \le \frac{1}{2}h_{X_k}$.*

**Proof:** Start with a set of centers $X_0$, which we assume to be in the nonempty family $\mathcal{F}$, and suppose that we have constructed the requisite sets up to $X_k$. Let the mesh norm, separation radius, and mesh ratio be $h_k$, $q_k$, and $\rho_k \le \rho$, respectively. We need to show that we are able to find a set $X_{k+1} \in \mathcal{F}$, such that $X_{k+1} \supset X_k$, $\frac{1}{4}h_k < h_{k+1} \le \frac{1}{2}h_k$. The clearest way to present the proof is to describe it in terms of steps in an algorithm. We initialize the algorithm by setting $X = X_k$, $h = h_k$, etc.

1. *Find $y \in \mathbb{S}^n$ such that $h = \mathrm{dist}(y, X)$.* The compactness of $\mathbb{S}^n$ allows us to do this.

2. *Form the set $X' = X \cup \{y\}$.*

3. *Compute the mesh norm $h'$, separation radius $q'$, and mesh ratio $\rho'$ for $X'$.* Each of these can be estimated in terms of corresponding quantities for $X$. Because we have added a point to $X$, the mesh norm

of $X'$ is smaller than that of $X$, so $h' \leq h$. In addition, the midpoint of the great circle joining $y$ to its nearest neighbor in $X$ is at a distance $h/2$ from $y$, and thus $h' \geq \frac{1}{2}h$. The separation radius $q'$ is half of the separation distance for points in $X'$. If the two closest points in $X'$ come from the set $X$, then $q' = q$. The only other possibility is if one of the two closest points is $y$ and the other is from $X$. In that case, $q' = \frac{1}{2}\mathrm{dist}(y, X) = \frac{1}{2}h$. Thus, $q' = \min\{q, \frac{1}{2}h\}$, and so it follows that $h'/q' \leq \max\{h/q, 2\} \leq \rho$. Hence, $X' \in \mathcal{F}$.

4. *If $h' > \frac{1}{2}h_k$, then go back to Step 1 and use $X'$ as the new $X$. If $h' \leq \frac{1}{2}h_k$, then stop; set $X_{k+1} = X'$.* First of all, the algorithm must stop after a finite number of iterations. Suppose it does not stop. A point is added to $X$ each time the algorithm runs, and the new set $X$ has a separation radius that is at least $(\frac{1}{2}h_k)/\rho$. Balls centered at points in the set and having this radius do not intersect. Thus, the total volume occupied by these balls is less than the volume of $\mathbb{S}^n$. On the other hand, because the number of points in the set $X$ is growing without bound, the volume associated with each set would become arbitrarily large as the algorithm ran on; in particular, larger than the finite volume of $\mathbb{S}^n$ – a contradiction. When the algorithm *does* stop, we have $h' = h_{k+1} \leq \frac{1}{2}h_k$. In addition, $h' \geq \frac{1}{2}h$, where $h$ comes from the next to last step. Since the algorithm did *not* stop at that step, $h > \frac{1}{2}h_k$. Putting these inequalities together, we see that $\frac{1}{4}h_k < h_{k+1} \leq \frac{1}{2}h_k$. □

The proof does not involve any property special to $\mathbb{S}^n$, other than that it has a metric, finite volume, and a few other things associated with compact, connected $C^\infty$ Riemannian manifolds. Thus it holds for these spaces. We summarize these observations below.

**Remark 2.2.** *If the appropriate metrics are used and if $\mathcal{F}$ is not empty for some $\rho \geq 2$, then Proposition 2.1 holds for any compact, connected $C^\infty$ Riemannian manifold.*

# 3 Spherical Basis Functions & Associated Spaces

**Spherical harmonics.** We will let $L^2(\mathbb{S}^n)$ be the Hilbert space equipped with the inner product

$$\langle f, g\rangle := \int_{\mathbb{S}^n} f(x)\overline{g(x)}d\mu(x),$$

where $d\mu$ is the standard volume element for $\mathbb{S}^n$. The $Y_{\ell,m}$'s will be taken to be the usual orthonormal basis of spherical harmonics [14], which we may assume to be real. For $\ell$ fixed, these span the eigenspace of the Laplace-Beltrami operator on $\mathbb{S}^n$ corresponding to the eigenvalue $\lambda_\ell = \ell(\ell+n-1)$. Here, $m = 1, \ldots, d_\ell$, where $d_\ell$ is the dimension of the eigenspace corresponding to $\lambda_\ell$ and is given by [14, p. 4]

$$d_\ell = \begin{cases} 1, & \ell = 0, \\ \dfrac{(2\ell+n-1)\Gamma(\ell+n-1)}{\Gamma(\ell+1)\Gamma(n)}, & \ell \geq 1. \end{cases} \tag{3.1}$$

For large $\ell$, $d_\ell = \mathcal{O}\big(\ell^{n-1}\big)$. We will let $\mathcal{H}_L$ be the span of the spherical harmonics of order $\ell \leq L$. If $f \in L^2(\mathbb{S}^n)$, then we may expand it in a series of spherical harmonics,

$$f = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \hat{f}(\ell,m) Y_{\ell,m}, \text{ where } \hat{f}(\ell,m) := \langle f, Y_{\ell,m} \rangle .$$

We will make use of one more fact concerning the spherical harmonics. If we let $x \cdot y$ to be the "dot" product in $\mathbb{R}^{n+1}$, then spherical harmonics satisfy the addition formula [14]:

$$\sum_{m=1}^{d_\ell} Y_{\ell,m}(x) Y_{\ell,m}(y) = \frac{d_\ell}{\omega_n} P_\ell(n+1; x \cdot y). \tag{3.2}$$

Here $P_\ell(n+1;t)$ denotes the degree $\ell$ Legendre polynomial in $n+1$ variables and $\omega_n$ is the volume of $\mathbb{S}^n$.

**Sobolev spaces on $\mathbb{S}^n$.** Let $\mathcal{D}'(\mathbb{S}^n)$ be the space of distributions on $\mathbb{S}^n$. Because each $Y_{\ell,m}$ is in $C^\infty(\mathbb{S}^n)$, the coefficient $\hat{f}(\ell,m) = \langle f, Y_{\ell,m} \rangle$ is defined for every distribution $f \in \mathcal{D}'(\mathbb{S}^n)$. The Sobolev space $H_\tau(\mathbb{S}^n)$, $\tau \in \mathbb{R}$, is then given by [7, 11]

$$H_\tau(\mathbb{S}^n) := \{ f \in \mathcal{D}'(\mathbb{S}^n) \, : \, \|f\|^2_{H_\tau(\mathbb{S}^n)} := \sum_{\ell,m} (\lambda_\ell + 1)^\tau |\hat{f}(\ell,m)|^2 < \infty \} ,$$

with the inner product being

$$\langle f, g \rangle_{H_\tau(\mathbb{S}^n)} := \sum_{\ell,m} (\lambda_\ell + 1)^\tau \hat{f}(\ell,m) \overline{\hat{g}(\ell,m)}.$$

The Sobolev imbedding theorem [7, p. 35] implies that if $\tau > \frac{n}{2}$, then $H_\tau(\mathbb{S}^n)$ is continuously imbedded in $C(\mathbb{S}^n)$, and so $H_\tau(\mathbb{S}^n)$ is a reproducing kernel Hilbert space, with the reproducing kernel being

$$\begin{aligned}\varphi_\tau(x\cdot y) &= \sum_{\ell=0}^{\infty} \underbrace{(\lambda_\ell+1)^{-\tau}}_{\hat{\varphi}_\tau(\ell)} \sum_{m=1}^{d_\ell} Y_{\ell,m}(x)Y_{\ell,m}(y) \\ &= \sum_{\ell=0}^{\infty} \hat{\varphi}_\tau(\ell)\frac{d_\ell}{\omega_n} P_\ell(n+1; x\cdot y). \qquad (3.3)\end{aligned}$$

There is a useful inequality for norms for these spaces; it is an easy consequence of Hölder's inequality applied to the sequences defining the norms involved. Namely, let $s, t$ be nonnegative and satisfy $1/s + 1/t = 1$. If $\alpha \geq \beta$ are also nonnegative, then for every $f \in H_\alpha(\mathbb{S}^n)$, we have

$$\|f\|_{H_{\alpha/s+\beta/t}(\mathbb{S}^n)} \leq \|f\|_{H_\alpha(\mathbb{S}^n)}^{1/s} \|f\|_{H_\beta(\mathbb{S}^n)}^{1/t}. \qquad (3.4)$$

**Spherical basis functions.** We will say that a continuous function $\phi : [-1,1] \to \mathbb{R}$ is *positive definite* on $\mathbb{S}^n$ if for every possible finite set $X$ of distinct points in $S^n$ the matrix $A_{j,k} := \phi(\cos(d(x_j, x_k)))$, $x_j$ and $x_k$ in $X$, is positive semi-definite. We will say that $\phi$ is *strictly positive definite* on $\mathbb{S}^n$ if these matrices are all positive definite. Here, $d(x,y)$ is the geodesic distance between $x$ and $y$ on $\mathbb{S}^n$, and is precisely the angle between the vectors $x$ and $y$ in $\mathbb{R}^{n+1}$. since $|x| = |y| = 1$, we have $x\cdot y = \cos(d(x,y))$. Thus, the matrix $A$ above has entries $A_{j,k} = \phi(x_j \cdot x_k)$.

Positive definite functions on spheres were introduced and characterized by Schoenberg [26]. In our notation, what he showed was that a function $\phi$ is positive definite if its expansion in $(n+1)$ variable Legendre polynomials,

$$\phi(x\cdot y) = \sum_{\ell=0}^{\infty} \hat{\phi}(\ell)\frac{d_\ell}{\omega_n} P_\ell(n+1; x\cdot y), \qquad (3.5)$$

has all Fourier-Legendre coefficients $\hat{\phi}(\ell) \geq 0$ and $\sum_\ell d_\ell \hat{\phi}(\ell) < \infty$.

*Strictly* positive definite functions are especially important. Because the matrix $A$ corresponding to any $X$ is positive definite, and hence invertible, one can always use an interpolant of the form $\sum_{j=1}^{N} \alpha_j \phi(x\cdot x_j)$ to solve a scattered-data interpolation problem where a value $d_j$, which may be real or complex, is given at each $x_j$ in $X$. Recently, for $n \geq 2$, Chen, Menegatto, and Sun [2] gave necessary and sufficient conditions for functions to be strictly positive definite, and, in the case of the circle $\mathbb{S}^1$, Pinkus [21] has characterized them. A useful sufficient, but not necessary, condition is that the Fourier-Legendre coefficients are all positive [23, 30]; that is, $\hat{\phi}(\ell) > 0$ for all $\ell$. This has the added advantage of allowing one to solve generalized Hermite interpolation problems [15]. We will say that $\phi$ is a *spherical basis function* (SBF) if $\hat{\phi}(\ell) > 0$ for all $\ell$.

The kernel $\varphi_\tau$ defined by (3.3) is an SBF. Indeed, restricting an order 0 radial basis function (RBF) on $\mathbb{R}^{n+1}$ to $\mathbb{S}^n$ is a way to generate SBFs, as the result below shows.

**Theorem 3.1 ([17, Theorem 4.1 & Corollary 4.3]).** *Let $\Phi$ be a positive definite radial function having a nonnegative Fourier transform $\widehat{\Phi} \in L^1(\mathbb{R}^{n+1})$, and let $\phi(x \cdot y) := \Phi(x-y)|_{x,y\in\mathbb{S}^n}$. For $\ell \geq 0$, we have that*

$$\hat{\phi}(\ell) = \int_0^\infty t\widehat{\Phi}(t)J_\nu^2(t)dt, \quad \nu := \ell + \frac{n-1}{2}, \tag{3.6}$$

*where $J_\nu$ is the order $\nu$ Bessel function of the first kind. Moreover, if $\widehat{\Phi}$ is nontrivial – i.e., positive on a set of nonzero measure –, then $\hat{\phi}(\ell) > 0$ for all $\ell$ and $\phi$ is an SBF.*

A more general version of this result, which applies to RBFs of order 1 or more, is given in [16, Proposition 3.1]. Similar results were given in [1].

Just as $\varphi_\tau$ is the reproducing kernel for the Sobolev space $H_\tau(\mathbb{S}^n)$, every SBF $\phi$ is itself a reproducing kernel for a Hilbert space $\mathcal{N}_\phi$, its *native space*. This space and its inner product are defined below:

$$\begin{aligned}
\mathcal{N}_\phi &:= \{f \in \mathcal{D}'(\mathbb{S}^n) \,:\, \|f\|_\phi^2 := \sum_{\ell,m} \frac{|\hat{f}(\ell,m)|^2}{\hat{\phi}(\ell)} < \infty\}, \\
\langle f, g\rangle_\phi &:= \sum_{\ell,m} \frac{\hat{f}(\ell,m)\overline{\hat{g}(\ell,m)}}{\hat{\phi}(\ell)}.
\end{aligned}$$

Of course, $H_\tau(\mathbb{S}^n)$ is exactly the same space as $\mathcal{N}_{\varphi_\tau}$, with $\varphi_\tau$ given by (3.3).

An order 0 RBF $\Phi$ on $\mathbb{R}^{n+1}$ for which $\hat{\Phi}(\xi) > 0$ has a native space $\mathcal{N}_\Phi$ associated with it. This is a reproducing kernel Hilbert space having a convolution kernel $\Phi(x-y)$. The norm and inner product for it are

$$\|f\|_\Phi^2 := \int_{\mathbb{R}^{n+1}} \frac{|\hat{f}(\xi)|^2}{\widehat{\Phi}(\xi)} d\xi \quad \text{and} \quad \langle f, g\rangle_\Phi := \int_{\mathbb{R}^{n+1}} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{\widehat{\Phi}(\xi)} d\xi\,.$$

# 4 Lifting

We are interested in looking at the relationship among various norms of functions related to interpolants produced with an SBF $\phi$. We begin our discussion with this lemma.

**Lemma 4.1.** *If $X = \{x_j\}_{j=1}^N$ is a finite set of distinct points in $\mathbb{S}^n$ and $\alpha \in \mathbb{C}^N$, then*

$$\|\sum_j \alpha_j \phi(x \cdot x_j)\|_\phi^2 = \sum_{j,k} \overline{\alpha}_k \alpha_j \phi(x_j \cdot x_k), \tag{4.1}$$

*and also*

$$\|\sum_j \alpha_j \phi(x \cdot x_j)\|_\phi^2 = \sum_{\ell=0}^{\infty} \hat{\phi}(\ell) \sum_{m=1}^{d_\ell} |\sum_j \alpha_j Y_{\ell,m}(x_j)|^2. \tag{4.2}$$

*Moreover, if $\phi_L(x \cdot y) = \sum_{\ell=0}^{L} \hat{\phi}(\ell) \frac{d_\ell}{\omega_n} P_\ell(n+1; x \cdot y)$, where $L$ is a nonnegative integer, then*

$$\|\sum_j \alpha_j (\phi - \phi_L)(x \cdot x_j)\|_\phi^2 = \sum_{\ell=L+1}^{\infty} \hat{\phi}(\ell) \sum_{m=1}^{d_\ell} |\sum_j \alpha_j Y_{\ell,m}(x_j)|^2. \tag{4.3}$$

**Proof:** The first equation (4.1) follows by virtue of $\phi$ being a reproducing kernel. To get the second, we let $g = \sum_j \alpha_j \phi(x \cdot x_j)$ and note that, from (3.5) and the addition theorem (3.2), we have

$$\hat{g}(\ell, m) = \hat{\phi}(\ell) \sum_j \alpha_j Y_{\ell,m}(x_j)$$

Calculate $\|g\|_\phi^2$ using its series definition and the expression above for $\hat{g}(\ell, m)$. Simplifying the terms in the resulting series then gives us (4.2). The last series follows via similar steps. □

We now want to "lift" norms involving SBFs on $\mathbb{S}^n$ to ones for RBFs on $\mathbb{R}^{n+1}$. Doing that requires that we have a connection between SBFs and RBFs in the first place. Suppose that $\psi(x \cdot y) = \Psi(x - y)|_{x,y \in \mathbb{S}^n}$, where $\Psi$ is an RBF, with $\widehat{\Psi} \in L^1(\mathbb{R}^{n+1})$. Then, by Theorem 3.1, $\psi$ is an SBF. For $\sigma > 0$, define the function

$$\widehat{\Psi}_\sigma(\xi) := \begin{cases} \widehat{\Psi}(\xi) & |\xi| \le \sigma, \\ 0 & |\xi| > \sigma. \end{cases} \tag{4.4}$$

Let $\Psi_\sigma$ be its (radial) inverse Fourier transform. Again by Theorem 3.1, the function $\psi_\sigma := \Psi_\sigma(x-y)|_{x,y\in\mathbb{S}^n}$ is an SBF with Fourier-Legendre coefficients given by

$$\hat{\psi}_\sigma(\ell) = \int_0^\sigma t\widehat{\Psi}(t)J^2_{\ell+\frac{n-1}{2}}(t)dt.$$

The following result holds.

**Proposition 4.2.** *With the notation introduced above, if an SBF $\phi$ satisfies*

$$a\hat{\psi}(\ell) \le \hat{\phi}(\ell) \le b\hat{\psi}(\ell) \tag{4.5}$$

*for all $\ell \ge 0$, where $a$ and $b$ are positive constants independent of $\ell$, then we have the inequality,*

$$a\Big\|\sum_j \alpha_j\Psi(x-x_j)\Big\|^2_\Psi \le \Big\|\sum_j \alpha_j\phi(x\cdot x_j)\Big\|^2_\phi \le b\Big\|\sum_j \alpha_j\Psi(x-x_j)\Big\|^2_\Psi. \tag{4.6}$$

*Moreover, if there is a constant $c > 0$ such that when $\sigma \le cL$ we have $\hat{\psi}_\sigma(\ell) < \frac{1}{2}\hat{\psi}(\ell)$ for all $\ell > L$, then*

$$\Big\|\sum_j \alpha_j(\phi-\phi_L)(x\cdot x_j))\Big\|^2_\phi \le 2b\|\sum_j \alpha_j(\Psi-\Psi_\sigma)(x-x_j)\|^2_\Psi. \tag{4.7}$$

**Proof:** From (4.2) and (4.5), we have that

$$a\Big\|\sum_j \alpha_j\psi(x\cdot x_j)\Big\|^2_\psi \le \Big\|\sum_j \alpha_j\phi(x\cdot x_j)\Big\|^2_\phi \le b\Big\|\sum_j \alpha_j\psi(x\cdot x_j)\Big\|^2_\psi.$$

Now, from (4.1), we also have that

$$\begin{aligned}\Big\|\sum_j \alpha_j\psi(x\cdot x_j)\Big\|^2_\psi &= \sum_{j,k}\overline{\alpha}_k\alpha_j\psi(x_j\cdot x_k)\\ &= \sum_{j,k}\overline{\alpha}_k\alpha_j\Psi(x_j-x_k)\\ &= \Big\|\sum_j \alpha_j\Psi(x-x_j)\Big\|^2_\Psi,\end{aligned}$$

where the last equation follows from $\Psi$ being a reproducing kernel for $\mathcal{N}_\Psi$. Combining this with what we did above yields (4.6). For $\sigma \le cL$ and all

$\ell > L$ we have $\hat{\psi}_\sigma(\ell) \le \frac{1}{2}\hat{\psi}(\ell)$, so that $\hat{\psi}(\ell) \le 2\big(\hat{\psi}(\ell) - \hat{\psi}_\sigma(\ell)\big)$. Using this and (4.5) in conjunction with (4.3), we obtain

$$\begin{aligned}
\Big\| \sum_j \alpha_j (\phi - \phi_L)(x\cdot x_j)\Big\|_\phi^2 &\le 2b \sum_{\ell=L+1}^{\infty} (\hat{\psi} - \hat{\psi}_\sigma)(\ell) \sum_{m=1}^{d_\ell} \Big| \sum_j \alpha_j Y_{\ell,m}(x_j)\Big|^2 \\
&\le 2b \sum_{\ell=0}^{\infty} (\hat{\psi} - \hat{\psi}_\sigma)(\ell) \sum_{m=1}^{d_\ell} \Big| \sum_j \alpha_j Y_{\ell,m}(x_j)\Big|^2 \\
&\le 2b \sum_{j,k} \overline{\alpha}_k \alpha_j (\psi - \psi_\sigma)(x_j\cdot x_k)\,.
\end{aligned}$$

Now, since $\psi_\sigma(x\cdot y) = \Psi_\sigma(x-y)|_{x,y\in\mathbb{S}^n}$, we arrive at

$$\Big\| \sum_j \alpha_j (\phi - \phi_L)(x\cdot x_j))\Big\|_\phi^2 \le 2b \sum_{j,k} \overline{\alpha}_k \alpha_j (\Psi - \Psi_\sigma)(x_j - x_k).$$

Writing out the functions involved on the right above using Fourier transforms, we can easily show that

$$\sum_{j,k} \overline{\alpha}_k \alpha_j (\Psi - \Psi_\sigma)(x_j - x_k) = \| \sum_j \alpha_j (\Psi - \Psi_\sigma)(x - x_j)\|_\Psi^2.$$

Using this in the previous inequality then gives us (4.7). $\square$

We will apply these results to specific kernels that generate Sobolev spaces on $\mathbb{S}^n$ and $\mathbb{R}^{n+1}$. Before we do that, we will obtain this simple, general estimate on $\psi_\sigma(\ell)$.

**Lemma 4.3.** *Let $\widehat{\Psi} \in L^1(\mathbb{R}^{n+1})$ be radial and nonnegative, and let $\ell \ge 0$. If $\sigma > 0$ and $\nu = \ell + \frac{n-1}{2}$, then*

$$\hat{\psi}_\sigma(\ell) = \int_0^\sigma t\widehat{\Psi}(t) J_\nu^2(t)dt \le \frac{2e\,\Psi(0)(e\pi)^n}{\omega_n(\nu+1)^n}\left(\frac{e\sigma}{2(\nu+1)}\right)^{2\nu-n+1}. \tag{4.8}$$

**Proof:** We begin with the inequality [27, (1) §3.31],

$$|J_\nu(z)| \le \frac{2^{-\nu}|z|^\nu e^{|\Im(z)|}}{\Gamma(\nu+1)},\ |z|>0,\ \nu > -\frac{1}{2}\,. \tag{4.9}$$

Using this in our integral and making a few obvious estimates, we obtain

$$\begin{aligned}
\int_0^\sigma t\widehat{\Psi}(t) J_\nu^2(t)dt &\le \frac{2^{-2\nu}}{\Gamma^2(\nu+1)} \int_0^\sigma t^{2\nu-n+1}\widehat{\Psi}(t) t^n dt \\
&\le \frac{2^{-2\nu}\sigma^{2\nu-n+1}}{\Gamma^2(\nu+1)} \int_0^\sigma \widehat{\Psi}(t) t^n dt.
\end{aligned}$$

We also have that

$$\int_0^\sigma \widehat{\Psi}(t)t^n dt \le \int_0^\infty \widehat{\Psi}(t)t^n dt = \frac{1}{\omega_n}\int_{\mathbb{R}^{n+1}} \widehat{\Psi}(\xi)d\xi = \frac{(2\pi)^{n+1}}{\omega_n}\Psi(0).$$

Using this in the previous inequality and simplifying, we obtain this:

$$\hat{\psi}_\sigma(\ell) = \int_0^\sigma t\widehat{\Psi}(t)J_\nu^2(t)dt \le \frac{4\pi^{n+1}(\sigma/2)^{2\nu-n+1}}{\omega_n\Gamma^2(\nu+1)}\Psi(0). \tag{4.10}$$

We will now employ a sharp version of Stirling's formula for the Gamma function [29, §12.33, pg. 253]: If $x > 0$, then $\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x+\theta/(12x)}$, where $\theta = \theta(x)$ and $0 < \theta < 1$. Consequently, we have that $\Gamma(\nu+1) > \sqrt{2\pi}(\nu+1)^{\nu+\frac{1}{2}}e^{-\nu-1}$, so that after doing a little algebra we obtain

$$\frac{4\pi^{n+1}(\sigma/2)^{2\nu-n+1}}{\omega_n\Gamma^2(\nu+1)}\Psi(0) \le \frac{2e\,\Psi(0)(e\pi)^n}{\omega_n(\nu+1)^n}\left(\frac{e\sigma}{2(\nu+1)}\right)^{2\nu-n+1}.$$

Replacing the right side in (4.10) with the right side above yields (4.8). □

For fixed $s > \frac{n+1}{2}$, we have from [16, Proposition 4.1] that if there are constants $C_1$ and $C_2$ such that

$$C_1(1+|\xi|^2)^{-s} \le \widehat{\Psi}(\xi) \le C_2(1+|\xi|^2)^{-s} \tag{4.11}$$

then as $\ell \to \infty$,

$$C_1'\ell^{-2s+1}\left(1+\mathcal{O}(\ell^{-1})\right) \le \hat{\psi}(\ell) \le C_2'\ell^{-2s+1}\left(1+\mathcal{O}(\ell^{-1})\right) \tag{4.12}$$

where $C_1' = C_1\frac{\Gamma(2s-1)}{2^{4s-3}\Gamma^2(s)}$ and $C_2' = C_2\frac{\Gamma(2s-1)}{2^{2s-3}\Gamma^2(s)}$. If we divide (4.8) in Lemma 4.3 by $\hat{\psi}(\ell)$, use (4.12), and replace $\nu$ by $\ell + \frac{n-1}{2}$, then there is an $\ell_0$ such that when $\ell \ge \ell_0$ we have

$$\frac{\hat{\psi}_\sigma(\ell)}{\hat{\psi}(\ell)} \le C_{s,n}\ell^{2s-1-n}\left(\frac{e\sigma}{2\ell}\right)^{2\ell}.$$

¿From this it follows that we can find $L_0 \ge \ell_0$ such that if $\ell \ge L \ge L_0$ and $\sigma \le e^{-1}L$, then

$$\frac{\hat{\psi}_\sigma(\ell)}{\hat{\psi}(\ell)} \le C_{s,n}\ell^{2s-1-n}\,2^{-2\ell} \le \frac{1}{2}. \tag{4.13}$$

Let $\tau = s - \frac{1}{2}$. Recall that the reproducing kernel for the Sobolev space $H_\tau(\mathbb{S}^n)$ is $\varphi_\tau$, which is defined in (3.3). Now, $\hat{\varphi}_\tau(\ell) = (\lambda_\ell + 1)^{-\tau} =$

$\ell^{-2\tau}(1+\mathcal{O}(\ell^{-1})) = \ell^{-2s+1}(1+\mathcal{O}(\ell^{-1}))$. This and (4.12) imply that for all $\ell$ sufficiently large we have

$$C_2'^{-1}\left(1+\mathcal{O}(\ell^{-1})\right) \le \frac{\hat{\varphi}_\tau(\ell)}{\hat{\psi}(\ell)} \le C_1'^{-1}\left(1+\mathcal{O}(\ell^{-1})\right).$$

Since both $\hat{\psi}(\ell)$ and $\varphi_\tau(\ell)$ are positive for all $\ell$, it follows that there are constants $a$ and $b$ such that for all $\ell$ we have

$$a \le \frac{\hat{\varphi}_\tau(\ell)}{\hat{\psi}(\ell)} \le b \tag{4.14}$$

Applying Proposition 4.2 then proves the following result:

**Proposition 4.4.** *Let $s > \frac{n+1}{2}$ and $\tau := s - \frac{1}{2}$. If $\widehat{\Psi}$ satisfies (4.11) and $\varphi_\tau$ is as in (3.3), then there are positive constants $a$, $b$ such that*

$$a\Big\|\sum_j \alpha_j \Psi(x-x_j)\Big\|_\Psi^2 \le \Big\|\sum_j \alpha_j \varphi_\tau(x\cdot x_j)\Big\|_{\varphi_\tau}^2 \le b\Big\|\sum_j \alpha_j \Psi(x-x_j)\Big\|_\Psi^2.$$

*Also, there is a constant $L_0 > 0$ such that if $L \ge L_0$ and $\sigma \le e^{-1}L$, then*

$$\Big\|\sum_j \alpha_j(\varphi_\tau - \varphi_{\tau,L}(x\cdot x_j))\Big\|_{\varphi_\tau}^2 \le 2b\|\sum_j \alpha_j(\Psi-\Psi_\sigma)(x-x_j)\|_\Psi^2.$$

# 5 Interpolation and Near-Best Approximation

## 5.1 Existence of a polynomial interpolant

Our aim in this section is to prove the theorem below. This asserts that we can use a spherical polynomial of order $L$ to both interpolate a sufficiently smooth function $f : \mathbb{S}^n \to \mathbb{C}$ on a set $X \subset \mathbb{S}^n$ of distinct, scattered points *and* simultaneously get a near-best approximant to $f$ in a Sobolev norm, provided $L$ is large enough.

**Theorem 5.1.** *Let $q = q_X$ be the separation radius for $X = \{x_j\}_{j=1}^N \subset \mathbb{S}^n$ and let $\tau > n/2$. There exists a constant $\kappa$, which depends only on $n$ and $\tau$, such that if $L \ge \kappa/q$, then for every $f \in H_\tau(\mathbb{S}^n)$ there is a spherical polynomial $p \in \mathcal{H}_L$ such that $p|_X = f|_X$ and $\|f-p\|_{H_\tau(\mathbb{S}^n)} \le 5\,\mathrm{dist}_{H_\tau(\mathbb{S}^n)}(f,\mathcal{H}_L)$.*

We will postpone the proof until we establish certain preliminary results. In preparation for them, we state this proposition, whose proof can be found in [18].

**Proposition 5.2.** *Let $\mathcal{Y}$ be a (possibly complex) Banach space, $\mathcal{V}$ a subspace of $\mathcal{Y}$, and $Z^*$ a finite dimensional subspace of $\mathcal{Y}^*$, the dual of $\mathcal{Y}$. If for every $z^* \in Z^*$ and some $\gamma > 1, \gamma$ independent of $z^*$,*

$$\|z^*\|_{\mathcal{Y}^*} \leq \gamma \|z^*|_{\mathcal{V}}\|_{\mathcal{V}^*}, \tag{5.1}$$

*then for $y \in \mathcal{Y}$ there exists $v \in \mathcal{V}$ such that $v$ interpolates $y$ on $Z^*$; that is, $z^*(y) = z^*(v)$ for all $z^* \in Z^*$. In addition, $v$ approximates $y$ in the sense that $\|y - v\| \leq (1 + 2\gamma)\mathrm{dist}(y, \mathcal{V})$.*

We want to apply this result to the case in which the underlying space is the Sobolev space, $\mathcal{Y} = H_\tau(\mathbb{S}^n)$, with $\tau > n/2$. The reproducing kernel $\varphi_\tau$ for this space is defined in (3.3). Since point evaluations are continuous, we may take $Z^* = \mathrm{span}\{\delta_{x_j}\}_{j=1}^N$, where the points are distinct and come from a finite set $X = \{x_j\}_{j=1}^N \subset \mathbb{S}^n$. Finally, we will also take $\mathcal{V} = \mathcal{H}_L$, the span of the spherical harmonics of order $\ell \leq L$.

The quantities in Proposition 5.2 can be put in terms of the reproducing kernel $\varphi_\tau$. First, we observe that

$$\Big\| \sum_j \alpha_j \delta_{x_j} \Big\|_{H_\tau(\mathbb{S}^n)^*} = \Big\| \sum_j \alpha_j \varphi_\tau(x \cdot x_j) \Big\|_{H_\tau(\mathbb{S}^n)}. \tag{5.2}$$

Second, with $\varphi_{\tau,L}(x \cdot y) = \sum_{\ell=0}^{L} (\lambda_\ell + 1)^{-\tau} \sum_{m=1}^{d_\ell} Y_{\ell,m}(x) Y_{\ell,m}(y)$, we obtain

$$\begin{aligned}
\Big\| \sum_j \alpha_j \delta_{x_j}|_{\mathcal{H}_L} \Big\|_{\mathcal{H}_L^*} &= \sup_{p \in \mathcal{H}_L,\, \|p\|=1} \Big| \langle p(x), \sum_j \alpha_j \varphi_\tau(x \cdot x_j) \rangle_{H_\tau(\mathbb{S}^n)} \Big| \\
&= \sup_{p \in \mathcal{H}_L,\, \|p\|=1} \Big| \langle p(x), \sum_j \alpha_j \varphi_{\tau,L}(x \cdot x_j) \rangle_{H_\tau(\mathbb{S}^n)} \Big| \\
&= \Big\| \sum_j \alpha_j \varphi_{\tau,L}(x \cdot x_j) \Big\|_{H_\tau(\mathbb{S}^n)}. \qquad (5.3)
\end{aligned}$$

Moreover, since $\varphi_{\tau,L}(x \cdot x_j)$ and $\varphi_\tau(x \cdot x_k) - \varphi_{\tau,L}(x \cdot x_k)$ are orthogonal in $H_\tau(\mathbb{S}^n)$ for all $j, k$, we can use the Pythagorean theorem to obtain

$$\begin{aligned}
\Big\| \sum_j \alpha_j \varphi_{\tau,L}(x \cdot x_j) \Big\|_{H_\tau(\mathbb{S}^n)}^2 &= \Big\| \sum_j \alpha_j \varphi_\tau(x \cdot x_j) \Big\|_{H_\tau(\mathbb{S}^n)}^2 \\
&\quad - \Big\| \sum_j \alpha_j \big(\varphi_\tau(x \cdot x_j) - \varphi_{\tau,L}(x \cdot x_j)\big) \Big\|_{H_\tau(\mathbb{S}^n)}^2.
\end{aligned}$$

¿From this and the quantities above, it follows that $\gamma$ in Proposition 5.2 satisfies the inequality below, which is equivalent to (5.1):

$$\frac{\big\| \sum_j \alpha_j \big(\varphi_\tau(x \cdot x_j) - \varphi_{\tau,L}(x \cdot x_j)\big) \big\|_{H_\tau(\mathbb{S}^n)}^2}{\big\| \sum_j \alpha_j \varphi_\tau(x \cdot x_j) \big\|_{H_\tau(\mathbb{S}^n)}^2} \leq 1 - \frac{1}{\gamma^2}. \tag{5.4}$$

**Proof of Theorem 5.1.** We begin by "lifting" the problem into $\mathbb{R}^{n+1}$. Set $s = \tau + \frac{1}{2}$ and let $\widehat{\Psi}(\xi) = (1 + |\xi|^2)^{-s}$. The function $\Psi$ is the reproducing kernel for $H_s(\mathbb{R}^{n+1})$. If $\tau > \frac{n}{2}$, then $s > \frac{n+1}{2}$ and the conditions of Proposition 4.4 are met. Using the inequalities and notation there, we have

$$\frac{\left\| \sum_j \alpha_j(\varphi_\tau(x \cdot x_j) - \varphi_{\tau,L}(x \cdot x_j))\right\|^2_{H_\tau(\mathbb{S}^n)}}{\left\| \sum_j \alpha_j \varphi_\tau(x \cdot x_j)\right\|^2_{H_\tau(\mathbb{S}^n)}} \le \frac{2b\| \sum_j \alpha_j(\Psi - \Psi_\sigma)(x - x_j)\|^2_\Psi}{a\left\| \sum_j \alpha_j \Psi(x - x_j)\right\|^2_\Psi},$$

where we take $\sigma = e^{-1}L$ and $L \ge L_0$. Of course, $\|\cdot\|_\Psi = \|\cdot\|_{H_s(\mathbb{R}^{n+1})}$.

The ratio of the squares of the norms on the right above was estimated in proving [19, Lemma 3.3]. In our notation, the result obtained there was

$$\frac{\| \sum_j \alpha_j(\Psi - \Psi_\sigma)(x - x_j)\|^2_\Psi}{\left\| \sum_j \alpha_j \Psi(x - x_j)\right\|^2_\Psi} \le C(\sigma Q_X)^{n+1-2s},$$

where $C = C(s, n+1)$ and $Q_X$ is the *Euclidean* separation radius for $X$ as a subset of $\mathbb{R}^{n+1}$. For a set $X$ with any appreciable number of points, $Q_X \approx q_X$. At the very worst, $Q_X \ge (2/\pi)q_X$. Also, $\sigma = e^{-1}L$. Combining all of these things yields

$$\frac{\left\| \sum_j \alpha_j(\varphi_\tau(x \cdot x_j) - \varphi_{\tau,L}(x \cdot x_j))\right\|^2_{H_\tau(\mathbb{S}^n)}}{\left\| \sum_j \alpha_j \varphi_\tau(x \cdot x_j)\right\|^2_{H_\tau(\mathbb{S}^n)}} \le C''(Lq_X)^{n-2\tau}, \tag{5.5}$$

where $C'' = \frac{2^{n+1-2\tau}bC}{a(e\pi)^{n-2\tau}}$. Choose $\kappa > 0$ so that $C''\kappa^{n-2\tau} \le \frac{3}{4}$. Since $\tau > \frac{n}{2}$, the exponent $n - 2\tau < 0$. If $Lq_X \ge \kappa$, then $C''(Lq_X)^{n-2\tau} \le \frac{3}{4}$. Thus, (5.4) holds with $\gamma = 2$ when $Lq_X \ge \kappa$. Applying Proposition 5.2 then finishes the proof. □

## 5.2 Sobolev error estimates for the polynomial interpolant

We want obtain Sobolev error estimates for the polynomial interpolant $p$ constructed previously. We begin by stating a result of Hubbert and Morton [10], but with our notation and for the case of interest here. Let $\tau > n/2$ and again take $\varphi_\tau$ to be the SBF in (3.3). In addition, recall that $\hat{\varphi}_\tau(\ell) = (\lambda_\ell + 1)^{-\tau}$ is the Fourier-Legendre coefficient for $\varphi_\tau$. Finally, let $\phi$ be an SBF that satisfies

$$a\hat{\varphi}_\tau(\ell) \le \hat{\phi}(\ell) \le b\hat{\varphi}_\tau(\ell), \tag{5.6}$$

where $a, b$ are positive constants independent of $\ell$. By our earlier remarks in Section 3, the native spaces for $\phi$ and $\varphi_\tau$, $\mathcal{N}_\phi$, $\mathcal{N}_{\varphi_\tau}$, coincide with the Sobolev space $H_\tau(\mathbb{S}^n)$, up to norm equivalence.

**Proposition 5.3 ([10, Theorem 3.4]).** *Let the SBF $\phi$ satisfy (5.6) with $\tau > n/2$, and let $X$ be a finite set of distinct points on $\mathbb{S}^n$ having mesh norm $h$. If $f \in \mathcal{N}_\phi$ and $I_{\phi,X}f$ is the $\phi$-interpolant for $f$, then for all $h$ sufficiently small there is a constant $C > 0$ that is independent of $f$ and $h$ such that*

$$\|f - I_{X,\phi}f\|_{L^2(\mathbb{S}^n)} \leq Ch^\tau \|f - I_{X,\phi}f\|_\phi . \tag{5.7}$$

**Proof:** We just need to reconcile our notation with that of [10, Theorem 3.4]. Our $n$ corresponds to their $d-1$, and our $\tau$ corresponds to $\frac{1}{2}(\alpha+d-1)$. Also, we have taken $p = 2$. $\square$

Our aim is to prove the following result.

**Corollary 5.4.** *Let the assumptions and notation of Theorem 5.1 hold, and let $0 \leq \mu \leq \tau$. If the mesh norm $h$ of $X$ is sufficiently small, then there is a constant $C > 0$ that is independent of $h$, $p$, and $f$ for which the interpolating polynomial $p \in \mathcal{H}_L$ satisfies*

$$\|f - p\|_{H_\mu(\mathbb{S}^n)} \leq Ch^{\tau-\mu}\|f\|_{H_\tau(\mathbb{S}^n)}. \tag{5.8}$$

**Proof:** Apply the estimate in (5.7) with $\phi = \varphi_\tau$. Using the minimizing property of the interpolant $I_{X,\varphi_\tau}f$ together with the fact that $\mathcal{N}_{\varphi_\tau}$ and $H_\tau(\mathbb{S}^n)$ are the same, we obtain

$$\|f - I_{X,\varphi_\tau}f\|_{L^2(\mathbb{S}^n)} \leq Ch^\tau\|f - I_{X,\varphi_\tau}f\|_{\varphi_\tau} \leq Ch^\tau\|f\|_{H_\tau(\mathbb{S}^n)} .$$

By Theorem 5.1, there exists a constant $\kappa > 0$ such that we can find a polynomial $p \in \mathcal{H}_L$ for which $f|_X = p|_X$ and $\|f-p\|_{H_\tau(\mathbb{S}^n)} \leq 5\,\mathrm{dist}_{H_\tau(\mathbb{S}^n)}(f, \mathcal{H}_L)$ hold, provided $L \geq \kappa/q$.

We can easily derive estimates on $\|p\|_{H_\tau(\mathbb{S}^n)}$. Just use $\mathrm{dist}_{H_\tau(\mathbb{S}^n)}(f, \mathcal{H}_L) \leq \|f\|_{H_\tau(\mathbb{S}^n)}$ together with the the estimate on $\|f-p\|_{H_\tau(\mathbb{S}^n)}$ to get

$$\|f - p\|_{H_\tau(\mathbb{S}^n)} \leq 5\|f\|_{H_\tau(\mathbb{S}^n)}. \tag{5.9}$$

Next, employ the triangle inequality; the result is

$$\|p\|_{H_\tau(\mathbb{S}^n)} \leq 6\|f\|_{H_\tau(\mathbb{S}^n)}. \tag{5.10}$$

In addition, we have that $I_{X,\varphi_\tau}f = I_{X,\varphi_\tau}p$, because $f|_X = p|_X$ and the interpolants only depend on function values from $X$. Making use of these facts and applying the inequality (5.7) to $p$, we get the following:

$$\begin{aligned}
\|f - p\|_{L^2(\mathbb{S}^n)} &= \|f - I_{X,\varphi_\tau}f + I_{X,\varphi_\tau}p - p\|_{L^2(\mathbb{S}^n)} \\
&\leq \|f - I_{X,\varphi_\tau}f\|_{L^2(\mathbb{S}^n)} + \|I_{X,\varphi_\tau}p - p\|_{L^2(\mathbb{S}^n)} \\
&\leq Ch^\tau\|f\|_{H_\tau(\mathbb{S}^n)} + Ch^\tau\|p\|_{H_\tau(\mathbb{S}^n)} \\
&\leq C'h^\tau\|f\|_{H_\tau(\mathbb{S}^n)}. \qquad (5.11)
\end{aligned}$$

Next, let $\alpha = \tau$, $\beta = 0$, $s = \tau/\mu$, $1/t = 1 - \mu/\tau$, and apply the inequality in (3.4) to $\|f - p\|_{H_\mu(\mathbb{S}^n)}$. This gives us

$$\|f - p\|_{H_\mu(\mathbb{S}^n)} \le \|f - p\|_{H_\tau(\mathbb{S}^n)}^{\mu/\tau} \|f - p\|_{L^2(\mathbb{S}^n)}^{1-\mu/\tau} .$$

Employing the bounds given in (5.9) and (5.11) in connection with the inequality above yields (5.8), after simplifying the exponents involved. □

### 5.3 Sobolev error estimates for the SBF interpolant

Our goal is to derive Sobolev-type error estimates for SBF interpolation when the target function $f$ is not smooth enough to be in the native space of the SBF. Specifically, we will prove the result below.

**Theorem 5.5.** *Let $\tau \ge \beta > n/2$ and let $\phi$ be an SBF satisfying (5.6). Also, take $X$ to be a finite set of distinct points on $\mathbb{S}^n$ having mesh norm $h_X$, separation radius $q_X$ and mesh ratio $\rho_X = h_X/q_X$. If $f \in H_\beta(\mathbb{S}^n)$, then for $0 \le \mu \le \beta$ we have*

$$\|f - I_{X,\phi} f\|_{H_\mu(\mathbb{S}^n)} \le C \rho_X^{\tau-\beta} h_X^{\beta-\mu} \|f\|_{H_\beta(\mathbb{S}^n)} . \tag{5.12}$$

**Proof:** Let $q = q_X$ and $h = h_X$ By Corollary 5.4, with $\tau$ replaced by $\beta$ and $0 \le \mu \le \beta$, we have a constant $\kappa > 0$ such that for any integer $L \ge \kappa/q$, there is a polynomial $p \in \mathcal{H}_L$ for which $p|_X = f|_X$,

$$\|f - p\|_{H_\mu(\mathbb{S}^n)} \le C h^{\beta-\mu} \|f\|_{H_\beta(\mathbb{S}^n)} \quad \text{and} \quad \|p\|_{H_\beta(\mathbb{S}^n)} \le 6 \|f\|_{H_\beta(\mathbb{S}^n)} . \tag{5.13}$$

The second inequality above is (5.10) with $\tau$ replaced by $\beta$. Since $p|_X = f|_X$, we also have $I_{X,\phi} p = I_{X,\phi} f$, because the interpolants only depend on the values of $f$ on $X$. Applying Proposition 5.3 to $p$, with the same $\tau$ used here, gives us

$$\|p - I_{X,\phi} f\|_{L^2(\mathbb{S}^n)} = \|p - I_{X,\phi} p\|_{L^2(\mathbb{S}^n)} \le C h^\tau \|p - I_{X,\phi} p\|_\phi .$$

Using the variational properties of the SBF interpolant and the bounds in (5.6), we have the inequality,

$$b^{-1/2} \|p - I_{X,\phi} p\|_{H_\tau(\mathbb{S}^n)} \le \|p - I_{X,\phi} p\|_\phi \le \|p\|_\phi \le a^{-1/2} \|p\|_{H_\tau(\mathbb{S}^n)} .$$

Putting this inequality and the previous one together, we have that

$$\begin{aligned} \|p - I_{X,\phi} p\|_{L^2(\mathbb{S}^n)} &\le C a^{-1/2} h^\tau \|p\|_{H_\tau(\mathbb{S}^n)} , \\ \|p - I_{X,\phi} p\|_{H_\tau(\mathbb{S}^n)} &\le (b/a)^{1/2} \|p\|_{H_\tau(\mathbb{S}^n)} . \end{aligned}$$

As in the proof of Corollary 5.4, we apply the inequality in (3.4), with $\alpha = \tau$, $\beta = 0$, $s = \tau/\mu$, $1/t = 1 - \mu/\tau$, along with these inequalities to obtain this:

$$\|p - I_{X,\phi}p\|_{H_\mu(\mathbb{S}^n)} \leq C' h^{\tau-\mu} \|p\|_{H_\tau(\mathbb{S}^n)}\,, \tag{5.14}$$

where $C' = b^{\mu/(2\tau)} C^{1-\mu/\tau} a^{-1/2}$. We now apply a well-known $L^2$ Bernstein theorem for spherical polynomials: If $p \in \mathcal{H}_L$ and $\lambda_L = L(L+n-1)$, where $\lambda_L$ is an eigenvalue of the Laplace-Beltrami operator for $\mathbb{S}^n$, then

$$\|p\|_{H_\tau(\mathbb{S}^n)} \leq (1+\lambda_L)^{\frac{\tau-\beta}{2}} \|p\|_{H_\beta(\mathbb{S}^n)}. \tag{5.15}$$

If we choose $L = \lceil \kappa/q \rceil$ and employ the second inequality in (5.13), then

$$\|p\|_{H_\tau(\mathbb{S}^n)} \leq C q^{\beta-\tau} \|f\|_{H_\beta(\mathbb{S}^n)}.$$

Using this inequality in conjunction with (5.14) results in

$$\|p - I_{X,\phi}p\|_{H_\mu(\mathbb{S}^n)} \leq C'' h^{\tau-\mu} q^{\beta-\tau} \|f\|_{H_\beta(\mathbb{S}^n)}\,. \tag{5.16}$$

Next, from the triangle inequality and the identity $I_{X,\phi}f = I_{X,\phi}p$, we see that

$$\|f - I_{X,\phi}f\|_{H_\mu(\mathbb{S}^n)} \leq \|f-p\|_{H_\mu(\mathbb{S}^n)} + \|p - I_{X,\phi}p\|_{H_\mu(\mathbb{S}^n)}.$$

If we now use (5.13) and (5.16) in conjunction with the inequality above, we obtain

$$\|f - I_{X,\phi}f\|_{H_\mu(\mathbb{S}^n)} \leq h^{\beta-\mu}(C + C''(h/q)^{\tau-\beta})\|f\|_{H_\beta(\mathbb{S}^n)}\,.$$

Finally, noting that $\rho_X = h/q \geq 1$ and $\tau - \beta \geq 0$, we can lump the constants involved into a single "$C$" to get (5.12). ☐

There is an immediate corollary to the previous theorem, one that deals with a $\rho$-uniform family $\mathcal{F}$ of centers. Recall that a set $X$ of centers being in $\mathcal{F}$ means that $\rho_X \leq \rho$. Thus, in the inequality in the theorem we can replace $\rho_X$ by $\rho$ to get the result below. The importance of this lay in its defining the idea of *convergence* in the context of SBF interpolation and approximation. We will turn to these topics in the next section.

**Corollary 5.6.** *Let $\mathcal{F}$ be a family of $\rho$-uniform sets. If $X \in \mathcal{F}$, then*

$$\|f - I_{X,\phi}f\|_{H_\mu(\mathbb{S}^n)} \leq C_{\mathcal{F}} h_X^{\beta-\mu} \|f\|_{H_\beta(\mathbb{S}^n)},$$

*where $C\rho_X^{\tau-\beta} \leq C\rho^{\tau-\beta} =: C_{\mathcal{F}}$ is fixed for all $X \in \mathcal{F}$.*

## 6 A Bernstein Inequality and an Inverse Theorem

In this section we want to establish two theorems. The first deals with bounding smoother norms of interpolants by weaker norms, similar to the classical Bernstein inequality bounding the derivative of a polynomial by its $L^\infty$ norm. The second result is an inverse theorem, dealing with the characterization of smooth functions by convergence orders of SBF interpolants.

Let $\tau > n/2$. For any SBF $\phi$ satisfying (5.6) and any finite set $X$ of distinct points on the sphere, define the space

$$V_{\phi,X} := \text{span}\{\phi(x \cdot x_j) : x_j \in X\}. \tag{6.1}$$

Of course, since $\phi(x \cdot x_j) \in H_\tau(\mathbb{S}^n)$, it follows that $V_{\phi,X}$ is a finite dimensional subspace of $H_\tau(\mathbb{S}^n)$. Indeed, set $g_\phi(x) = \sum_j \alpha_j \phi(x \cdot x_j)$ and $g_{\varphi_\tau}(x) = \sum_j \alpha_j \varphi_\tau(x \cdot x_j)$, where $\varphi_\tau$ is defined in (3.3). From

$$\|g_\phi\|^2_{H_\mu(\mathbb{S}^n)} = \sum_{\ell,m} (1+\lambda_\ell)^\mu \hat{\phi}(\ell)^2 \sum_j |\alpha_j Y_{\ell,m}(x_j)|^2$$

and (5.6), it follows that

$$a\|g_{\varphi_\tau}\|_{H_\mu(\mathbb{S}^n)} \le \|g_\phi\|_{H_\mu(\mathbb{S}^n)} \le b\|g_{\varphi_\tau}\|_{H_\mu(\mathbb{S}^n)}\,.$$

Bearing this in mind, we have the following Bernstein-type inequality.

**Theorem 6.1.** *Let $V_{\phi,X}$ be as in (6.1), with $q_X$ being the separation radius of $X$. For $0 \le \mu \le \tau$, if $g \in V_{\phi,X}$, then there is a constant $C = C_{\phi,\tau,\mu,n}$ that is independent of $X$ and $g$ such that*

$$\|g\|_{H_\mu(\mathbb{S}^n)} \le C q_X^{-\mu} \|g\|_{L^2(\mathbb{S}^n)}.$$

**Proof:** We will work with $\phi = \varphi_\tau$. By the preceding remarks, there is no loss of generality in doing so. Note that the $\mu = 0$ case is trivial. To get the $\mu = \tau$ case, we turn to equation (5.5), where we again require $Lq_X \ge \kappa$ to get (5.4) to hold with $\gamma = 2$. In the notation here, the equation (5.4) becomes

$$\frac{\|g - g_L\|^2_{H_\tau(\mathbb{S}^n)}}{\|g\|^2_{H_\tau(\mathbb{S}^n)}} \le 3/4,$$

where $g(x) = \sum_j \alpha_j \varphi_\tau(x \cdot x_j)$ and $g_L(x) = \sum_j \alpha_j \varphi_{\tau,L}(x \cdot x_j)$. Of course, the $\alpha_j$'s are arbitrary and so $g$ is an arbitrary function in $V_{\varphi_\tau,X}$. Also, since $\varphi_{\tau,L}(x \cdot x_j)$ is the orthogonal projection of the $\varphi_\tau(x \cdot x_j)$ onto $\mathcal{H}_L$, the span of the spherical harmonics of order $\ell \le L$, the function $g_L$ is the orthogonal projection of $g$ onto $\mathcal{H}_L$; thus, $g_L \in \mathcal{H}_L$. Standard orthogonality in $H_\tau(\mathbb{S}^n)$ implies that $\|g - g_L\|^2_{H_\tau(\mathbb{S}^n)} = \|g\|^2_{H_\tau(\mathbb{S}^n)} - \|g_L\|^2_{H_\tau(\mathbb{S}^n)}$. Using this in conjunction with the inequality above, we obtain

$$\|g\|_{H_\tau(\mathbb{S}^n)} \le 2\|g_L\|_{H_\tau(\mathbb{S}^n)}.$$

The $L^2$ Bernstein inequality in (5.15) holds for any polynomial in $\mathcal{H}_L$, not just the particular one used there. We can thus apply it to $g_L$, with $\beta = 0$, to get $\|g_L\|_{H_\tau(\mathbb{S}^n)} \le (1 + \lambda_L)^{\tau/2}\|g_L\|_{L^2(\mathbb{S}^n)} \le C_{\tau,n}L^\tau\|g_L\|_{L^2(\mathbb{S}^n)}$. Since $g_L$ is the orthogonal projection of $g$ onto $\mathcal{H}_L$, we have $\|g_L\|_{L^2(\mathbb{S}^n)} \le \|g\|_{L^2(\mathbb{S}^n)}$ and, consequently,

$$\|g\|_{H_\tau(\mathbb{S}^n)} \le 2C_{\tau,n}L^\tau\|g\|_{L^2(\mathbb{S}^n)}.$$

This holds if $Lq_X \ge \kappa$. In particular, we will take $L = \lceil \kappa/q_X \rceil < 2\kappa/q_X$, to arrive at

$$\|g\|_{H_\tau(\mathbb{S}^n)} \le C'_{\tau,n}q_X^{-\tau}\|g\|_{L^2(\mathbb{S}^n)}.$$

Finally, we apply the inequality in (3.4) to $\|g\|_{H_\mu(\mathbb{S}^n)}$, with $\alpha = \tau$, $\beta = 0$, $s = \tau/\mu$, $1/t = 1 - \mu/\tau$, to get

$$\|g\|_{H_\mu(\mathbb{S}^n)} \le \|g\|^{\mu/\tau}_{H_\tau(\mathbb{S}^n)}\|g\|^{1-\mu/\tau}_{L^2(\mathbb{S}^n)} \le Cq_X^{-\mu}\|g\|_{L^2(\mathbb{S}^n)}\,.$$

□

Inverse theorems give indications of rates of approximation being best, or nearly best, possible. The two results below, which involve slightly different conditions on a target function $f$, are inverse theorems for the approximation rates derived in Section 5.3.

**Theorem 6.2.** *Let $\tau > n/2$ and $\phi$ be an SBF satisfying (5.6). In addition, let $\mathcal{F}$ be a $\rho$-uniform family, $\rho \ge 2$. If for some $f \in C(\mathbb{S}^n)$ there are constants $0 < \mu \le \tau$ and $c_f > 0$ such that*

$$\|f - I_{X,\phi}f\|_{L^2(\mathbb{S}^n)} \le c_f h_X^\mu \tag{6.2}$$

*holds for all $X \in \mathcal{F}$, then, for every $0 \le \beta < \mu$, $f \in H_\beta(\mathbb{S}^n)$.*

**Proof:** By Proposition 2.1, we can find a nested sequence $X_k \in \mathcal{F}$, each $X_k$ having mesh norm $h_k := h_{X_k}$ that satisfies $\frac{1}{4}h_k < h_{k+1} \le \frac{1}{2}h_k \le \frac{1}{2^k}h_0$. Let $f_k := I_{X_k,\phi}f \in V_{\phi,X_k} \subset H_\tau$. In fact, because $X_k \subset X_{k+1}$, we have $f_k \in V_{\phi,X_j}$ for all $j \ge k$. We want to show that $f_k$ is a Cauchy sequence in $H_\beta$. From the Bernstein estimate in Theorem 6.1 and the inequality $h_{k+1}/q_{k+1} \le \rho$, we have

$$\|f_{k+1} - f_k\|_{H_\beta(\mathbb{S}^n)} \le C\rho^\beta h_{k+1}^{-\beta}\|f_{k+1} - f_k\|_{L^2(\mathbb{S}^n)}.$$

And by (6.2), we also have

$$\begin{aligned}\|f_{k+1}-f_k\|_{H_\beta(\mathbb{S}^n)} &\le C\rho^\beta h_{k+1}^{-\beta}\|f_{k+1}-f_k\|_{L^2(\mathbb{S}^n)}\\ &\le C\rho^\beta h_{k+1}^{-\beta}\big(\|f_{k+1}-f\|_{L^2(\mathbb{S}^n)}+\|f-f_k\|_{L^2(\mathbb{S}^n)}\big)\\ &\le Cc_f\rho^\beta h_{k+1}^{-\beta}(h_{k+1}^\mu+h_k^\mu)\le C'\,2^{-(\mu-\beta)k},\end{aligned}$$

where $C'$ is independent of $k$. Take $j>k$. Using the previous inequality, a standard telescoping-series argument, and the sum of terms from a geometric series, we arrive at this:

$$\|f_j-f_k\|_{H_\beta(\mathbb{S}^n)}\le C'\frac{2^{-(\mu-\beta)j}-2^{-(\mu-\beta)k}}{1-2^{-(\mu-\beta)}}.$$

Letting $j,k\to\infty$, we see $\|f_j-f_k\|_{H_\beta(\mathbb{S}^n)}\to 0$. Thus, $f_k$ is a Cauchy sequence in $H_\beta(\mathbb{S}^n)$ and is therefore convergent to $\tilde{f}\in H_\beta(\mathbb{S}^n)$. Moreover, by (6.2) with $X=X_k$, we see that $f_k\to f$ in $L^2(\mathbb{S}^n)$, so $\tilde{f}=f$ almost everywhere. Hence, we have $f\in H_\beta(\mathbb{S}^n)$. □

The space $V_{\phi,X}$ is finite dimensional, because $X$ is a finite set. Thus for $X_k$ in the sequence and any $f\in L^2(\mathbb{S}^n)$, we can find $f_k\in V_{\phi,X_k}$ such that

$$\mathrm{dist}_{L^2(\mathbb{S}^n}(f,V_{\phi,X_k})=\|f-f_k\|_{L^2(\mathbb{S}^n)}\,.$$

By replacing (6.2) with $\mathrm{dist}_{L^2(\mathbb{S}^n}(f,V_{\phi,X})\le c_f h_X^\mu$ in the theorem, we can weaken the condition $f\in C(\mathbb{S}^n)$ to $f\in L^2(\mathbb{S}^n)$, and then use a nearly identical proof to reach the same conclusion. We state this result as a corollary.

**Corollary 6.3.** *Let the notation and assumptions of Theorem 6.2 hold, except that we suppose for some $f\in L^2(\mathbb{S}^n)$,*

$$\mathrm{dist}_{L^2(\mathbb{S}^n)}(f,V_{\phi,X})\le c_f h_X^\mu$$

*holds for all $X\in\mathcal{F}$. Then, $f\in H_\beta(\mathbb{S}^n)$ whenever $0\le\beta<\mu$.*

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