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Sobolev Error Estimates and a Bernstein Inequality for Scattered Data Interpolation via Radial Basis Functions *†

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Abstract

Error estimates for scattered-data interpolation via radial basis functions for target functions in the associated reproducing kernel Hilbert space have been known for a long time. Recently, these estimates have been extended to apply to certain classes of target functions generating the data which are outside of the associated RKHS. However, these classes of functions still were not "large" enough to be applicable to a number of practical situations. In this paper, we obtain Sobolev-type error estimates on compact regions of $\mathbb{R}^n$ when the RBFs have Fourier transforms that decay algebraically. In addition, we derive a Bernstein inequality for spaces of finite shifts of an RBF in terms of the minimal separation parameter.

* *Key words:* Radial basis functions, band-limited functions, error estimates, scattered data, Bernstein inequality.

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1 Introduction

The problem of effectively representing an underlying function based on its values sampled at finitely many distinct scattered sites $X = \{x_1, \ldots, x_N\}$ lying in a compact region $\Omega \subset R^n$ is important and arises in many applications – neural networks, computer aided geometric design, and grid-less methods for solving partial differential equations, to name a few. A good example of this type of problem is addressed in a recent paper of Carr et al. [4]. There, the authors used radial basis function (RBF) interpolation to reconstruct 3D objects from "clouds" of points. Handling the large numbers of points was aided by new, fast evaluation techniques for RBFs.

This problem of representing a multivariate function by interpolating at scattered values is a difficult one. RBFs were introduced as a means to attack this problem. An RBF is a radial function $\Phi(x) = \Phi(|x|)$ that is either positive definite or conditionally positive definite on $\mathbb{R}^n$. Interpolants for multivariate functions sampled at scattered sites are constructed from translates of RBFs, with the possible addition of a polynomial term. It was Duchon who introduced a type of RBF, the thin-plate spline, which he constructed via a variational technique similar to those used to obtain ordinary splines. The error analysis he provided for thin-plate splines involved reproducing kernel Hilbert space (RKHS) methods. Later, there were important contributions by Madych and Nelson [6] and Wu and Schaback [16], who also used RKHS methods to obtain scattered-data interpolation error estimates for a wide class of RBFS, including the Hardy multiquadrics and the Gaussians.

Important as these results are, they do suffer from a common difficulty. In all cases, convergence is proved only for functions in an RKHS that depends on Φ: the smoother the function Φ, the smaller the RKHS for which convergence estimates apply. This drawback had restricted the use of RBFs in applications and had seemed artificial, especially in light of both the lattice-based least-squares theory, which was completely and satisfactorily solved in [1] and the work of Schaback dealing with pure approximation by RBFs [14]. Indeed, Yoon also had noted this problem, and introduced scaled RBFs in which a parameter λ is required to depend on the spacing of the data. In effect, the radial function is changing with the data. [7]

This difficulty has been partially overcome. The first "escape" from native space came in connection with the n-sphere, $\mathbb{S}^n$, for spherical basis functions (SBFs) rather than RBFs [11]. By "escape" we mean that the RBF interpolant $I_X f$ for a function f, whose native space norm is infinite, is still an effective approximation to f. Put another way, RBF interpolants are smooth, but they still provide a good fit to less smooth functions. Other work [3, 12, 13] was directed at problems where the underlying space was a domain in $\mathbb{R}^n$. For a more complete description of these results, as well as other references, see [7].

The estimates in the escape theorems up to now still do not fully address the most common error estimates required in applications to PDE, learning theory, and so on. What's been lacking are estimates with appropriate norms on both sides of the inequality. In particular, for a quasi-uniform X, a main result of this paper will be to show that a Sobolev least-squares estimate of the form,

$$\|f - I_X f\|_{W_2^{|\alpha|}(\Omega)} \leq C h_X^{k-|\alpha|} \|f\|_{W_2^k(\Omega)}, \quad 0 \leq |\alpha| \leq k\,,$$

holds for f outside of the native space of Φ – at least in the case where $\widehat{\Phi}$ has an algebraic singularity at the origin and algebraic decay at ∞; e.g. thin-plate splines and Wendland functions. Other "escape" estimates involving $W_p^k(\Omega)$ will also be derived.

The main results of this paper may be viewed in two ways. From the theoretical point of view, they provide a much larger class of functions for which interpolation error estimates apply. From a practical point of view, they allow more flexibility in the choice of RBFs when applied for collocation purposes and faster convergence rates of interpolants away from singularities of the target function. This may even make possible using RBF methods for singularity detection.

This paper is organized as follows. Section 2 is devoted to notation and background information. Section 3 is key for obtaining our main results. In it, we prove a theorem constructing a band-limited function that is both an interpolant and nearly a best approximant in the Sobolev space W_2^β, $\beta > n/2$. The band length is proportional to the reciprocal of the minimum separation of data. As a corollary, we obtain a similar result, with $\mathbb{R}^n$ replaced by a compact region Ω that has a Lipschitz boundary. The main result, which is in section 4, concern Sobolev error estimates for RBF interpolation. The RBFs have native spaces equivalent to Sobolev spaces or Beppo-Levi spaces. A by-product is the interesting fact (cf. Corollary 4.3) that the interpolation map $I_X : W_2^\beta(\Omega) \to W_2^\beta(\Omega)$ is bounded under mild assumptions on X and Φ. Finally, in section 5, a Bernstein inequality is given for functions in spaces of the form $\mathrm{span}\{\Phi(\cdot - x_j) : x_j \in X\}$.

2 Notation

We will take Ω to be a compact set in $\mathbb{R}^n$. Unless we explicitly state otherwise, we will assume in addition that Ω satisfies an interior cone condition and has a Lipschitz boundary. The set $X = \{x_1, \ldots, x_N\} \subset \Omega$ will always be a finite subset of Ω, with the points all assumed to be distinct. There are two useful quantities associated with X. The first is the separation radius,

$$q_X := \frac{1}{2} \min_{j \neq k} \|x_j - x_k\|_2$$

which is half of the smallest distance between any two distinct points in X. The second is the mesh norm for X relative to Ω given by

$$h_{X,\Omega} := \sup_{x \in \Omega} \inf_{x_j \in X} \|x - x_j\|_2;$$

it measures the maximum distance any point in Ω can be from X. It is easy to see that $h_{X,\Omega} \geq q_X$; equality can only hold for a uniform distribution of points on an interval in $\mathbb{R}$. The mesh ratio $\rho = \rho_{X,\Omega} := h_{X,\Omega}/q_X \geq 1$ provides a measure of how uniformly points in X are distributed in Ω.

Our conventions for the Fourier transform and its inverse are

$$\widehat{f}(\omega) := \int_{\mathbb{R}^n} f(x) e^{-i\omega^T x} dx \quad \text{and} \quad f^\vee(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\omega) e^{i\omega^T x} d\omega.$$

We will make use of the Sobolev spaces $W_2^k(\mathbb{R}^n)$ and $W_2^k(\Omega)$. These spaces consist of all $f \in L^2$ having distributional derivatives $D^\alpha f$, $|\alpha| \leq k$ in L^2. In the case of $W_2^\beta(\mathbb{R}^n)$, where $\beta > n/2$ can also be a non integer, the norm that will be used here is

$$\|f\|_{W_2^\beta(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1 + \|\omega\|_2^2)^\beta |\widehat{f}(\omega)|^2 d\omega \right)^{1/2}.$$

Associated with $W_2^k(\Omega)$, are the (semi-) norms

$$|u|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p} \text{and } \|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p}.$$

We will also make use of some interpolation theory in Sobolev spaces. To be more precise, we will employ the following result, which is a direct consequence, for example, of [2, Prop. 12.1.5 and Thm. 12.2.7]).

Lemma 2.1. *Suppose $T : W_2^\tau(\Omega) \to W_2^\tau(\Omega)$ is a linear operator, where $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain. Suppose further that the operator is bounded in the following way:*

$$\|Tf\|_{L_2(\Omega)} \leq C_1 \|f\|_{W_2^\tau(\Omega)}, \quad \|Tf\|_{W_2^\tau(\Omega)} \leq C_2 \|f\|_{W_2^\tau(\Omega)}, \quad f \in W_2^\tau(\Omega).$$

Then, for every $0 < \beta < \tau$ we also have

$$\|Tf\|_{W_2^\beta(\Omega)} \leq C_1^{1-\beta/\tau} C_2^{\beta/\tau} \|f\|_{W_2^\tau(\Omega)}, \qquad f \in W_2^\tau(\Omega).$$

3 Band-Limited Functions

In this section, we review and establish certain interpolation and approximation results in the Paley-Wiener class of band-limited functions. Let $\sigma > 0$. We then define $\mathcal{B}_\sigma$ to be

$$\mathcal{B}_\sigma := \{f \in L^2(\mathbb{R}^n) : \text{ supp } \hat{f} \subseteq B(0, \sigma)\},$$

where $B(0, \sigma)$ is the (closed) ball in $\mathbb{R}^n$ having center 0 and radius σ.

Functions in $\mathcal{B}_\sigma$ are, of course, in L^2 and are analytic. Moreover, they satisfy the Bernstein inequality

$$\|D^\alpha f_\sigma\|_{L_2(\mathbb{R}^n)} \leq \sigma^{|\alpha|} \|f\|_{L_2(\mathbb{R}^n)}.$$

The results in this section are central for those that follow, and they concern simultaneous interpolation and approximation by functions in $\mathcal{B}_\sigma$. In preparation for them, we state the following proposition, whose proof can be found in [12].

Proposition 3.1. *Let $\mathcal{Y}$ be a (possibly complex) Banach space, $\mathcal{V}$ a subspace of $\mathcal{Y}$, and Z^* a finite dimensional subspace of $\mathcal{Y}^*$, the dual of $\mathcal{Y}$. If for every $z^* \in Z^*$ and some $\gamma > 1, \gamma$ independent of z^*,*

$$\|z^*\|_{\mathcal{Y}^*} \leq \gamma \|z^*|_\mathcal{V}\|_{\mathcal{V}^*},$$

then for $y \in \mathcal{Y}$ there exists $v \in \mathcal{V}$ such that v interpolates y on Z^*; that is, $z^*(y) = z^*(v)$ for all $z^* \in Z^*$. In addition, v approximates y in the sense that $\|y - v\| \leq (1 + 2\gamma)\text{dist}(y, \mathcal{V})$.

In addition to the proposition we just stated, we also need results involving the function space $W_2^\beta(\mathbb{R}^n)$. Recall that this space is a reproducing kernel Hilbert space for $\beta > n/2$. The kernel $\mathcal{K}_\beta$ is characterized by its

Fourier transform having the form $\widehat{\mathcal{K}}_\beta(\omega) = (1+\|\omega\|_2^2)^{-\beta}$, and it is given by [15, Theorem 6.13]

$$\mathcal{K}_\beta(x) = c_\beta \|x\|_2^{\beta-n/2} K_{n/2-\beta}(\|x\|_2) = c_\beta \|x\|_2^{\beta-n/2} K_{\beta-n/2}(\|x\|_2) \qquad (1)$$

where K_ν is the modified Bessel function of the second kind and c_β is a constant. The equality of the two expressions on the right follows from K_ν being even in its order ν. We will need the properties for $\mathcal{K}_\beta$ stated below.

Lemma 3.2. *The kernel $\mathcal{K}_\beta(x)$ is radial, positive, decreasing on $[0,\infty)$, and has the bound*

$$\mathcal{K}_\beta(x) \leq \sqrt{2\pi} c_\beta r^{\nu-\frac{1}{2}} e^{-r+\frac{\nu^2}{2r}}, \qquad r := \|x\|_2 > 0,$$

where $\nu = \beta - n/2$.

Proof. See Corollary 5.12 and Lemma 5.13 in [15] for proofs. □

Suppose that $X = \{x_1, \ldots, x_N\} \subset \Omega$ is a set of distinct points from a bounded set $\Omega \subset \mathbb{R}^n$, and that $c_1, \ldots, c_N$ are scalars in $\mathbb{R}$. If $\beta > n/2$ and $g := \sum c_j \mathcal{K}_\beta(\cdot - x_j)$, then it is straightforward to show that

$$\|g\|^2_{W_2^\beta(\mathbb{R}^n)} = (2\pi)^n \sum_{j,k} c_j c_k \mathcal{K}_\beta(x_j - x_k). \qquad (2)$$

Consequently, we have that

$$(2\pi)^n \lambda_X \|c\|_2^2 \leq \|g\|^2_{W_2^\beta(\mathbb{R}^n)} \leq (2\pi)^n \Lambda_X \|c\|_2^2,$$

where λ_X and Λ_X are, respectively, the minimum and maximum eigenvalues of the $N \times N$ matrix $(\mathcal{K}_\beta(x_j - x_k))$.

The minimum eigenvalues associated with such kernels were estimated from below in [9, 10]; those for the particular kernel $\mathcal{K}_\beta$ were dealt with in [15, Cor. 12.7 and 12.8]. The result is

$$\lambda_X \geq c_{\beta,n}\, q_X^{2\beta-n}, \qquad (3)$$

where $c_{\beta,n}$ depends only on $\mathcal{K}_\beta$ and n, but not on q_X or X. Upper estimates for the maximum eigenvalues of matrices for kernels related to the Gaussian kernel $F_\rho(x) = e^{-\rho\|x\|_2^2}$ were given in [8]. Now the properties of F_ρ used to obtain the results in [8] are just that F_ρ is positive, radial, decreasing on $[0,\infty)$, and decays fast enough for certain series to converge. These are the same as those given in Lemma 3.2 for $\mathcal{K}_\beta$. Thus, merely repeating the arguments used to establish [8, Theorem 2.2], with appropriate notational changes, gives us the following estimate,

$$\Lambda_X \leq \mathcal{K}_\beta(0) + \sum_{k=1}^{\infty} 3n(k+2)^{n-1}\mathcal{K}_\beta(kq_X). \tag{4}$$

Of course, by the bound in Lemma 3.2, the series on the right is uniformly convergent and decreasing in q_X, at least for values of q_X bounded away from 0. We now make use of these things to establish another lemma.

Lemma 3.3. *Let* $g := \sum_{j=1}^N c_j \mathcal{K}_\beta(\cdot - x_j)$ *and define* g_σ *by* $\widehat{g}_\sigma = \widehat{g}\chi_{B(0,\sigma)}$, *where* $\chi_{B(0,\sigma)}$ *is the characteristic function of the ball* $B(0,\sigma)$. *Then, there exists a constant* $\kappa > 0$, *which is independent of* X *and the* c_j*'s, such that for* $\sigma = \kappa/q_X$ *the following inequality holds*

$$I_\sigma := \|g - g_\sigma\|_{W_2^\beta(\mathbb{R}^n)} \leq \frac{1}{2}\|g\|_{W_2^\beta(\mathbb{R}^n)}. \tag{5}$$

Proof. From the definition of I_σ in (5) and a change of variables to $\omega = \sigma\xi$, we have that

$$I_\sigma^2 = \int\limits_{\|\omega\|_2 \geq \sigma} \frac{\left|\sum_j c_j e^{-i\omega^T x_j}\right|^2}{(1+\|\omega\|_2^2)^\beta} d^n\omega = \sigma^n \int\limits_{\|\xi\|_2 \geq 1} \frac{\left|\sum_j c_j e^{-i\xi^T \sigma x_j}\right|^2}{(1+\sigma^2\|\xi\|_2^2)^\beta} d^n\xi.$$

Since $\|\xi\|_2 \geq 1$, we have $\frac{1}{(1+\sigma^2\|\xi\|_2^2)^\beta} \leq \frac{2^\beta}{\sigma^{2\beta}} \frac{1}{\left(1+\|\xi\|_2^2\right)^\beta}$, so that

$$I_\sigma^2 \leq 2^\beta \sigma^{n-2\beta} \int\limits_{\|\xi\|_2 \geq 1} \frac{\left|\sum_j c_j e^{-i\xi^T \sigma x_j}\right|^2}{(1+\|\xi\|_2^2)^\beta} d^n\xi \leq 2^\beta \sigma^{n-2\beta} \int\limits_{\mathbb{R}^n} \frac{\left|\sum_j c_j e^{-i\xi^T \sigma x_j}\right|^2}{(1+\|\xi\|_2^2)^\beta} d^n\xi$$

Hence, we see that

$$I_\sigma^2 \leq (2\pi)^n 2^\beta \sigma^{n-2\beta} \sum_{j,k=1}^N c_j c_k \mathcal{K}_\beta(\sigma x_j - \sigma x_k) \leq (2\pi)^n 2^\beta \sigma^{n-2\beta} \Lambda_{\sigma X} \|c\|_2^2.$$

Combining (2) and (3), we also have that

$$(2\pi)^n c_{\beta,n} q_X^{2\beta-n} \|c\|_2^2 \leq \|g\|^2_{W_2^\beta(\mathbb{R}^n)},$$

and consequently we obtain

$$I_\sigma^2 \leq 2^\beta c_{\beta,n}^{-1} (\sigma q_X)^{n-2\beta} \Lambda_{\sigma X} \|g\|^2_{W_2^\beta(\mathbb{R}^n)}.$$

We now make two observations. The first is that the set σX has separation radius $q_{\sigma X} = \sigma q_X$, so if we choose $\sigma q_X \geq 1$, then (4) implies that

$$\Lambda_{\sigma X} \leq \mathcal{K}_\beta(0) + \sum_{k=1}^{\infty} 3n(k+2)^{n-1}\mathcal{K}_\beta(k) =: C_{\beta,n},$$

where $C_{\beta,n}$ depends only on n and β. From this, it follows that

$$I_\sigma^2 \leq 2^\beta C_{\beta,n} c_{\beta,n}^{-1} (\sigma q_X)^{n-2\beta} \|g\|^2_{W_2^\beta(\mathbb{R}^n)}.$$

The second is that we may now choose $\sigma q_X = \kappa$ so large that the factor multiplying $\|g\|^2_{W_2^\beta(\mathbb{R}^n)}$ is less than $1/4$. Taking square roots then completes the proof. $\square$

We may now combine Lemma 3.3 with Proposition 3.1 to arrive at the following conclusion.

Theorem 3.4. *Let $\beta, t \in \mathbb{R}$ satisfy $\beta > n/2$ and $t \geq 0$. If $f \in W_2^{\beta+t}(\mathbb{R}^n)$, then there exists an $f_\sigma \in \mathcal{B}_\sigma$ such that $f_\sigma|_X = f|_X$ and*

$$\|f - f_\sigma\|_{W_2^\beta(\mathbb{R}^n)} \leq 5 \cdot \mathrm{dist}_{W_2^\beta(\mathbb{R}^n)}(f, \mathcal{B}_\sigma) \leq 5 \cdot \kappa^{-t} q_X^t \|f\|_{W_2^{\beta+t}(\mathbb{R}^n)}, \qquad (6)$$

with $\sigma = \kappa/q_X$, where $\kappa \geq 1$ depends only on n and β.

Proof. We will apply Proposition 3.1. To do that, take $\mathcal{Y} = W_2^\beta(\mathbb{R}^n)$, $\mathcal{V} = \mathcal{B}_\sigma$, and $Z^* = \mathrm{span}\{\delta_{x_j} : x_j \in X\}$. Recall that we may identify $\mathcal{Y}^*$ with $\mathcal{Y}$. In particular, we may use the reproducing kernel $\mathcal{K}_\beta$ to identify Z^* with $\mathrm{span}\{\mathcal{K}_\beta(\cdot - x_j) : x_j \in X\}$. Thus $z^* = \sum_j c_j \delta(\cdot - x_j)$ corresponds to $g = \sum_j c_j \mathcal{K}_\beta(\cdot - x_j)$, and also $\|z^*\|_{\mathcal{Y}^*} = \|g\|_{W_2^\beta(\mathbb{R}^n)}$. It is easy to see that $z^*|_\mathcal{V}$ corresponds to taking the $W_2^\beta(\mathbb{R}^n)$ inner product of a function in $\mathcal{B}_\sigma$ with g_σ, where g_σ is defined in Lemma 3.3; consequently, $\|z^*|_\mathcal{V}\|_{\mathcal{V}^*} = \|g_\sigma\|_{W_2^\beta(\mathbb{R}^n)}$. Now, by Lemma 3.3 we have that for sufficiently large κ and all $\sigma = \kappa/q_X$,

$$\begin{aligned} \|g_\sigma\|_{W_2^\beta(\mathbb{R}^n)} &\geq \|g\|_{W_2^\beta(\mathbb{R}^n)} - \|g_\sigma - g\|_{W_2^\beta(\mathbb{R}^n)} \\ &\geq \|g\|_{W_2^\beta(\mathbb{R}^n)} - \frac{1}{2}\|g\|_{W_2^\beta(\mathbb{R}^n)} = \frac{1}{2}\|g\|_{W_2^\beta(\mathbb{R}^n)}. \end{aligned}$$

The conditions of Proposition 3.1 are thus satisfied, with the parameter $\gamma = 2$. It follows that for $f \in W_2^\beta(\mathbb{R}^n)$ there will be an $f_\sigma \in \mathcal{B}_\sigma$, with $\sigma = \kappa/q_X$, that interpolates f on X – $f_\sigma|_X = f|_X$ – and approximates it, in the sense that

$$\|f - f_\sigma\|_{W_2^\beta(\mathbb{R}^n)} \leq 5 \cdot \mathrm{dist}_{W_2^\beta(\mathbb{R}^n)}(f, \mathcal{B}_\sigma)$$

To finish the proof, we note that if $f \in W_2^{\beta+t}(\mathbb{R}^n)$, then we have

$$\begin{aligned}
\operatorname{dist}_{W_2^\beta(\mathbb{R}^n)}(f,\mathcal{B}_\sigma)^2 &= \int\limits_{\|\omega\|_2\geq\sigma} (1+\|\omega\|_2^2)^\beta \big|\widehat{f}(\omega)\big|^2 d^n\omega \\
&= \int\limits_{\|\omega\|_2\geq\sigma} \frac{(1+\|\omega\|_2^2)^{\beta+t}}{(1+\|\omega\|_2^2)^t}\big|\widehat{f}(\omega)\big|^2 d^n\omega \\
&\leq \sigma^{-2t}\|f\|^2_{W_2^{\beta+t}(\mathbb{R}^n)}
\end{aligned}$$

Taking square roots, using $\sigma = \kappa/q_X$, and combining the resulting inequality with the estimate from the first part of the proof, we obtain (6). □

We have assumed that Ω is compact, has a Lipschitz boundary, and satisfies an interior cone condition. These assumptions are sufficient to ensure the existence of a continuous extension operator $E_\Omega : W_2^{\beta+t}(\Omega) \to W_2^{\beta+t}(\mathbb{R}^n)$, as we noted in the proof of [13, Lemma 3.1]. Since $X \subset \Omega$, any function $f \in W_2^\beta(\Omega)$ coincides with its extension $E_\Omega f$ on X. Hence, if we extend the function $f \in W_2^{\beta+t}(\Omega)$ to $E_\Omega f \in W_2^{\beta+t}(\mathbb{R}^n)$ and choose $f_\sigma \in \mathcal{B}_\sigma$ as before, but this time for $E_\Omega f$, then we find $f_\sigma|_X = E_\Omega f|_X = f|_X$ and

$$\begin{aligned}
\|f-f_\sigma\|_{W_2^\beta(\Omega)} &= \|E_\Omega f - f_\sigma\|_{W_2^\beta(\Omega)} \\
&\leq \|E_\Omega f - f_\sigma\|_{W_2^\beta(\mathbb{R}^n)} \leq Cq_X^t\|E_\Omega f\|_{W_2^{\beta+t}(\mathbb{R}^n)} \qquad (7)\\
&\leq Cq_X^t\|f\|_{W_2^{\beta+t}(\Omega)} \qquad (8)
\end{aligned}$$

by Theorem 3.4. The middle equation (7) has another interesting consequence. Set $t=0$ above and note that

$$\begin{aligned}
\|f_\sigma\|_{W_2^\beta(\mathbb{R}^n)} &\leq \|E_\Omega f - f_\sigma\|_{W_2^\beta(\mathbb{R}^n)} + \|E_\Omega f\|_{W_2^\beta(\mathbb{R}^n)} \\
&\leq C_0\|E_\Omega f\|_{W_2^\beta(\mathbb{R}^n)} \leq C\|f\|_{W_2^\beta(\Omega)} \qquad (9)
\end{aligned}$$

We collect these remarks in the corollary below.

Corollary 3.5. *With the assumptions and notation of Theorem 3.4, we have that f_σ also satisfies*

$$\|f-f_\sigma\|_{W_2^\beta(\Omega)} \leq Cq_X^t\|f\|_{W_2^{\beta+t}(\Omega)}$$

In addition, we have

$$\|f_\sigma\|_{W_2^\beta(\mathbb{R}^n)} \leq C\|f\|_{W_2^\beta(\Omega)}$$

4 Radial Basis Functions

In the following we will assume that the native space $\mathcal{N}_\Phi$ is isomorphic to either $W_2^\tau(\mathbb{R}^n)$ or $\mathrm{BL}_\tau(\mathbb{R}^n)$, the Beppo–Levi space; that is, we assume for the rest of this section that our (conditionally) positive definite kernel Φ has either a classical Fourier transform that satisfies

$$c_1(1+\|\omega\|_2^2)^{-\tau} \le \widehat{\Phi}(\omega) \le c_2(1+\|\omega\|_2^2)^{-\tau}, \qquad \omega \in \mathbb{R}^n, \tag{10}$$

or a generalized Fourier transform that satisfies

$$c_1\|\omega\|_2^{-2\tau} \le \widehat{\Phi}(\omega) \le c_2\|\omega\|_2^{-2\tau}, \qquad \tau \in 2\mathbb{N},\ \omega \in \mathbb{R}^n \setminus \{0\}, \tag{11}$$

where we take $\tau > n/2$. We note that the functions involved include both the Wendland functions and the thin-plate splines. Finally, given such a Φ and a finite set X, we will denote the associated interpolant for a continuous function f by $I_X f$.

The proofs for the error estimates we provide in this section depend upon the Sobolev space interpolation result stated in Lemma 2.1. We will use this lemma several times below. Our first error estimate, which we need in the sequel, concerns functions which are in the Sobolev space $W_2^\tau(\Omega)$, which is essentially the native space for both types of RBFs.

Lemma 4.1. *Let $\tau = k+s$ with $0 < s \le 1$ and $k \in \mathbb{N}$ with $k > n/2$. If $f \in W_2^\tau(\Omega)$, then*

$$\|f - I_X f\|_{W_2^\beta(\Omega)} \le C h_{X,\Omega}^{\tau-\beta}\|f\|_{W_2^\tau(\Omega)}, \qquad 0 \le \beta \le \tau.$$

Proof. We have $\|f - I_X f\|_{L_2(\Omega)} \le C h_{X,\Omega}^{\tau}\|f\|_{W_2^\tau(\Omega)}$ by [13, Prop. 3.2 and Cor. 3.6]. Moreover, by the best approximation property of the interpolant, we can conclude $\|f - I_X f\|_{W_2^\tau(\Omega)} \le C\|f\|_{W_2^\tau(\Omega)}$. Hence, Lemma 2.1 yields

$$\|f - I_X f\|_{W_2^\beta(\Omega)} \le C h_{X,\Omega}^{\tau(1-\beta/\tau)}\|f\|_{W_2^\tau(\Omega)},$$

which is the inequality we wished to obtain. □

We now come to the case of interest, namely to error estimates for RBF approximation of $f \in W_2^\beta(\Omega)$, with $\beta \le \tau$. The function f is *not* in the native space, and so the traditional rkhs techniques do not apply.

Theorem 4.2. *If $\tau \ge \beta$, $\beta = k+s$ with $0 < s \le 1$ and $k > n/2$, and if $f \in W_2^\beta(\Omega)$, then*

$$\|f - I_X f\|_{W_2^\mu(\Omega)} \le C h_{X,\Omega}^{\beta-\mu}\rho_{X,\Omega}^{\tau-\mu}\|f\|_{W_2^\beta(\Omega)}, \qquad 0 \le \mu \le \beta.$$

Proof. The function $f - I_X f$ vanishes on X. Hence, by [13, Theorem 2.12] there exists a constant $C > 0$ such that if $0 \leq m < k - n/2$, then

$$|f - I_X f|_{W_2^m(\Omega)} \leq C h_{X,\Omega}^{\beta-m} \|f - I_X f\|_{W_2^\beta(\Omega)}.$$

To apply Lemma 2.1 to interpolate the operator norms involved, we need to prove that

$$\|f - I_X f\|_{W_2^\beta(\Omega)} \leq C \rho_{X,\Omega}^{\tau-\beta} \|f\|_{W_2^\beta(\Omega)} \tag{12}$$

holds. According to Theorem 3.4, there exists $\kappa > 0$ such that with $\sigma = \kappa/q_X$, we may select $f_\sigma \in W_2^\beta(\mathbb{R}^n)$ so that $f_\sigma|_X = f|_X$ and

$$\|f - f_\sigma\|_{W_2^\beta(\Omega)} \leq C\|f\|_{W_2^\beta(\Omega)}.$$

The fact that f_σ interpolates f on X implies that $I_X f = I_X f_\sigma$. Putting this together with the inequality above yields

$$\|f - I_X f\|_{W_2^\beta(\Omega)} \leq C\|f\|_{W_2^\beta(\Omega)} + \|f_\sigma - I_X f_\sigma\|_{W_2^\beta(\Omega)}. \tag{13}$$

To bound the second term on the right in (13), we apply Lemma 4.1 to f_σ to obtain

$$\|f_\sigma - I_X f_\sigma\|_{W_2^\beta(\Omega)} \leq C h_{X,\Omega}^{\tau-\beta} \|f_\sigma\|_{W_2^\tau(\Omega)}.$$

Additionally, since $\sigma = \kappa/q_X$, from the Bernstein Theorem for band-limited functions, one gets $\|f_\sigma\|_{W_2^\tau(\mathbb{R}^n)} \leq C q_X^{\beta-\tau} \|f_\sigma\|_{W_2^\beta(\mathbb{R}^n)}$ so that

$$\|f_\sigma - I_X f_\sigma\|_{W_2^\beta(\Omega)} \leq C h_{X,\Omega}^{\tau-\beta} q_X^{\beta-\tau} \|f_\sigma\|_{W_2^\beta(\mathbb{R}^n)} = C \rho_{X,\Omega}^{\tau-\beta} \|f_\sigma\|_{W_2^\beta(\mathbb{R}^n)}.$$

¿From Corollary 3.5, we have $\|f_\sigma\|_{W_2^\beta(\mathbb{R}^n)} \leq C\|f\|_{W_2^\beta(\Omega)}$, with $C > 0$ independent of f, (12) holds. Applying Lemma 2.1 then completes the proof. $\square$

We will say that a set of centers $X \subset \Omega$ is ρ uniform if the mesh ratio $\rho_{X,\Omega} \leq \rho$. In a ρ-uniform set of centers, the mesh norm $h_{X,\Omega}$ and separation radius q_X are comparable. Of course, we must have $\rho > 1$. The smallest value it can take for an Ω isn't known. For a hypercubic grid, $\rho = \sqrt{n}$.

Corollary 4.3. *Let $\tau \geq \beta = k + s$ with $0 < s \leq 1$ and $k > n/2$ and assume that Φ satisfies (10) or (11). For all ρ-uniform $X \subset \Omega$, the interpolation map $I_X\colon\ W_2^\beta(\Omega) \to W_2^\beta(\Omega)$ is bounded uniformly in X.*

Proof. Let $f \in W_2^\beta(\Omega)$ be given. By Theorem 4.2 we find

$$\begin{aligned}\|I_X f\|_{W_2^\beta(\Omega)} &\leq \|f\|_{W_2^\beta(\Omega)} + \|f - I_X f\|_{W_2^\beta(\Omega)} \\ &\leq (1 + C\rho_X^{\tau-\beta})\|f\|_{W_2^\beta(\Omega)} \\ &\leq (1 + C\rho^{\tau-\beta})\|f\|_{W_2^\beta(\Omega)}.\end{aligned}$$

This implies that $\|I_X\| \leq 1 + C\rho^{\tau-\beta}$, which is independent of X. □

We point out that, under the assumptions listed in the corollary, RBF interpolation is comparable to best approximation.

5 A Bernstein Theorem for RBFs

In this section we wish to establish a Bernstein Theorem for certain RBFs. More specifically, the next theorem deals with bounding stronger norms of interpolants by weaker ones.

Throughout this section we assume that our basis function Φ has a Fourier transform that satisfies (10), so that its associated reproducing kernel Hilbert space coincides with $W_2^\tau(\mathbb{R}^n)$.

We are now interested in bounding the norms of functions from the following space:

$$V_{\Phi,X} = \operatorname{span}\{\Phi(\cdot - x_j) \colon \ x_j \in X\},$$

where $X = \{x_1, \ldots, x_N\} \subset \Omega$. Naturally, $V_{\Phi,X}$ is a subspace of $W_2^\tau(\mathbb{R}^n)$.

Theorem 5.1. *Suppose that Φ satisfies (10) and that $0 \leq \mu \leq \tau$. Then there is a constant C_Φ such that*

$$\|f\|_{W_2^\mu(\mathbb{R}^n)} \leq C_\Phi q_X^{-\mu}\|f\|_{L^2(\mathbb{R}^n)}, \quad f \in V_{\Phi,X}.$$

Proof. Since (10) holds for Φ, $f = \sum_j c_j\Phi(\cdot - x_j)$ satisfies

$$c_1\Big\|\sum_j c_j\mathcal{K}_\tau(\cdot - x_j)\Big\|_{W_2^\mu(\mathbb{R}^n)} \leq \|f\|_{W_2^\mu(\mathbb{R}^n)} \leq c_2\Big\|\sum_j c_j\mathcal{K}_\tau(\cdot - x_j)\Big\|_{W_2^\mu(\mathbb{R}^n)} \tag{14}$$

where $\mathcal{K}_\tau$ is the reproducing kernel for $W_2^\tau(\mathbb{R}^n)$ that is given in (1), with β instead of τ. This inequality means that there is no loss of generality in making the simplifying assumption that $\Phi = \mathcal{K}_\tau$, and we now do so.

We begin with a simple observation that follows easily from $\widehat{\mathcal{K}}_\beta(\omega) = (1+\|\omega\|_2^2)^{-\beta}$; namely, we have that

$$\Big\|\sum_j c_j \mathcal{K}_\tau(\cdot - x_j)\Big\|_{W_2^\mu(\mathbb{R}^n)} = \Big\|\sum_j c_j \mathcal{K}_{2\tau-\mu}(\cdot - x_j)\Big\|_{W_2^{2\tau-\mu}(\mathbb{R}^n)} .$$

Next, let $\beta = 2\tau - \mu$ in Lemma 3.3. In the notation of Lemma 3.3, there is a constant $\kappa = \kappa_\beta > 1$ such that when $\sigma = \kappa/q_X$, we have $\|g - g_\sigma\|_{W_2^\beta(\mathbb{R}^n)} \le \frac{1}{2}\|g\|_{W_2^\beta(\mathbb{R}^n)}$, where of course $g_\sigma \in \mathcal{B}_\sigma$. From this inequality it easily follows that $\|g\|_{W_2^\beta} \le 2\|g_\sigma\|_{W_2^\beta(\mathbb{R}^n)}$. Consequently, we have

$$\begin{aligned}
\|g\|_{W_2^\beta}^2 &\le \int\limits_{\|\omega\|_2\le\sigma} 4\frac{\big|\sum_j c_j e^{i\omega^T x_j}\big|^2}{(1+\|\omega\|_2^2)^{2\tau}}(1+\|\omega\|_2^2)^\mu d^n\omega \\
&\le 4(1+\sigma^2)^\mu \int\limits_{\|\omega\|_2\le\sigma} \frac{\big|\sum_j c_j e^{i\omega^T x_j}\big|^2}{(1+\|\omega\|_2^2)^{2\tau}} d^n\omega \\
&\le 2^{2+\mu}\sigma^{2\mu}\Big\|\sum_j c_j \mathcal{K}_\tau(\cdot - x_j)\Big\|_{L_2(\mathbb{R}^n)}^2
\end{aligned}$$

Using the connection between σ and q_X, we arrive at this inequality:

$$\Big\|\sum_j c_j \mathcal{K}_\tau(\cdot - x_j)\Big\|_{W_2^\mu(\mathbb{R}^n)} \le 2^{\frac{2+\mu}{2}}\kappa^\mu q_X^{-\mu}\Big\|\sum_j c_j \mathcal{K}_\tau(\cdot - x_j)\Big\|_{L_2(\mathbb{R}^n)} .$$

Since $\kappa > 1$ and $\mu \le \tau$, the constant multiplying $q_X^{-\mu}$ may be replaced by one depending on τ or, equivalently, Φ. The equivalence of norms from (14) then implies the desired result. $\square$

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