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Abstract

In this paper, we study the stability of symmetric collocation methods for boundary value problems using certain positive definite kernels. We derive lower bounds on the smallest eigenvalue of the associated collocation matrix in terms of the separation distance. Comparing these bounds to the well-known error estimates shows that another trade-off appears, which is significantly worse than the one known from classical interpolation. Finally, we show how this new trade-off can be overcome as well as how the collocation matrix can be stabilized by smoothing.

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1 Introduction

Meshless collocation methods for the numerical solution of partial differential equations have recently become more and more popular. They provide, for example, a greater flexibility when it comes to adaptivity and time-dependent changes of the underlying region.

Radial basis functions or, more generally, (conditionally) positive definite kernels are one of the main stream methods in the field of meshless collocation. There are, in principle, two different approaches to collocation using radial basis functions. The *unsymmetric* approach by Kansa ([10, 9]), which has the advantage that less derivatives have to be formed but has the drawback of an unsymmetric collocation matrix, which can even be singular ([8]). Despite this drawback unsymmetric collocation has been used frequently and successfully in several applications.

In this paper, however, we will concentrate on *symmetric* collocation methods based on radial basis functions, as they have been introduced in the context of *generalized interpolation* in [22, 13] and used for elliptic problems in [3, 4, 6, 5].

Radial basis functions, in general, are a powerful tool for reconstruction processes from scattered data (see for example [2, 20]).

We consider boundary value problems of the following form:

$$Lu = f \quad \text{in } \Omega \tag{1}$$
$$Bu = g \quad \text{on } \partial\Omega \tag{2}$$

where L is an partial differential operator of order m of the form

$$Lu(x) = \sum_{|\alpha| \leq m} c_\alpha(x) D^\alpha u(x)$$

and B is a typical boundary operator. In our analysis later on, we will restrict ourselves to $B = I$, i.e. we will only deal with *Dirichlet* boundary conditions, though *Neumann* or *mixed* boundary conditions can be treated similarly.

While error estimates for a discretization of (1) and (2) by symmetric collocation have been investigated in [6, 5, 20, 7], the other important question concerning the stability of the associated collocation matrix has not yet been addressed.

It is the goal of this paper, to give first results on the condition number of the collocation matrix, if specific kernels, including the Wendland kernels, are employed. To this end we use a simplified version of results for the classical interpolation problem (see for example [11, 12, 14, 16]).

This paper is organized as follows. In the next section, we will shortly review the meshless, symmetric collocation method. The third section is devoted to specific kernels for which our analysis applies. The fourth section gives the main result on the stability by bounding the smallest eigenvalue of the discrete collocation matrix from below. The final section deals with regularization and stabilization of the collocation problem.

2 Meshless symmetric collocation

The symmetric collocation approach to discretize (1) and (2) based upon positive definite kernels can be described as follows.

First, two sets of discrete points $X_1 = \{x_1, \ldots, x_N\} \subseteq \partial\Omega$ and $X_2 = \{x_{N+1}, \ldots, x_M\}$ are chosen. Then, a positive definite kernel $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is fixed and the approximate solution is formed as

$$s(x) = \sum_{j=1}^{N} \alpha_j (\delta_{x_j} \circ B^y)\Phi(x, y) + \sum_{j=N+1}^{M} \alpha_j (\delta_{x_j} \circ L^y)\Phi(x, y). \tag{3}$$

Here, $(\delta_x \circ B)u$ is defined to be $(\delta_x \circ B)u = (Bu)(x)$ and the additional superscript in $(\delta_{x_j} \circ B)^y \Phi(x, y)$ indicates to which argument of $\Phi : \Omega \times \Omega \to \mathbb{R}$ the linear functional $\delta_{x_j} \circ B$ is applied.

The unknown coefficients in the representation (3) are determined by the collocation conditions

$$\begin{aligned} Bs(x_j) &= g(x_j), \qquad 1 \leq j \leq N, \\ Ls(x_j) &= f(x_j), \qquad N+1 \leq j \leq M. \end{aligned}$$

This results into the linear system

$$\begin{pmatrix} A & C \\ C^T & D \end{pmatrix} \alpha = \begin{pmatrix} g|X_1 \\ f|X_2 \end{pmatrix} \tag{4}$$

with the block matrices given by

$$\begin{aligned} A &= (\delta_{x_i} \circ B^x)(\delta_{x_j} \circ B^y)\Phi(x, y) \\ C &= (\delta_{x_i} \circ B^x)(\delta_{x_j} \circ L^y)\Phi(x, y) \\ D &= (\delta_{x_i} \circ L^x)(\delta_{x_j} \circ L^y)\Phi(x, y). \end{aligned}$$

It is well-known (cf. [6, 5, 22, 20]) that such a reconstruction problem can be put in the more general framework of *generalized interpolation*. Defining the functionals

$$\lambda_j := \begin{cases} \delta_{x_j} \circ B, & 1 \leq j \leq N, \\ \delta_{x_j} \circ L, & N+1 \leq j \leq M, \end{cases} \tag{5}$$

shows that the interpolant s in (3) can be written in the unified form

$$s(x) = \sum_{j=1}^{M} \alpha_j \lambda_j^y \Phi(\cdot, y). \tag{6}$$

Moreover, the collocation matrix in (4) simply becomes

$$A_\Lambda = (\lambda_i^x \lambda_j^y \Phi(x, y)). \tag{7}$$

In the above cited sources, it is shown, that this matrix is a Gramian matrix and hence positive semi-definite. Moreover, if the functionals λ_j are linearly independent over the reproducing kernel Hilbert space associated to Φ, then the matrix is even positive definite. The functionals defined in (5) are known to be linearly independent (see for example [7]) for all relevant kernels provided that the centers x_j are not singular points of the differential (or boundary) operator.

Definition 2.1 *The point $x \in \mathbb{R}^d$ is called a* singular point of L *if $\delta_x \circ L = 0$, i.e. if $c_\alpha(x) = 0$ for all $|\alpha| \le m$.*

In this paper we are mainly concerned with estimating the smallest eigenvalue of the collocation matrix A_Λ.

3 Some typical kernels

For the rest of this paper, we will assume that the kernel Φ is translation invariant and hence defined on all of $\mathbb{R}^d$, i.e. of the form

$$\Phi(x, y) = \Phi(x - y).$$

The function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is assumed to be integrable and to act as a reproducing kernel of a Sobolev space $W_2^\tau(\mathbb{R}^d)$ with $\tau > d/2$.

Sobolev spaces are introduced in the usual way. Let $\Omega \subseteq \mathbb{R}^d$ be a domain. For $k \in \mathbb{N}_0$, and $1 \le p < \infty$, we define the Sobolev spaces $W_p^k(\Omega)$ to consist of all u with weak derivatives $D^\alpha u \in L_p(\Omega)$, $|\alpha| \le k$. Associated with these spaces are the (semi-)norms

$$|u|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p} \text{ and } \|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p}.$$

The case $p = \infty$ is defined in the obvious way:

$$|u|_{W_\infty^k(\Omega)} = \sup_{|\alpha|=k} \|D^\alpha u\|_{L_\infty(\Omega)} \text{ and } \quad \|u\|_{W_\infty^k(\Omega)} = \sup_{|\alpha|\le k} \|D^\alpha u\|_{L_\infty(\Omega)}.$$

To deal also with fractional Sobolev spaces, let $1 \le p < \infty$ and let $\tau = k + s$ with $k \in \mathbb{N}_0$, and $0 < s < 1$. We define the fractional order Sobolev spaces $W_p^\tau(\Omega)$ to be all u for which the norms below are finite.

$$\begin{aligned} |u|_{W_p^{k+s}(\Omega)} &:= \left(\sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{\|x - y\|_2^{d+ps}} dx dy \right)^{1/p} \\ \|u\|_{W_p^{k+s}(\Omega)} &:= \left(\|u\|_{W_p^k(\Omega)}^p + |u|_{W_p^{k+s}(\Omega)}^p \right)^{1/p}. \end{aligned}$$

Here, $\|\cdot\|_2$ denotes the Euclidean distance on $\mathbb{R}^d$.

In the case $p = 2$ the Sobolev spaces $W_2^\tau(\Omega)$ are Hilbert spaces. Moreover, the fractional Sobolev space $W_2^\tau(\mathbb{R}^d)$ can also be introduced employing Fourier transforms. If the Fourier transform of a function f is defined by

$$\widehat{f}(\omega) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix^T\omega} dx,$$

then

$$W_2^\tau(\mathbb{R}^d) = \{f \in L_2(\mathbb{R}^d) : (1 + \|\cdot\|_2^2)^{\tau/2} \widehat{f} \in L_2(\mathbb{R}^d)\}.$$

This, on the other hand, leads to a simple characterization of kernels Φ, which generate $W_2^\tau(\mathbb{R}^d)$ by linear combinations of their translates. For us, the following result will be of importance, which can be found, for example, in [20].

Proposition 3.1 *Suppose $\Phi : \mathbb{R}^d \to \mathbb{R}$ is integrable with a Fourier transform satisfying*

$$c_1(1 + \|\omega\|_2^2)^{-\tau} \le \widehat{\Phi}(\omega) \le c_2(1 + \|\omega\|_2^2)^{-\tau}, \qquad \omega \in \mathbb{R}^d, \tag{8}$$

with $\tau > d/2$. Then, Φ generates an inner product on $W_2^\tau(\mathbb{R}^d)$ by

$$(f, g)_\Phi := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\widehat{f}(\omega)\overline{\widehat{g}(\omega)}}{\widehat{\Phi}(\omega)} d\omega,$$

which leads to a norm equivalent to the Sobolev norm

$$\|f\|_{W_2^\tau(\mathbb{R}^d)} := \|(1 + \|\cdot\|_2^2)^{\tau/2} \widehat{f}\|_{L_2(\mathbb{R}^d)}.$$

Moreover, Φ is the reproducing kernel for $W_2^\tau(\mathbb{R}^d)$ with respect to this inner product, i.e. for every $f \in W_2^\tau(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$, the reproduction

$$f(x) = (f, \Phi(\cdot - x))_\Phi$$

is satisfied.

Because of this proposition we will call Φ with (8) a reproducing kernel of $W_2^\tau(\mathbb{R}^d)$, relaxing in this way the fact that a reproducing kernel is otherwise uniquely determined.

The following result contains the details to what we have mentioned at the end of the last section. Its proof can be found in [7].

Proposition 3.2 *Suppose $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a reproducing kernel of $W_2^\tau(\mathbb{R}^d)$ with $\tau > m + d/2$. Let L be a linear differential operator of degree m. Let $X_1 = \{x_1, \dots, x_N\} \subseteq \partial\Omega$ and $X_2 = \{x_{N+1}, \dots, x_M\} \subseteq \Omega$ be two sets of pairwise distinct points such that X_2 contains no singular point of L Then, the functionals $\Lambda = \{\lambda_1, \dots, \lambda_M\}$ with $\lambda_j = \delta_{x_j}$, $1 \le j \le N$, and $\lambda_j = \delta_{x_j} \circ L$ for $N + 1 \le j \le M$ are linearly independent over $W_2^\tau(\mathbb{R}^d)$.*

For our analysis in the next section, it will be crucial to have a compactly supported function Φ that satisfies the decay condition (8).

Definition 3.3 *We say that the Sobolev space $W_2^\tau(\mathbb{R}^d)$ and hence every associated reproducing kernel Φ is feasible, if there exists at least one compactly supported reproducing kernel for $W_2^\tau(\mathbb{R}^d)$.*

Typical examples of feasible kernels are the *Wendland kernels* ([18, 19]). They are radial kernels $\Phi_{d,k} \in C^{2k}(\mathbb{R}^d)$, which lead to Sobolev spaces $W_2^\tau(\mathbb{R}^d)$ with $\tau = d/2 + k + 1/2$. Hence, in odd space dimensions, integer order Sobolev spaces are feasible, while in even space dimensions, Sobolev spaces of order "integer plus a half" are feasible. However, using the fact that the Fourier transform of the convolution of two functions is the product of the two Fourier transforms, we see that, for example, also all Sobolev spaces of order $\tau = d+2k+1$ are feasible.

It is still an open question, whether any Sobolev space $W_2^\tau(\mathbb{R}^d)$ of order $\tau > d/2$ is feasible in this sense.

When we consider Sobolev spaces over domains $\Omega \subseteq \mathbb{R}^d$, we will always assume that Ω has a sufficiently nice boundary, such that there exists an extension operator, i.e. a mapping $E : W_2^\tau(\Omega) \to W_2^\tau(\mathbb{R}^d)$ with $Ef(x) = f(x)$ for $f \in W_2^\tau(\Omega)$ and $x \in \Omega$ and $\|Ef\|_{W_2^\tau(\mathbb{R}^d)} \le C\|f\|_{W_2^\tau(\Omega)}$. Such extensions are well studied, in particular for integer order Sobolev spaces, but they also exist for fractional order Sobolev spaces, if the domain Ω satisfies an interior cone condition and has a Lipschitz boundary, see for example [1]. This assumption allows us to work always with a function from $W_2^\tau(\mathbb{R}^d)$ instead of $W_2^\tau(\Omega)$.

4 Bounding the smallest eigenvalue

Since numerical tests show and the Gerschgorin theorem implies that the maximum eigenvalue of the collocation matrix A_Λ is uncritical, we will concentrate on the smallest eigenvalue to measure the ill-conditioning of the collocation matrix.

We consider the following simplifications of (1) and (2). From now on we will assume that $B = I$ is the identity on $\partial\Omega$, i.e. we deal with Dirichlet boundary conditions, only. Furthermore, we will assume that L has only *constant* coefficients c_α.

It is now time to derive our main result. We will measure the smallest eigenvalue in terms of the separation distance. Let $X = X_1 \cup X_2$. Then, the separation distance of X is defined as

$$q_X = \frac{1}{2}\min_{j\neq k} \|x_j - x_k\|_2.$$

Theorem 4.1 *Suppose Φ is a positive definite kernel function satisfying (8) with $\tau > d/2+m$ for a feasible Sobolev space $W_2^\tau(\mathbb{R}^d)$. Suppose L is an mth-order differential operator with constant coefficients and $B = I$. Then, the smallest eigenvalue of the collocation matrix can be bounded by*

$$\lambda_{\min}(A_\Lambda) \ge C q_X^{2\tau-d},$$

with a constant $C > 0$ independent of X.

Proof: We use again the functional notation, which now becomes

$$\lambda_j = \begin{cases} \delta_{x_j} \circ L & \text{for } 1 \le j \le N \\ \delta_{x_j} & \text{for } N+1 \le j \le M. \end{cases}$$

Then, we have to establish the following bound

$$\sum_{j,k=1}^{M} \beta_j\beta_k\lambda_j^x\lambda_k^y\Phi(x-y) \ge C q_X^{2\tau-d}\|\beta\|_2^2, \qquad \beta \in \mathbb{R}^M.$$

To achieve this, we introduce the symbol of the functionals λ_j defined by

$$\sigma_j(\omega) := \lambda_j^x(e^{ix^T\omega}),$$

which is simply $\sigma_j(\omega) = e^{ix_j^T\omega}$ in the case of the boundary functionals and

$$\sigma_j(\omega) = \sum_{|\alpha|\le m} c_\alpha e^{ix_j^T\omega}(i\omega)^\alpha$$

in the case of the interior functionals. With this notation, the quadratic form becomes

$$\sum_{j,k=1}^{M} \beta_j \beta_k \lambda_j^x \lambda_k^y \Phi(x-y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \left| \sum_{j=1}^{M} \beta_j \sigma_j(\omega) \right|^2 \widehat{\Phi}(\omega) d\omega.$$

Since $W_2^\tau(\mathbb{R}^d)$ is feasible, we can choose a compactly supported function Ψ with support in the unit ball $B(0,1)$ satisfying the decay condition (8) and set $\Psi_\delta = \Psi(\cdot/\delta)$.

Next, let us assume that $\delta \le 1$. Then,

$$\widehat{\Psi_\delta}(\omega) = \delta^d \widehat{\Psi}(\delta\omega) \le c_2(\Psi)\delta^d (1 + \|\delta\omega\|_2^2)^{-\tau} \le c_2(\Psi)\delta^{d-2\tau}(1+\|\omega\|_2^2)^{-\tau}$$

obviously yields

$$\widehat{\Phi}(\omega) \ge c_1(\Phi)(1+\|\omega\|_2^2)^{-\tau} \ge \frac{c_1(\Phi)}{c_2(\Psi)} \delta^{2\tau-d} \widehat{\Psi_\delta}(\omega),$$

which establishes

$$\sum_{j,k=1}^{M} \beta_j \beta_k \lambda_j^x \lambda_k^y \Phi(x-y) \ge \frac{c_1(\Phi)}{c_2(\Psi)} \delta^{2\tau-d} \sum_{j,k=1}^{M} \beta_j \beta_k \lambda_j^x \lambda_k^y \Psi_\delta(x-y).$$

If we choose $\delta = q_X$ as the separation distance and assume without restriction $q_X \le 1$, the last estimate reduces to

$$\begin{aligned} \sum_{j,k=1}^{M} \beta_j \beta_k \lambda_j^x \lambda_k^y \Phi(x-y) &\ge \frac{c_1(\Phi)}{c_2(\Psi)} \delta^{2\tau-d} \sum_{j=1}^{M} \beta_j^2 \lambda_j^x \lambda_j^y \Psi_\delta(x-y) \\ &= \frac{c_1(\Phi)}{c_2(\Psi)} \delta^{2\tau-d} \left[\sum_{j=1}^{N} \beta_j^2 \Psi(0) + \sum_{j=N+1}^{M} \beta_j^2 \lambda_j^x \lambda_j^y \Psi_\delta(x-y) \right]. \end{aligned}$$

Finally, since the matrix $(\lambda_j^x \lambda_k^y \Psi_\delta(x-y))$ is positive definite, we see that the function

$$F(\delta) := \lambda_j^x \lambda_j^y \Psi_\delta(x-y) = \sum_{|\alpha|,|\beta| \le m} c_\alpha c_\beta (-1)^{|\beta|} \delta^{-|\alpha|-|\beta|} D^{\alpha+\beta}\Psi(0)$$

is positive for all $\delta > 0$ and independent of $N+1 \le j \le M$. Since $F(\delta) \to \infty$ with $\delta \to 0$, the function F attains its minimum in $(0,1]$. Thus, we have $F(q_X) > \widetilde{c}$ with an appropriate constant $\widetilde{c} > 0$, which is independent of the data sites. This shows

$$\sum_{j,k=1}^{M} \beta_j \beta_k \lambda_j^x \lambda_k^y \Phi(x-y) \ge \frac{c_1(\Phi)}{c_2(\Psi)} \min\{\Psi(0), \widetilde{c}\}\, q_X^{2\tau-d} \|\beta\|_2^2,$$

which immediately gives the stated lower bound. □

Note that for non-constant coefficients we still have

$$\lambda_j^x \lambda_j^y \Psi_{q_X}(x-y) = \sum_{|\alpha|,|\beta| \le m} c_\alpha(x_j) c_\beta(x_j) (-1)^{|\beta|} q_X^{-|\alpha|-|\beta|} D^{\alpha+\beta}\Psi(0) > 0,$$

but this time, the right hand side still depends on j. Nonetheless, this technique can be extended to more general situations if appropriate conditions on the coefficients of the operator L are imposed.

Moreover, for specific operators L, the constant $\widetilde{c}$ can be determined more explicitly. Consider, for example, the Poisson equation

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{aligned}$$

Then, using the notation of the proof, we find

$$\lambda_j^x \lambda_j^y \Psi_\delta(x-y) = \delta^{-4}\Delta^2\Psi(0)$$

for $N+1 \le j \le M$. Hence we have

$$\begin{aligned} \beta^T A_\Lambda \beta &\geq \frac{c_1(\Phi)}{c_2(\Psi)} q_X^{2\tau-d}\left[\sum_{j=1}^N \beta_j^2\Psi(0) + \sum_{j=N+1}^M \beta_j^2 q_X^{-4}\Delta^2\Psi(0)\right] \\ &\geq \frac{c_1(\Phi)}{c_2(\Psi)} q_X^{2\tau-d} \min\left\{\Psi(0), q_X^{-4}\Delta^2\Psi(0)\right\} \|\beta\|_2^2. \end{aligned}$$

Unfortunately, the q_X^{-4} factor, which would result into a better conditioning is overruled by the constant $\Psi(0)$ part.

Note, however, that, in the case of only interior functionals, i.e. in the case of a collocation matrix of the form

$$\widetilde{A} = (\Delta^2\Phi(x_i - x_j))$$

the proof just established shows that

$$\lambda_{\min}(\widetilde{A}) \geq C q_X^{2\tau-d-4}$$

and leads hence to a better conditioned matrix. The same holds true for any other differential operator L.

As a conclusion we can state:

Corollary 4.2 *The collocation matrix for a mixed problem is as badly conditioned as a pure interpolation matrix would be.*

On the other hand, the approximation error is determined by the derivative part. We will make this more precise now, to work out a new trade-off problem and to establish a remedy in the next section.

We introduce the *fill distance* for the region Ω and the discrete point set $X_2 \subseteq \Omega$ as usual by

$$h_{X_2,\Omega} := \sup_{x\in\Omega} \inf_{x_j \in X_2} \|x - x_j\|_2.$$

To define the fill distance for the boundary, we have to be more specific. We will assume that the bounded region $\Omega \subseteq \mathbb{R}^d$ has a $C^{k,s}$-boundary $\partial\Omega$, where $\tau = k+s$ with $k \in \mathbb{N}_0$ and $s \in [0,1)$. This means in particular, that $\partial\Omega$ is a $d-1$ dimension $C^{k,s}$-sub-manifold of $\mathbb{R}^d$. It also means that $\partial\Omega$ is Lipschitz continuous and satisfies the cone condition.

We will represent the boundary $\partial\Omega$ by a finite atlas consisting of $C^{k,s}$-diffeomorphisms with a slightly abuse of terminology. To be more precise, we assume that $\partial\Omega \subseteq \cup_{j=1}^K V_j$, where $V_j \subseteq \mathbb{R}^d$ are open sets. Moreover, the set V_j are images of $C^{k,s}$-diffeomorphism

$$\varphi_j : B \to V_j,$$

where $B = B(0,1)$ denotes the unit sphere in $\mathbb{R}^{d-1}$. Finally, suppose $\{w_j\}$ is a partition of unity with respect to $\{V_j\}$. Then, the Sobolev norms on $\partial\Omega$ can be defined via

$$\|u\|_{W_p^\mu(\partial\Omega)}^p = \sum_{j=1}^K \|(uw_j)\circ\varphi_j\|_{W_p^\mu(B)}^p.$$

It is well known that this norm is independent of the chosen atlas $\{V_j, \varphi_j\}$ but this is of less importance here, since we will assume that the atlas is fixed. For us, it is more important that we have the trace theorem, which states that the restriction of $u \in W_2^\tau(\Omega)$ to $\partial\Omega$ is well defined, belongs to $W_2^{\tau-1/2}(\partial\Omega)$, and satisfies

$$\|u\|_{W_2^{\tau-1/2}(\partial\Omega)} \le \|u\|_{W_2^\tau(\Omega)}.$$

Using the fixed atlas $\{V_j, \varphi_j\}$, we can now define the mesh norm be

$$h_{X_1,\partial\Omega} := \max_{1\le j\le K} h_{T_j,B}$$

with $T_j = \varphi_j^{-1}(X_1 \cap V_j) \subseteq B$. As mentioned before, we will assume the atlas is fixed and hence do not have to care about the dependence of $h_{X_2,\partial\Omega}$ on the atlas. However, it is interesting to see that

$$\begin{aligned} h_{T_j,B} = \sup_{x\in B}\min_{t\in T_j}\|x-t\|_2 &= \sup_{y\in V_j}\min_{s\in \widetilde{X}_j}\|\varphi_j^{-1}(y) - \varphi^{-1}(s)\|_2 \\ &\le \sup_{y\in V_j}\min_{s\in \widetilde{X}_j}\|\nabla\varphi^{-1}(\xi_{y,s})\|_2\|y-s\|_2 \\ &\le C_j h_{V_j,\widetilde{X}_j}, \end{aligned}$$

where $\widetilde{X}_j = X_1 \cap V_j$. Hence, the fill distance on the boundary is comparable to the fill distances of the boundary points measured in the Euclidean norm in $\mathbb{R}^d$.

The following result comes from [7].

Theorem 4.3 *Suppose Φ is a reproducing kernel of $W_2^\tau(\mathbb{R}^d)$ with $k := \lfloor\tau\rfloor > m + d/2$. Let $\Omega \subseteq \mathbb{R}^d$ be bounded satisfying an interior cone condition and having a $C^{k,s}$- boundary. Let L be a linear differential operator of order m with coefficients c_α in $W_\infty^{k-m+1}(\Omega)$ Finally, let s be the generalized interpolant to $u \in W_2^\tau(\Omega)$. If the data sets have sufficiently small mesh norms then the error estimates*

$$\|Lu - Ls\|_{L_\infty(\Omega)} \le Ch_{X_2,\Omega}^{\tau-m-d/2}\|u\|_{W_2^\tau(\Omega)} \tag{9}$$

$$\|u - s\|_{L_\infty(\partial\Omega)} \le Ch_{X_1,\partial\Omega}^{\tau-d/2}\|u\|_{W_2^\tau(\Omega)} \tag{10}$$

are satisfied.

For simplicity, let us assume that $h = h_{X_2,\Omega} \approx q_X$ and that $h_{X_1,\partial\Omega} \approx h^{1-4/(2\tau-d)}$. Then, Theorem 4.3 means for second order elliptic problems, using the maximum principle, that

$$\|u - s_u\|_{L_\infty(\Omega)} \le Ch^{\tau-2-d/2}\|u\|_{W_2^\tau(\Omega)},$$

while the smallest eigenvalue of the collocation matrix behaves like

$$\lambda_{\min}(A_\Lambda) \ge Ch^{2\tau-d}.$$

Hence, in this situation the trade-off principle (cf. [16]) is even worse than in the situation of pure interpolation. This is a consequence of the behavior of the Fourier transform. While the Fourier transform of the differential operator applied to the basis function leads to a slower decay of the resulting Fourier transform and hence to a slower convergence, the Fourier transform of the pure function is responsible for the smallest eigenvalue.

5 Smoothing and overcoming the new trade-off

The estimates in Theorem 4.3 were derived from a general result in [15]. This result states that a function $u \in W_2^\tau(\Omega)$, which vanishes on a discrete set $X \subseteq \Omega$ satisfies the estimate

$$|u|_{W_p^m(\Omega)} \leq C h_{X,\Omega}^{\tau - m - (d/q - d/p)_+} |u|_{W_q^\tau(\Omega)}. \tag{11}$$

Obviously, (9) follows immediately from this, while for (10) the result has to be combined with and applied to an appropriate atlas of the boundary.

A generalization of (11) has been done in [21], it states that

$$|u|_{W_p^m(\Omega)} \leq C \left\{ h_{X,\Omega}^{\tau - m - (d/q - d/p)_+} |u|_{W_q^\tau(\Omega)} + \max_{x \in X} |u(x)| \right\}. \tag{12}$$

This result has then, among other things, been applied to spline smoothing. We want to use this result here also in the context of (generalized) spline smoothing.

Sticking to the functional notation introduced in the second section, we will now not enforce collocation, but solve the minimization problem

$$\min \left\{ \sum_{j=1}^{M} [\lambda_j(s) - \lambda_j(u)]^2 + \epsilon \|s\|_\Phi^2 : s \in W_2^\tau(\mathbb{R}^d) \right\}. \tag{13}$$

It is well known (see for example [17]), that the solution to this problem also has a representation of the form (6), but this time the coefficient vector $\alpha \in \mathbb{R}^M$ is determined by the linear system

$$(A_\Lambda + \epsilon I)\alpha = \begin{pmatrix} g|X_1 \\ f|X_2 \end{pmatrix},$$

where A_Λ is the collocation matrix from (7) and I is the identity matrix.

Lemma 5.1 *Suppose the linear functionals $\lambda_1, \ldots, \lambda_N$ are linearly independent over $W_2^\tau(\Omega)$. Then, the solution s_ϵ to (13) satisfies*

- $|\lambda_j(u - s_\epsilon)| \leq \sqrt{\epsilon}\|u\|_\Phi \leq C\sqrt{\epsilon}\|u\|_{W_2^\tau(\Omega)}$
- $\|u - s_\epsilon\|_\Phi \leq \|u\|_\Phi \leq C\|u\|_{W_2^\tau(\Omega)}$

Proof: This follows immediately from

$$\begin{aligned} \max\{|\lambda_j(u - s_\epsilon)|^2, \epsilon\|s_\epsilon\|_\Phi^2\} &\leq \sum_{j=1}^{M} [\lambda_j(s) - \lambda_j(u)]^2 + \epsilon\|s\|_\Phi^2 \\ &\leq \epsilon\|u\|_\Phi^2 \leq C\epsilon\|u\|_{W_2^\tau(\Omega)}^2, \end{aligned}$$

since u is a feasible candidate for the minimization problem (6). In the last estimate we have used the fact that the Φ-norm is equivalent to the $W_2^\tau(\mathbb{R}^d)$-norm and that we can replace $u \in W_2^\tau(\Omega)$ by its extension $Eu \in W_2^\tau(\mathbb{R}^d)$ without changing s_ϵ. □

Hence, combining this result with (12) shows for the solution s_ϵ of (13) the estimate

$$\begin{aligned} \|Lu - Ls_\epsilon\|_{L_p(\Omega)} &\leq C \left\{ h_{X_2,\Omega}^{\tau - m - (d/2 - d/p)_+} \|Lu - Ls_\epsilon\|_{W_2^{\tau - m}(\Omega)} + \max_{N+1 \leq j \leq M} |\lambda_j(u - s_\epsilon)| \right\}. \\ &\leq C \left\{ h_{X_2,\Omega}^{\tau - m - (d/2 - d/p)_+} + \sqrt{\epsilon} \right\} \|u\|_{W_2^\tau(\Omega)}, \end{aligned}$$

since $\|Lu-Ls_\epsilon\|_{W_2^{\tau-m}(\Omega)} \le C\|u-s_\epsilon\|_\Phi \le C\|u\|_{W_2^\tau(\Omega)}$. Moreover, turning to the boundary part, we set $u_k := ((u-s_\epsilon)w_k)\circ\varphi_k$, which belongs to $W_2^{\tau-1/2}(B)$ and satisfies on $T_j = \varphi^{-1}(X_1\cap V_j) \subseteq B$:

$$|u_k(t_j)| = |(u-s_\epsilon)(x_j)w_k(x_j)| \le |(u-s_\epsilon)(x_j)| \le C\sqrt{\epsilon}\|u\|_{W_2^\tau(\Omega)}.$$

Hence, we have for $1 \le p < \infty$

$$\begin{aligned}
\|u-s_\epsilon\|^p_{L_p(\partial\Omega)} &= \sum_{k=1}^K \|u_k\|^p_{L_p(B)} \\
&\le C\sum_{k=1}^K \left\{h_{T_k,B}^{p(\tau-1/2-(d-1)(1/2-1/p)_+)} + \sqrt{\epsilon}\right\} \|u_k\|^p_{W_2^{\tau-1/2}(B)} \\
&\le C\left\{h_{X_1,\partial\Omega}^{p(\tau-1/2-(d-1)(1/2-1/p)_+)} + \sqrt{\epsilon}\right\} \|u-s_\epsilon\|^p_{W_2^{\tau-1/2}(\partial\Omega)} \\
&\le C\left\{h_{X_1,\partial\Omega}^{p(\tau-1/2-(d-1)(1/2-1/p)_+)} + \sqrt{\epsilon}\right\} \|u\|^p_{W_2^\tau(\Omega)}
\end{aligned}$$

and the case $p=\infty$ is, of course, treated in the same fashion.

In conclusion, we have for an mth order differential operator L, the following error estimates for the smoothing solution:

Theorem 5.2 *Under the assumptions of Theorem 4.3 on the domain, the Sobolev space, and the operator, let s_ϵ be the solution of (13). Then, the error between $u \in W_2^\tau(\Omega)$ and s_ϵ can be bounded by:*

$$\begin{aligned}
\|Lu-Ls_\epsilon\|_{L_p(\Omega)} &\le C\left\{h_{X_2,\Omega}^{\tau-m-(d/2-d/p)_+} + \sqrt{\epsilon}\right\}\|u\|_{W_2^\tau(\Omega)}, \\
\|u-s_\epsilon\|_{L_p(\partial\Omega)} &\le C\left\{h_{X_1,\partial\Omega}^{\tau-1/2-(d-1)(1/2-1/p)_+} + \sqrt{\epsilon}\right\}\|u\|_{W_2^\tau(\Omega)}.
\end{aligned}$$

Returning to the case of a second order elliptic partial differential operator and $p=\infty$:

$$\begin{aligned}
\|Lu-Ls_\epsilon\|_{L\infty(\Omega)} &\le C\left\{h_{X_2,\Omega}^{\tau-2-d/2} + \sqrt{\epsilon}\right\}\|u\|_{W_2^\tau(\Omega)}, \\
\|u-s_\epsilon\|_{L_\infty(\partial\Omega)} &\le C\left\{h_{X_1,\partial\Omega}^{\tau-d/2} + \sqrt{\epsilon}\right\}\|u\|_{W_2^\tau(\Omega)},
\end{aligned}$$

which gives by the maximum principle

$$\|u-s_\epsilon\|_{L_\infty(\Omega)} \le C\left\{h_{X_2,\Omega}^{\tau-2-d/2} + h_{X_1,\partial\Omega}^{\tau-d/2} + \sqrt{\epsilon}\right\}\|u\|_{W_2^\tau(\Omega)}.$$

On the other hand, the smallest eigenvalue of the matrix $A_\Lambda + \epsilon I$ has the lower bound

$$\lambda_{\min}(A_\Lambda+\epsilon I) = \lambda_{\min}(A_\Lambda) + \epsilon \ge Cq_X^{2\tau-d} + \epsilon.$$

In the situation of $h = h_{X_2,\Omega}$, $h_{X_1,\partial\Omega} \approx h^{1-4/(2\tau-d)}$, and an *arbitrary* $q_X > 0$, this reduces to

$$\begin{aligned}
\|u-s_\epsilon\|_{L_\infty(\Omega)} &\le C\left\{h^{\tau-2-d/2} + \sqrt{\epsilon}\right\}\|u\|_{W_2^\tau(\Omega)} \\
\lambda_{\min}(A_\Lambda+\epsilon I) &\ge Cq_X^{2\tau-d} + \epsilon.
\end{aligned}$$

Hence, the choice $\epsilon \approx h^{2\tau-4-d}$, gives the symmetric bounds

$$\begin{aligned}
\|u-s_\epsilon\|_{L_\infty(\Omega)} &\le Ch^{\tau-2-d/2}\|u\|_{W_2^\tau(\Omega)} \\
\lambda_{\min}(A_\Lambda+\epsilon I) &\ge Ch^{2(\tau-2-d/2)}.
\end{aligned}$$

This means, that, without changing the order of convergence, we have improved the behavior of the smallest eigenvalue by a power of 4. Note again, that this remains true, even if the separation distance q_X is much smaller than the fill distances.

Corollary 5.3 *Suppose the discretization on the interior and the boundary have a fill distance proportional to* h *and* $h^{1-4/(2\tau-d)}$*, respectively. Selecting* $\epsilon = h^{2(\tau-2-d/2)}$ *for the smoothing parameter in the collocation solution has the following effects:*

- *The discretization error keeps its optimal convergence order.*
- *The smallest eigenvalue can be bounded from below in terms of* h *instead of* q_X*.*
- *The exponent in this lower bound improves by four orders and matches now the convergence order.*

We end this paper by remarking that most of our analysis applies also to the family of thin-plate or surface splines, but leave the details to the reader.

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