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# Meshless Collocation: Error Estimates with Application to Dynamical Systems

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**Abstract**

In this paper, we derive error estimates for generalized interpolation, in particular collocation, in Sobolev spaces. We employ our estimates to collocation problems using radial basis functions and extend and improve previously known results for elliptic problems. Finally, we use meshless collocation to approximate Lyapunov functions for dynamical systems.

**Key words**: partial differential equation, radial basis function, error estimates, Lyapunov function.

## 1 Introduction

Meshless collocation methods for the numerical solution of partial differential equations have recently become more and more popular. They provide a greater flexibility when it comes to adaptivity and time-dependent changes of the underlying region.

Radial basis functions or, more generally, (conditionally) positive definite kernels are one of the main stream methods in the field of meshless collocation. There are, in principle, two different approaches to collocation using radial basis functions. The *unsymmetric* approach by Kansa ([12, 11]) has the advantage that less derivatives have to be formed but has the drawback of an unsymmetric collocation

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matrix, which can even be singular ([10]). Despite this drawback unsymmetric collocation has been used frequently and successfully in several applications.
In this paper, however, we will concentrate on *symmetric* collocation methods based on radial basis functions, as they have been introduced in the context of *generalized interpolation* in [24, 14] and used for elliptic problems in [2, 3, 5, 4]. Radial basis functions, in general, are a powerful tool for reconstruction processes from scattered data (see for example [1, 22]).
In this paper, we study a general linear partial differential equation of the form

$$Lu = f \text{ on } \Omega, \tag{1}$$

where $\Omega$ is a domain in $\mathbb{R}^n$ and $L$ is a linear differential operator of the form

$$Lu(x) = \sum_{|\alpha|\le m} c_\alpha(x) D^\alpha u(x), \tag{2}$$

where the coefficients have a certain smoothness $c_\alpha \in C^\sigma(\overline{\Omega}, \mathbb{R})$, i.e. the derivatives of order $\beta$ with $|\beta| \le \sigma$ exist and are continuous on $\overline{\Omega}$.
Moreover, we consider boundary value problems, where additionally to (1), $u$ is required to satisfy the following boundary condition

$$u(x) = F(x) \text{ for } x \in \partial\Omega. \tag{3}$$

The numerical solution of such boundary value problems by collocation using radial basis functions has been studied by several authors. First error estimates have been given in [5, 4]. However, despite following a rather general approach, the authors of those papers show that the problems are well-posed and provide error estimates only for differential operators with *constant coefficients* $c_\alpha$. A generalization to non-constant coefficients without zeros including also a more thorough discussion of the boundary estimates can be found in [22]. However, in that book the approximation orders are, to a certain extent, not optimal. Moreover, the restriction to nonzero coefficients is not sufficient for our applications in dynamical systems.
It is the goal of this paper to investigate well-posedness of the collocation problem for the differential operator (2) with *non-constant* coefficients and to state error estimates with optimal orders in Sobolev spaces. To this end we will put the setting in the general framework of generalized interpolation in reproducing kernel Hilbert spaces and then use a recent result [16] on error estimates in Sobolev spaces for *arbitrary* scattered data reconstruction methods.
Next, we will apply the general estimates to derive error estimates in Sobolev spaces for elliptic partial differential equations. Another major and new application will be the approximation of Lyapunov functions in dynamical systems. Here, the differential operator is given by the *orbital derivative* of a function $u$ with respect to the ordinary differential equation $\dot{x} = g(x)$, i.e. by

$$Lu(x) := \langle \nabla u(x), g(x) \rangle = \sum_{j=1}^{n} g_j(x) \partial_j u(x).$$

This operator $L$ is a first-order differential operator of the form (2) with $c_{e^j}(x) = g_j(x)$. The approximation of the orbital derivative for Lyapunov functions has been studied in [7, 6, 8]. However, the approximation orders of those results can be improved significantly with the results of this paper.

This paper is organized as follows: in the rest of this section we will introduce notation which is necessary throughout the paper. Section 2 deals with generalized interpolation and is mainly a collection of known results, which will be helpful in this paper. In Section 3 we investigate collocation by radial basis functions, derive our new estimates and apply these results to elliptic problems. The final section deals with applications to dynamical systems. In particular, we describe a method to calculate Lyapunov functions and thus to calculate the basin of attraction of an equilibrium.

### 1.1 Notation

We will need to work with a variety of Sobolev spaces. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. For $k \in \mathbb{N}_0$, and $1 \leq p < \infty$, we define the Sobolev spaces $W_p^k(\Omega)$ to be all $u$ with distributional derivatives $D^\alpha u \in L_p(\Omega)$, $|\alpha| \leq k$. Associated with these spaces are the (semi-)norms

$$|u|_{W_p^k(\Omega)} = \left( \sum_{|\alpha|=k} \|D^\alpha u\|^p_{L_p(\Omega)} \right)^{1/p} \text{ and } \|u\|_{W_p^k(\Omega)} = \left( \sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L_p(\Omega)} \right)^{1/p}.$$

The case $p = \infty$ is defined in the obvious way:

$$|u|_{W_\infty^k(\Omega)} = \sup_{|\alpha|=k} \|D^\alpha u\|_{L_\infty(\Omega)} \text{ and } \quad \|u\|_{W_\infty^k(\Omega)} = \sup_{|\alpha|\leq k} \|D^\alpha u\|_{L_\infty(\Omega)}.$$

We also need fractional order Sobolev spaces. Let $1 \leq p < \infty$ and let $\tau = k + s$ with $k \in \mathbb{N}_0$, and $0 < s < 1$. We define the fractional order Sobolev spaces $W_p^\tau(\Omega)$ to be all $u$ for which the norms below are finite:

$$\begin{aligned} |u|_{W_p^{k+s}(\Omega)} &:= \left( \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{\|x-y\|_2^{n+ps}} dxdy \right)^{1/p}, \\ \|u\|_{W_p^{k+s}(\Omega)} &:= \left( \|u\|^p_{W_p^k(\Omega)} + |u|^p_{W_p^{k+s}(\Omega)} \right)^{1/p}. \end{aligned}$$

Here, $\|\cdot\|_2$ denotes the Euclidean distance on $\mathbb{R}^n$.

Let $X := \{x_1, \ldots, x_N\}$ be a finite, discrete subset of $\Omega$, which we now assume to be bounded. There are two quantities that we associate with $X$: the *separation radius* and the *mesh norm* or *fill distance*. Respectively, these are given by

$$q_X := \frac{1}{2} \min_{j \neq k} \|x_j - x_k\|_2, \quad h_{X,\Omega} := \sup_{x \in \Omega} \min_{x_j \in X} \|x - x_j\|_2.$$

The first is half the smallest distance between points in $X$, the second measures the maximum distance a point in $\Omega$ can be from any point in $X$. Frequently, when it is clear from the context, what the set $\Omega$ (or $X$) is, we will drop subscripts and write $h_X$ or $h$. Other notation will be introduced along the way.

# 2 Generalized Interpolation

## 2.1 Reproducing Kernel Hilbert Spaces

Let $H \subseteq C(\Omega)$ be a Hilbert space of functions $f : \Omega \to \mathbb{R}$ and let $H^*$ be its dual. We consider a generalized interpolation problem of the following form:

**Definition 2.1** *Given $N$ linearly independent functionals $\lambda_1, \ldots, \lambda_N \in H^*$ and $N$ function values $f_1, \ldots, f_N \in \mathbb{R}$, a generalized interpolant is a function $s \in H$ satisfying $\lambda_j(s) = f_j$, $1 \leq j \leq N$. The norm-minimal interpolant is the interpolant that minimizes in addition the norm of the Hilbert space, i.e. $s^*$ is the norm-minimal interpolant if it is the solution of*

$$\min\{\|s\|_H : \lambda_j(s) = f_j, 1 \leq j \leq N\}. \tag{4}$$

It is well known that the norm-minimal generalized interpolant is a linear combination of the Riesz representer of the functionals and that the coefficients can be computed by solving a linear system. Such problems can best be solved in reproducing kernel Hilbert spaces.

**Definition 2.2** *A* reproducing kernel Hilbert space *$H$ is a Hilbert space of functions $f : \Omega \to \mathbb{R}$, which has a unique kernel $\Phi : \Omega \times \Omega \to \mathbb{R}$, satisfying*

1. *$\Phi(\cdot, x) \in H$ for all $x \in \Omega$,*
2. *$f(x) = (f, \Phi(\cdot, x))_H$ for all $x \in \Omega$ and all $f \in H$.*

Here, the Riesz representer of a functional $\lambda \in H^*$ is simply given by applying it to one argument of the kernel, i.e. by $\lambda^y \Phi(\cdot, y)$.

**Lemma 2.3 ([22, Theorem 16.1])** *If $H$ is a reproducing kernel Hilbert space then the solution $s^*$ of (4) is given by*

$$s^* = \sum_{j=1}^{N} \alpha_j \lambda_j^y \Phi(\cdot, y),$$

*where $\alpha \in \mathbb{R}^N$ is the solution of the linear system*

$$A_{\Lambda,\Phi}\alpha = f$$

*with $A_{\Lambda,\Phi} = (\lambda_i^x \lambda_j^y \Phi(x, y))$ and $f = (f_j)$.*

Note that the Matrix $A_{\Lambda,\Phi} = (a_{ij})$ is a Gramian matrix because of

$$a_{ij} = \lambda_i^x \lambda_j^y \Phi(x,y) = (\lambda_i^x \Phi(\cdot,x), \lambda_j^y \Phi(\cdot,y))_H = (\lambda_i, \lambda_j)_{H^*}$$

and hence positive semi-definite. In the last equation we have used the fact that the Riesz representer of a functional $\lambda$ is given by $\lambda^y \Phi(\cdot,y)$. Since the functionals are supposed to be linearly independent the matrix is even positive definite.
Looking at point evaluations $\lambda_j(f) = \delta_{x_j}(f) = f(x_j)$ alone, shows that the kernel of a reproducing kernel Hilbert space is *positive definite* in the sense that all the matrices

$$(\Phi(x_i,x_j))_{1\le i,j\le N}$$

are positive definite, provided that point evaluation functionals are linearly independent.
Now, it is easy to see that the kernel of a reproducing kernel Hilbert space is uniquely determined. On the other hand, also the Hilbert space is uniquely determined by the kernel. Moreover, every positive definite kernel generates a unique Hilbert space to which it is the reproducing kernel. This can simply be achieved by completing the pre-Hilbert space

$$F_\Phi(\Omega) = \text{span}\{\Phi(\cdot,x) : x \in \Omega\}$$

with respect to the inner product defined by

$$(\Phi(\cdot,x), \Phi(\cdot,y)) := \Phi(x,y).$$

More details about this fact and the construction of such *native* function spaces can be found in [22]. Here, the only thing that matters is that two *different* kernels can generate the *same* function Hilbert space $H$ but with different, but equivalent inner products.
In such a situation we will say that both kernels are reproducing kernels of $H$, thus relaxing Definition 2.2.
Moreover, it will be helpful to consider kernels defined on all $\mathbb{R}^n$ instead of only $\Omega \subseteq \mathbb{R}^n$. Such kernels are often *translation-invariant* meaning $\Phi(x,y) = \Phi(x-y)$ and often even *radial* meaning $\Phi(x,y) = \Phi(\|x-y\|_2)$.
This will be very useful when it comes to Sobolev spaces. Remember, that the Sobolev embedding theorem states that $W_2^\tau(\mathbb{R}^n)$ can be embedded into $C(\mathbb{R}^n)$ provided that $\tau > n/2$. Hence, in this situation $W_2^\tau(\mathbb{R}^n)$ is a reproducing kernel Hilbert space. Unfortunately, the reproducing kernel involves some modified Bessel functions of the third kind.
However, it is well known that other reproducing kernels of $W_2^\tau(\mathbb{R}^n)$ can be characterized by their Fourier transform

$$\widehat{\Phi}(\omega) = (2\pi)^{-n} \int_{\mathbb{R}^n} \Phi(x) e^{-ix^T\omega} dx.$$

To be more precise, the following result holds:

**Lemma 2.4 ([22, Corollary 10.13])** *Let $\tau > n/2$. Suppose the Fourier transform of an integrable function $\Phi : \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$c_1(1 + \|\omega\|_2^2)^{-\tau} \le \widehat{\Phi}(\omega) \le c_2(1 + \|\omega\|_2^2)^{-\tau}, \qquad \omega \in \mathbb{R}^n, \tag{5}$$

*with two constants $c_2 \ge c_1 > 0$. Then, the kernel $\Phi$ is also a reproducing kernel of $W_2^\tau(\mathbb{R}^n)$ and the inner product defined by*

$$(f, g) := \int_{\mathbb{R}^n} \frac{\widehat{f}(\omega)\overline{\widehat{g}(\omega)}}{\widehat{\Phi}(\omega)} d\omega$$

*is equivalent to the usual inner product on $W_2^\tau(\mathbb{R}^n)$.*

The following observation will be of use. It follows straight-forward from the Fourier inversion theorem.

**Remark 2.5** *If $\Phi \in L_1(\mathbb{R}^n)$ satisfies (5) with $\tau > m + n/2$, then $\Phi \in C^{2m}(\mathbb{R}^n)$.*

The most prominent examples of kernels satisfying (5) are the Wendland functions ([20, 21]). They are positive definite and radial functions with compact support. On their support they can be represented by univariate polynomials. They are defined by the following recursion with respect to the parameter $k$, cf. also Table 4.2.

**Definition 2.6 (Wendland functions)** *Let $\ell \in \mathbb{N}$, $k \in \mathbb{N}_0$. We define by recursion*

$$\psi_{\ell,0}(r) = (1 - r)_+^\ell \tag{6}$$

$$\text{and } \psi_{\ell,k+1}(r) = \int_r^1 t\psi_{\ell,k}(t)\, dt \tag{7}$$

*for $r \in \mathbb{R}_0^+$ where $(x)_+ = \max\{x, 0\}$.*

We fix the parameter $\ell$ depending on the space dimension $n$ and the smoothness parameter $k$. Then we have the following properties for the function $\Phi(x) = \psi_{\ell,k}(c\|x\|_2)$ with scaling parameter $c > 0$.

**Proposition 2.7 ([20, 21])** *Let $k \in \mathbb{N}$ and $\ell := \left\lfloor \frac{n}{2} \right\rfloor + k + 1$. Let $\Phi(x) := \psi_{\ell,k}(c\|x\|_2)$ with $c > 0$. Then*

1. *$\psi_{\ell,k}(cr)$ is a polynomial of degree $\left\lfloor \frac{n}{2} \right\rfloor + 3k + 1$ for $r \in \left[0, \frac{1}{c}\right]$.*
2. *$\Phi \in C_0^{2k}(\mathbb{R}^n, \mathbb{R})$, where $C_0^{2k}$ denotes the functions with compact support, which are $2k$ times continuously differentiable.*
3. *The Fourier transform $\widehat{\Phi}$ is an analytic function. It satisfies the decay condition (5) with $\tau = k + (n + 1)/2$.*

Note that these functions generate integer order Sobolev spaces in odd space dimensions, while for even space dimensions the order is integer plus a half.
Though most kernels, which generate Sobolev spaces, are radial, there exist also kernels, which are not even translation invariant, cf. [18, 17]. Our results will hold regardless whether the kernels are translation invariant or not.
We end this section by citing a general convergence result from [16] in its improved form (see the remarks in [15]).

**Theorem 2.8** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a Lipschitz continuous boundary, satisfying an interior cone condition. Let $1 \le p < \infty$, $1 \le q \le \infty$, and let $m \in \mathbb{N}_0$ and $\tau \in \mathbb{R}$ satisfying $\lfloor \tau \rfloor > m + n/p$ if $p > 1$, or $\lfloor \tau \rfloor \ge m + n$ if $p = 1$. Also, let $X \subseteq \Omega$ be a discrete set with sufficiently small mesh norm $h$. If $u \in W_p^\tau(\Omega)$ satisfies $u|X = 0$, then*

$$|u|_{W_q^m(\Omega)} \le C h^{\tau - m - n(1/p - 1/q)_+} |u|_{W_p^\tau(\Omega)}, \tag{8}$$

*where $(x)_+ = \max\{x, 0\}$.*

# 3 Partial Differential Equations

## 3.1 General PDE operators

It is now time to look at specific collocation problems. We start with the partial differential equation (1), i.e. we want to solve

$$Lu(x) := \sum_{|\alpha| \le m} c_\alpha(x) D^\alpha u(x) = f(x)$$

numerically on a bounded region $\Omega \subseteq \mathbb{R}^n$. Following the general approach of the previous section, we define functionals

$$\lambda_j(f) := \delta_{x_j} \circ L(f) = (Lf)(x_j)$$

with scattered points $X = \{x_1, \ldots, x_N\} \subseteq \Omega$. Hence, employing a sufficiently smooth kernel $\Phi : \Omega \times \Omega \to \mathbb{R}$ results in the approximating function

$$s = \sum_{k=1}^N \alpha_k (\delta_{x_k} \circ L)^y \Phi(\cdot, y), \tag{9}$$

and the interpolation conditions become

$$\begin{aligned} f(x_j) &= (\delta_{x_j} \circ L)^x u(x) = (\delta_{x_j} \circ L)^x s(x) \\ &= \sum_{k=1}^N \alpha_k (\delta_{x_j} \circ L)^x (\delta_{x_k} \circ L)^y \Phi(x, y). \end{aligned}$$

We summarize the interpolation problem.

**Definition 3.1 (Interpolation problem, operator)** *Let $X = \{x_1, \ldots, x_N\}$ be a set of pairwise distinct points in $\Omega \subseteq \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. Let $L$ be a linear differential operator.*
*The interpolation matrix $A = (a_{jk})_{j,k=1,\ldots,N}$ is given by*

$$a_{jk} = (\delta_{x_j} \circ L)^x (\delta_{x_k} \circ L)^y \Phi(x, y). \tag{10}$$

*The* reconstruction $s$ of $u$ with respect to the set $X$ and the operator $L$ *is given by*

$$s(x) \quad = \quad \sum_{k=1}^{N} \alpha_k (\delta_{x_k} \circ L)^y \Phi(x, y),$$

*where $\alpha$ is the solution of $A\alpha = f = (f_j)$ with $f_j = (\delta_{x_j} \circ L)^x u(x) = f(x_j)$.*

According to Lemma 2.3, the generalized interpolation matrix is positive definite, provided that the involved functionals are linearly independent.

**Definition 3.2 (Singular points of $L$)** *The point $x \in \mathbb{R}^n$ is called a* singular point of $L$ *if $\delta_x \circ L = 0$, i.e. $c_\alpha(x) = 0$ for all $|\alpha| \leq m$.*

**Proposition 3.3** *Suppose $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a reproducing kernel of $W_2^\tau(\mathbb{R}^n)$ with $\tau > m + n/2$. Let $L$ be a linear differential operator of degree $m$. Let $X = \{x_1, \ldots, x_N\}$ be a set of pairwise distinct points, which are no singular points of $L$. Then, the functionals $\lambda_j = \delta_{x_j} \circ L$ are linearly independent over $W_2^\tau(\mathbb{R}^n)$.*

PROOF: First of all note that, according to Remark 2.5, (10) is well defined for reproducing kernels of $W_2^\tau(\mathbb{R}^n)$ even with $\tau > m+n/2$. Moreover, the functionals are indeed in the dual space to $W_2^\tau(\mathbb{R}^n)$.
Next, suppose that

$$\sum_{k=1}^{N} d_k \lambda_k = 0 \tag{11}$$

on $W_2^\tau(\mathbb{R}^n)$ with certain coefficients $d_1, \ldots, d_N$.
Then, we choose a flat bump function $g \in C_0^\infty(\mathbb{R}^n)$, i.e. a nonnegative, compactly supported function with support $B(0, 1)$ which is non-vanishing and satisfies $g(x) = 1$ on $B(0, 1/2)$. Fix $1 \leq j \leq N$. Since $x_j$ is not a singular point of $L$, there exists a $\beta \in \mathbb{N}_0^n$ with minimal $|\beta| \leq m$ such that $c_\beta(x_j) \neq 0$. Employing the separation radius $q_X$, the function

$$g_j(x) = \frac{1}{\beta!}(x - x_j)^\beta g((x - x_j)/q_X)$$

then satisfies $D^\alpha g_j(x_k) = 0$ for all $|\alpha| \leq m$ and $x_k \neq x_j$. Furthermore, we have $D^\alpha g_j(x_j) = 0$ if $\alpha \neq \beta$ and $D^\beta g_j(x_j) = 1$. Hence, (11) gives in particular

$$0 = \sum_{k=1}^{N} d_k \lambda_k(g_j) = \sum_{|\alpha| \leq m} \sum_{k=1}^{N} d_k c_\alpha(x_k) D^\alpha g_j(x_k) = d_j c_\beta(x_j),$$

which implies $d_j = 0$. Since $j$ was chosen arbitrarily, this shows that the functionals are linearly independent. $\square$

This proposition is a generalization of the results in [4], where only constant coefficients have been allowed and of the results in [22], where also variable coefficients without zeros were treated.
Note also that the reproducing kernel Hilbert space does not have to be a Sobolev space at all. It is only necessary that the Hilbert space contains bump functions of the described form. Hence, the results remain true, if, for example, function spaces associated to Gaussians or (inverse) multiquadrics are considered.
Next we turn to error estimates. We need a simple auxiliary result.

**Lemma 3.4** *Fix $\tau \in \mathbb{R}$ with $k = \lfloor \tau \rfloor > n/2 + m$, where $m$ is the order of the differential operator $L$. Suppose that the coefficients $c_\alpha$ of the differential operator $L$ belong to $W_\infty^{k-m+1}(\Omega)$. Then, $L$ is a bounded operator from $W_2^\tau(\Omega)$ to $W_2^{\tau-m}(\Omega)$, i.e.*

$$\|Lu\|_{W_2^{\tau-m}(\Omega)} \leq C\|u\|_{W_2^\tau(\Omega)}, \qquad u \in W_2^\tau(\Omega).$$

PROOF: Take a multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k + 1 - m$. Then,

$$\begin{aligned} |D^\alpha(Lu)| &= \left| \sum_{|\beta|\leq m} \sum_{\gamma \leq \alpha} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} (D^{\alpha-\gamma} c_\beta)(D^{\gamma+\beta} u) \right| \\ &\leq C \sum_{|\beta|\leq m} \sum_{\gamma\leq\alpha} |D^{\gamma+\beta} u|, \end{aligned}$$

where we used the boundedness of the derivatives of the coefficients. This shows that

$$\|D^\alpha(Lu)\|_{L_2(\Omega)} \leq C\|u\|_{W_2^{m+|\alpha|}(\Omega)}$$

and hence

$$\begin{aligned} \|Lu\|_{W_2^{k-m}(\Omega)} &\leq C\|u\|_{W_2^k(\Omega)} \\ \|Lu\|_{W_2^{k+1-m}(\Omega)} &\leq C\|u\|_{W_2^{k+1}(\Omega)}. \end{aligned}$$

From this, the result for fractional order Sobolev spaces $W_2^\tau(\Omega)$ follows by interpolation theory. $\square$

**Theorem 3.5** *Suppose $\Phi$ is the reproducing kernel of $W_2^\tau(\mathbb{R}^n)$ with $k := \lfloor \tau \rfloor > m + n/2$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain satisfying an interior cone condition and having a Lipschitz boundary. Let $L$ be a linear differential operator of order $m$ with coefficients $c_\alpha$ in $W_\infty^{k-m+1}(\Omega)$ Finally, let $s$ be the generalized interpolant to $u \in W_2^\tau(\Omega)$ from Definition 3.1. If $X \subseteq \Omega$ has sufficiently small mesh norm $h_X$, then for $1 \leq p \leq \infty$, the error estimate*

$$\|Lu - Ls\|_{L_p(\Omega)} \leq C h_X^{\tau-m-n(1/2-1/p)_+} \|u\|_{W_2^\tau(\Omega)}$$

*is satisfied.*

PROOF: Note that $u \in W_2^\tau(\Omega) \subseteq C^m(\mathbb{R}^n)$ by assumption, while $s \in C^m(\mathbb{R}^n)$ by Remark 2.5. Hence, application of $L$ is feasible.
Since $Lu|X = Ls|X$ by definition, we can apply Theorem 2.8 to derive

$$\begin{aligned} \|Lu - Ls\|_{L_p(\Omega)} &\leq Ch_X^{\tau-m-n(1/2-1/p)_+}\|Lu - Ls\|_{W_2^{\tau-m}(\Omega)} \\ &\leq Ch_X^{\tau-m-n(1/2-1/p)_+}\|u - s\|_{W_2^\tau(\Omega)}, \end{aligned}$$

where we have also used Lemma 3.4.
Next, we follow the ideas in [16]. Our assumptions on the region $\Omega$ allow us to extend the function $u \in W_2^\tau(\Omega)$ to a function $Eu \in W_2^\tau(\mathbb{R}^n)$. Moreover, since $X \subseteq \Omega$ and $Eu|\Omega = u|\Omega$, the generalized interpolant $s = s_u$ to $u$ coincides with the generalized interpolant $s_{Eu}$ to $Eu$ on $\Omega$. Finally, the Sobolev space norm on $W_2^\tau(\mathbb{R}^n)$ is equivalent to the norm induced by the kernel $\Phi$ on $W_2^\tau(\mathbb{R}^n)$ (Lemma 2.4) and the generalized interpolant is norm-minimal (Lemma 2.3). This all gives

$$\begin{aligned} \|u - s\|_{W_2^\tau(\Omega)} &= \|Eu - s_{Eu}\|_{W_2^\tau(\Omega)} \leq \|Eu - s_{Eu}\|_{W_2^\tau(\mathbb{R}^n)} \\ &\leq C\|Eu\|_{W_2^\tau(\mathbb{R}^n)} \leq C\|u\|_{W_2^\tau(\Omega)}, \end{aligned}$$

and this establishes the stated error estimate. □

The most important choices of $p = 2$ and $p = \infty$ yield

$$\begin{aligned} \|Lu - Ls\|_{L_2(\Omega)} &\leq Ch_X^{\tau-m}\|u\|_{W_2^\tau(\Omega)} \\ \|Lu - Ls\|_{L_\infty(\Omega)} &\leq Ch_X^{\tau-m-n/2}\|u\|_{W_2^\tau(\Omega)}. \end{aligned}$$

As a consequence, using the compactly supported functions from Definition 2.6, we have to set $\tau = k+(n+1)/2$, where $k$ is the smoothness index of the compactly supported functions, i.e $\Phi = \psi_{\ell,k}(c\|\cdot\|_2) \in C^{2k}(\mathbb{R}^n)$. Note that this $k$ is different from the $k$ in Theorem 3.5. As a matter of fact the $k$ in that theorem is given by $\lfloor\tau\rfloor = k + \lfloor(n+1)/2\rfloor$.

**Corollary 3.6** *Denote by k the smoothness index of the compactly supported functions from Definition 2.6. Let $k > m - \frac{1}{2}$ if $n$ is odd or $k > m$ if $n$ is even. Let $c_\alpha \in W_\infty^{k-m+1+\lfloor\frac{n+1}{2}\rfloor}$. Suppose $u \in W_2^{k+(n+1)/2}(\Omega)$. Then, employing the basis functions from Definition 2.6 yields*

$$\|Lu - Ls\|_{L_\infty(\Omega)} \leq Ch_X^{k-m+\frac{1}{2}}\|u\|_{W_2^{k+(n+1)/2}(\Omega)}.$$

## 3.2 Boundary Value Problems

The collocation problem of the previous section will already be useful in its form in our application to dynamical systems; however, also boundary value problems will occur, cf. Section 4. For other applications like solving elliptic PDEs it is even crucial to incorporate also boundary values.

In order to solve a boundary value problem of the form (2), (3), we have two linear operators $L$ and $L^0 = \mathrm{id}$, the values of which are given on $\Omega$, $\partial\Omega$, respectively. The ansatz for the approximating function $s$ reflects this. We choose two sets of points, $X_1 := \{x_1, \ldots, x_N\} \subseteq \Omega$ and $X_2 := \{x_{N+1}, \ldots, x_{N+M}\} \subseteq \partial\Omega$ and define the functionals by

$$\lambda_j = \begin{cases} \delta_{x_j} \circ L, & \text{for } 1 \le j \le N, \\ \delta_{x_j} \circ L^0 & \text{for } N+1 \le j \le N+M. \end{cases} \tag{12}$$

The mixed ansatz for the approximant $s$ of the function $u$ is then given by

$$\begin{aligned} s(x) &= \sum_{k=1}^{N+M} \alpha_k \lambda_k^y \Phi(x,y) \\ &= \sum_{k=1}^{N} \alpha_k (\delta_{x_k} \circ L)^y \Phi(x,y) + \sum_{k=N+1}^{N+M} \alpha_k (\delta_{x_k} \circ L^0)^y \Phi(x,y), \end{aligned} \tag{13}$$

where we will assume that $L^0 = \mathrm{id}$. The coefficient vector $\alpha \in \mathbb{R}^{N+M}$ is determined by the interpolation conditions

$$(\delta_{x_j} \circ L)(s) = (\delta_{x_j} \circ L)(u) = f(x_j), \quad 1 \le j \le N \tag{14}$$

$$(\delta_{x_j} \circ L^0)(s) = (\delta_{x_j} \circ L^0)(u) = F(x_j), \quad N+1 \le j \le N+M. \tag{15}$$

Plugging the ansatz (13) into both (14) and (15) one obtains

$$\begin{aligned} (\delta_{x_j} \circ L)(u) &= \sum_{k=1}^{N} \alpha_k (\delta_{x_j} \circ L)^x (\delta_{x_k} \circ L)^y \Phi(x,y) \\ &\quad + \sum_{k=N+1}^{N+M} \alpha_k (\delta_{x_j} \circ L)^x (\delta_{x_k} \circ L^0)^y \Phi(x,y) \end{aligned}$$

for $1 \le j \le N$ and

$$\begin{aligned} (\delta_{x_j} \circ L^0)(u) &= \sum_{k=1}^{N} \alpha_k (\delta_{x_j} \circ L^0)^x (\delta_{x_k} \circ L)^y \Phi(x,y) \\ &\quad + \sum_{k=N+1}^{N+M} \alpha_k (\delta_{x_j} \circ L^0)^x (\delta_{x_k} \circ L^0)^y \Phi(x,y) \end{aligned}$$

for $N+1 \le j \le N+M$.

This is equivalent to the following system of linear equations

$$\widetilde{A}\alpha = \beta \text{ with } \widetilde{A} := \begin{pmatrix} A & C \\ C^T & A^0 \end{pmatrix} \in \mathbb{R}^{(N+M)\times(N+M)}. \tag{16}$$

For the definition of the matrices cf. the following definition, where we summarize the mixed interpolation problem.

**Definition 3.7 (Mixed interpolation problem)** *Let $X_1 = \{x_1, \dots, x_N\} \subseteq \Omega$ and $X_2 := \{x_{N+1}, \dots, x_{N+M}\} \subseteq \partial\Omega$ be two sets of pairwise distinct points and let $u\colon \Omega \to \mathbb{R}$ be the solution of (2), (3).*
*The* interpolation matrix *$\widetilde{A}$ from (16) has sub-matrices $A = (a_{ij}) \in \mathbb{R}^{N\times N}$, $C = (c_{ij}) \in \mathbb{R}^{N\times M}$ and $A^0 = (a^0_{ij}) \in \mathbb{R}^{M\times M}$ with elements*

$$\begin{aligned} a_{i,j} &= (\delta_{x_i} \circ L)^x (\delta_{x_j} \circ L)^y \Phi(x,y) \\ c_{i,\ell-N} &= (\delta_{x_i} \circ L)^x (\delta_{x_\ell} \circ L^0)^y \Phi(x,y) \\ a^0_{k-N,\ell-N} &= (\delta_{x_k} \circ L^0)^x (\delta_{x_\ell} \circ L^0)^y \Phi(x,y). \end{aligned}$$

*for $1 \le i,j \le N$, $N+1 \le k,\ell \le N+M$. The* reconstruction $s$ of $u$ with respect to the sets $X_1$ and $X_2$ and the operators $L$ and $L^0$ *is given by*

$$s(x) = \sum_{k=1}^{N} \alpha_k (\delta_{x_k} \circ L)^y \Phi(x,y) + \sum_{k=N+1}^{N+M} \alpha_k (\delta_{x_k} \circ L^0)^y \Phi(x,y),$$

*where $\alpha \in \mathbb{R}^{N+M}$ is the solution of $\widetilde{A}\alpha = \beta$ with $\beta_j = f(x_j)$ for $1 \le j \le N$ and $\beta_j = F(x_j)$ for $N+1 \le j \le N+M$, respectively.*

As in the case of one operator, it is easy to show that the functionals $\lambda_j$, this time defined by (12) are linearly independent.

**Proposition 3.8** *Suppose $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a reproducing kernel of $W_2^\tau(\mathbb{R}^n)$ with $\tau > m + n/2$. Let $L$ be a linear differential operator of degree $m$. Let $X_1 = \{x_1, \dots, x_N\} \subseteq \Omega$ and $X_2 = \{x_{N+1}, \dots, x_{N+M}\} \subseteq \partial\Omega$ be two sets of pairwise distinct points such that $X_1$ contains no singular point of $L$. Then, the functionals $\Lambda = \{\lambda_1, \dots, \lambda_{N+M}\}$ with $\lambda_j = \delta_{x_j} \circ L$, $1 \le j \le N$ and $\lambda_j = \delta_{x_j}$ for $N+1 \le j \le N+M$ are linearly independent over $W_2^\tau(\mathbb{R}^n)$.*

Next we turn to error estimates. To this end we have to make certain further assumptions on the boundary.
We will assume that the bounded region $\Omega \subseteq \mathbb{R}^n$ has a $C^{k,s}$-boundary $\partial\Omega$, where $\tau = k + s$ with $k \in \mathbb{N}_0$ and $s \in [0,1)$. This means in particular, that $\partial\Omega$ is a $n-1$ dimensional $C^{k,s}$-sub-manifold of $\mathbb{R}^n$. It also means that $\Omega$ is Lipschitz continuous and satisfies the cone condition. For details, we refer the reader to [23].
We will represent the boundary $\partial\Omega$ by a finite atlas consisting of $C^{k,s}$-diffeomorphisms with a slightly abuse of terminology. To be more precise, we assume that

$\partial\Omega \subseteq \cup_{j=1}^{K} V_j$, where $V_j \subseteq \mathbb{R}^n$ are open sets. Moreover, the set $V_j$ are images of $C^{k,s}$-diffeomorphism

$$\varphi_j : B \to V_j,$$

where $B = B(0,1)$ denotes the unit ball in $\mathbb{R}^{n-1}$. Finally, suppose $\{w_j\}$ is a partition of unity with respect to $\{V_j\}$. Then, the Sobolev norms on $\partial\Omega$ can be defined via

$$\|u\|_{W_p^\mu(\partial\Omega)}^p = \sum_{j=1}^{K} \|(uw_j) \circ \varphi_j\|_{W_p^\mu(B)}^p.$$

It is well known that this norm is independent of the chosen atlas $\{V_j, \varphi_j\}$ but this is of less importance here, since we will assume that the atlas is fixed. For us, the next also well known result will play a crucial role.

**Lemma 3.9** *Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded region with a $C^{k,s}$-boundary $\partial\Omega$. Then, the restriction of $u \in W_2^\tau(\Omega)$ with $\tau = k + s$ to $\partial\Omega$ is well defined, belongs to $W_2^{\tau-1/2}(\partial\Omega)$, and satisfies*

$$\|u\|_{W_2^{\tau-1/2}(\partial\Omega)} \le \|u\|_{W_2^\tau(\Omega)}.$$

Moreover, we now have two different mesh norms, $h_{X_1,\Omega}$ for the domain part and $h_{X_2,\partial\Omega}$ for the boundary part. Using the atlas $\{V_j, \varphi_j\}$, we simply define the latter to be

$$h_{X_2,\partial\Omega} := \max_{1\le j\le K} h_{T_j,B}$$

with $T_j = \varphi_j^{-1}(X_2 \cap V_j) \subseteq B$. As mentioned before, we will assume the atlas fixed and hence do not have to care about the dependence of $h_{X_2,\partial\Omega}$ on the atlas.

**Theorem 3.10** *Suppose $\Phi$ is the reproducing kernel of $W_2^\tau(\mathbb{R}^n)$ with $k := \lfloor \tau \rfloor > m + n/2$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain having a $C^{k,s}$-boundary. Let $L$ be a linear differential operator of order $m$ with coefficients $c_\alpha$ in $W_\infty^{k-m+1}(\Omega)$ Finally, let $s$ be the generalized interpolant to $u \in W_2^\tau(\Omega)$ from Definition 3.7. If the data sets have sufficiently small mesh norms then for $1 \le p \le \infty$, the error estimates*

$$\begin{aligned} \|Lu - Ls\|_{L_p(\Omega)} &\le Ch_{X_1,\Omega}^{\tau-m-n(1/2-1/p)_+} \|u\|_{W_2^\tau(\Omega)} & (17)\\ \|u - s\|_{L_p(\partial\Omega)} &\le Ch_{X_2,\partial\Omega}^{\tau-1/2-(n-1)(1/2-1/p)_+} \|u\|_{W_2^\tau(\Omega)} & (18) \end{aligned}$$

*are satisfied.*

PROOF: Estimate (17) follows as in Theorem 3.5. For the second estimate, note that the functions $u_j = ((u - s)w_j) \circ \varphi_j$ belong to $W_2^{\tau-1/2}(B)$ and vanish on $T_j$.

Hence, using the definition of the Sobolev norm on $\partial\Omega$ and Theorem 2.8 yields

$$\begin{aligned}
\|u-s\|^p_{L_p(\partial\Omega)} &= \sum_{j=1}^K \|u_j\|^p_{L_p(B)} \\
&\le C\sum_{j=1}^K h_{T_j,B}^{p(\tau-1/2-(n-1)(1/2-1/p)_+)}\|u_j\|^p_{W_2^{\tau-1/2}(B)} \\
&\le C h_{X_2,\partial\Omega}^{p(\tau-1/2-(n-1)(1/2-1/p)_+)}\|u-s\|^p_{W_2^{\tau-1/2}(\partial\Omega)} \\
&\le C h_{X_2,\partial\Omega}^{p(\tau-1/2-(n-1)(1/2-1/p)_+)}\|u-s\|^p_{W_2^{\tau}(\Omega)}
\end{aligned}$$

for $1 \le p < \infty$ and the case $p = \infty$ is treated in the same fashion. Finally, since $s$ is a norm-minimal interpolant, the norm in the last expression can again be bounded by the norm of $u$. □

The two most important estimates for the boundary part are hence

$$\begin{aligned}
\|u-s\|_{L_\infty(\partial\Omega)} &\le C h_{X_2\partial\Omega}^{\tau-n/2}\|u\|_{W_2^\tau(\Omega)}, \\
\|u-s\|_{L_2(\partial\Omega)} &\le C h_{X_2,\partial\Omega}^{\tau-1/2}\|u\|_{W_2^\tau(\Omega)}.
\end{aligned}$$

The proof of Theorem 3.10 shows, that the following alternative version of Theorem 3.10 is also true.

**Corollary 3.11** *Suppose $\Gamma \subseteq \partial\Omega$ is a part of the boundary satisfying*

$$\Gamma = \bigcup_{j=1}^L (V_j \cap \partial\Omega). \tag{19}$$

*This means, that the first $L$ charts $\{V_j, \varphi_j\}_{j=1}^L$ are exclusive for $\Gamma$, or that, for $1 \le j \le L$, $V_j \cap (\partial\Omega \setminus \Gamma) = 0$. Suppose further, that the boundary collocation points $X_2$ are chosen only on $\Gamma$, while the interior points are still chosen in $\Omega$, then estimate (17) remains valid and (18) becomes*

$$\|u-s\|_{L_p(\Gamma)} \le C h_{X_2,\Gamma}^{\tau-1/2-(n-1)(1/2-1/p)_+}\|u\|_{W_2^\tau(\Omega)}, \tag{20}$$

*where $h_{X_2,\Gamma} = \max_{1\le j\le L} h_{T_j,B}$ with $T_j$ defined as before.*

As a matter of fact, neither condition (19) nor the fact that $X_2 \subseteq \Gamma$ are necessary to derive (20). But if (19) is not satisfied, the fill distance $h_{X_2,\Gamma}$ might be larger than necessary if $X_2$ is only chosen from $\Gamma$. On the other hand, if $X_2$ is dense on all of $\partial\Omega$, than, of course, (18) implies (20).
Considering again the compactly supported functions $\Phi = \psi_{\ell,k}(\|\cdot\|_2)$, i.e. choosing $\tau = k + (n+1)/2$ gives this time

**Corollary 3.12** *Let $k > m - 1/2$ if $n$ is odd or $k > m$ if $n$ is even. Let $c_\alpha \in W_\infty^{k-m+1+\lfloor \frac{n+1}{2} \rfloor}$. Suppose $u \in W_2^{k+(n+1)/2}(\Omega)$. Then, employing the basis functions from Definition 2.6 yields*

$$\begin{aligned} \|Lu - Ls\|_{L_\infty(\Omega)} &\leq Ch_{X_1,\Omega}^{k-m+1/2} \|u\|_{W_2^{k+(n+1)/2}(\Omega)}, & (21) \\ \|u - s\|_{L_\infty(\partial\Omega)} &\leq Ch_{X_2,\partial\Omega}^{k+1/2} \|u\|_{W_2^{k+(n+1)/2}(\Omega)}. & (22) \end{aligned}$$

A similar statement holds also for $\Gamma \subset \partial\Omega$, cf. Corollary 3.11.

### 3.3 Elliptic PDEs

We now consider the following elliptic operator of second order in a bounded domain $\Omega \subset \mathbb{R}^n$ with a sufficiently smooth boundary

$$Lu(x) := \sum_{i,j=1}^n a_{ij}(x)\partial_{i,j}u(x) + \sum_{i=1}^n b_i(x)\partial_i u(x) + c(x)u(x) \qquad (23)$$

where $a$, $b$ and $c$ are bounded, $a_{ij}(x) = a_{ji}(x)$ (symmetry) and $c(x) \leq 0$ holds for all $x \in \Omega$. Moreover, let $L$ be strictly elliptic, i.e. there is a constant $\lambda > 0$ such that

$$\lambda \|\xi\|_2^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Then, if $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ is the solution of

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= F \text{ on } \partial\Omega, \end{aligned}$$

it enjoys the following estimate (see [9, Theorem 3.7])

$$\|u\|_{L_\infty(\Omega)} \leq \|F\|_{L_\infty(\partial\Omega)} + \frac{C}{\lambda}\|f\|_{L_\infty(\Omega)}, \qquad (24)$$

where the constant $C$ depends on the diameter of $\Omega$ and on $\|b\|_{L_\infty(\Omega)}/\lambda$.

**Corollary 3.13** *Assume that the solution $u$ belongs to $W_2^\tau(\Omega)$ with $\lfloor \tau \rfloor > 2 + n/2$. Then, the error between $u$ and its collocation approximation $s$ can be bounded by*

$$\begin{aligned} \|u - s\|_{L_\infty(\Omega)} &\leq C\left(h_{X_1,\Omega}^{\tau-2-n/2} + h_{X_2,\partial\Omega}^{\tau-n/2}\right)\|u\|_{W_2^\tau(\Omega)} \\ &\leq Ch_X^{\tau-2-n/2}\|u\|_{W_2^\tau(\Omega)}, \end{aligned}$$

*where $h_X = \max\{h_{X_1,\Omega}, h_{X_2,\partial\Omega}\}$.*

PROOF: Using Theorem 3.10 and (24) gives

$$\begin{aligned}\|u-s\|_{L_\infty(\Omega)} &\le \|u-s\|_{L_\infty(\partial\Omega)} + C\|Lu-Ls\|_{L_\infty(\Omega)}\\ &\le C\left(h_{X_1,\Omega}^{\tau-2-n/2} + h_{X_2,\partial\Omega}^{\tau-n/2}\right)\|u\|_{W_2^\tau(\Omega)}.\end{aligned}$$

□

Note that this result unfortunately means that we have to choose a higher data density in the interior than on the boundary.
The result for the compactly supported functions is

$$\|u-s\|_{L_\infty(\Omega)} \le C\left(h_{X_1,\Omega}^{k-3/2} + h_{X_2,\partial\Omega}^{k-1/2}\right)\|u\|_{W_2^{k+(n+1)/2}(\Omega)}.$$

In the case of constant coefficients, i.e. $a_{ij}(x) = a_{ij}$, $b_i(x) = b_i$ and $c(x) = c$ for all $x \in \Omega$, this result was obtained in [4] using a Transformation Theorem. Our result, however, also holds for non-constant coefficients and is mainly a simple application of Theorem 3.10.

# 4 Dynamical Systems

## 4.1 A Short Introduction

In the theory of dynamical systems given by an ordinary differential equation

$$\dot{x} = \frac{dx}{dt} = g(x),$$

where $x(t) \in \mathbb{R}^n$, one is interested, among other things, in the construction of Lyapunov functions. The definition goes back to Lyapunov, cf. [13]. These Lyapunov functions are a tool to determine the basin of attraction of equilibria or other invariant attracting sets. The main characteristic of a Lyapunov function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ is that its orbital derivative $V'(x)$, i.e. the derivative along solutions of the differential equation, is negative. The orbital derivative is given by

$$V'(x) = LV(x) = \sum_{i=1}^{n} g_i(x)\partial_i V(x)$$

and defines thus a first-order differential operator $L$ of the form (2).
It is well known that there exists a special Lyapunov function, which satisfies the linear partial differential equation of first order

$$LV(x) = V'(x) = \sum_{i=1}^{n} g_i(x)\partial_i V(x) = -\|x-x_0\|_2^2 = f(x),$$

which is of the form (1). We approximate this solution by $s$ and thus obtain a function $s$ with negative orbital derivative itself, i.e. a Lyapunov function. We will now explain the method in more detail.

Consider the ordinary differential equation

$$\dot{x} \quad = \quad g(x), \tag{25}$$

where $g \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma \geq 1$, and $x(t) \in \mathbb{R}^n$. We search for solutions $x(t)$, $t \geq 0$ of the initial value problem (25), $x(0) = \xi$. We denote these solutions also by $S_t\xi := x(t)$. Since $g$ is at least $C^1$, we have existence and uniqueness of solutions of this initial value problem locally in time.
Since one cannot determine the solutions of (25) in general, dynamical systems theory is interested in the qualitative long-time behavior of solutions. Therefore, one studies simple solutions such as equilibria, i.e. solutions which are constant in time.

**Definition 4.1** *$x_0 \in \mathbb{R}^n$ is called an equilibrium for (25) if $g(x_0) = 0$. Then $S_t x_0 = x_0$ for all $t \geq 0$, i.e. the constant function $x(t) = x_0$ is a solution of (25).*

The concept of stability describes the behavior of solutions near the equilibrium $x_0$. Stability can be analyzed using the linearization of $g$ at $x_0$.

**Proposition 4.2** *Let $x_0 \in \mathbb{R}^n$ be an equilibrium for (25). If all eigenvalues of the Jacobian $Dg(x_0)$ have negative real part, then $x_0$ is asymptotically stable, i.e. stable and attractive, which is defined as follows:*

- *Stability: for each $\epsilon > 0$ there is a $\delta > 0$ such that $\xi \in B(x_0, \delta)$ implies that $S_t\xi$ exists for all $t \geq 0$ and that $S_t\xi \in B(x_0, \epsilon)$ for all $t \geq 0$.*
- *Attractivity: there is a $\delta' > 0$ such that $\xi \in B(x_0, \delta')$ implies that $S_t\xi$ exists for all $t \geq 0$ and that $\lim_{t\to\infty} S_t\xi = x_0$.*

For asymptotically stable equilibria $x_0$ we can define the basin of attraction $A(x_0)$, which is the set of all initial conditions, for which the solution tends to $x_0$ as $t \to \infty$. The set $B(x_0, \delta')$, cf. the definition of attractivity, is a subset of $A(x_0)$. However, since the basin of attraction is a global object in contrast to the local character of the asymptotic stability, its determination cannot be obtained by linearization.

**Definition 4.3** *Let $x_0 \in \mathbb{R}^n$ be an asymptotically stable equilibrium for (25). Then we define the basin of attraction as*

$$A(x_0) := \{\xi \in \mathbb{R}^n \mid \lim_{t\to\infty} S_t\xi = x_0\}.$$

Note that $A(x_0) \neq \varnothing$ and $A(x_0)$ is open.
A method to determine subsets of the basin of attraction is the method of Lyapunov functions. The main characteristic of a Lyapunov function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ is that its orbital derivative $V'(x)$, i.e. the derivative along solutions of (25), is negative. The orbital derivative can be calculated by the chain rule:

$$\frac{d}{dt}V(x(t)) = \langle \nabla V(x(t)), \dot{x}(t)\rangle = \sum_{j=1}^{n} (\partial_j V)(x(t)) g_j(x(t)).$$

**Definition 4.4** *Given a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ its orbital derivative with respect to (25) is defined as*

$$V'(x) := \langle \nabla V(x), g(x) \rangle = \sum_{j=1}^{n} \partial_j V(x) g_j(x).$$

Note that we do not need to know the solution of (25) to calculate the orbital derivative. Moreover, the orbital derivative is a linear differential operator of first order of the form (2):

$$LV(x) = V'(x) = \sum_{i=1}^{n} g_i(x) \partial_i V(x).$$

Here, the singular points, i.e. those points where $(\delta_x \circ L) = 0$, are simply the equilibrium points, i.e. those points satisfying $g(x) = 0$.
The following theorem explains the use of Lyapunov functions for the determination of the basin of attraction.

**Theorem 4.5** *Let $s \in C^1(\mathbb{R}^n, \mathbb{R})$ and $K \subset \mathbb{R}^n$ be a compact set with neighborhood $B$ such that $x_0 \in \overset{\circ}{K}$. Furthermore, let*

1. *$K = \{x \in B \mid s(x) \leq R\}$ with an $R \in \mathbb{R}$, i.e. $K$ is a sublevel set of $s$.*
2. *$s'(x) < 0$ for all $x \in K \setminus \{x_0\}$, i.e. $s$ is decreasing along solutions in $K \setminus \{x_0\}$.*

*Then $K \subset A(x_0)$.*

Hence, a Lyapunov function provides information on the basin of attraction through its sublevel sets. However, it is not easy to find a Lyapunov function for a general system (25). Although existence of several types of Lyapunov functions is known, their construction is not easy.
For linear differential equations, i.e. $g(x)$ is linear, however, one can easily calculate a Lyapunov function. For a nonlinear system we consider the linearized system at the equilibrium point, namely $\dot{x} = Dg(x_0)(x - x_0)$. This is a linear system and, thus, one can easily calculate a Lyapunov function of the form $v(x) = (x - x_0)^T C(x - x_0)$, where the positive definite matrix $C$ is the unique solution of the matrix equation $Dg(x_0)^T C + CDg(x_0) = -I$, cf. [19]. The function $v$ is not only a Lyapunov function for the linearized system, but also for the nonlinear system in a neighborhood of $x_0$, for details cf. [7].

**Lemma 4.6 (Local Lyapunov function)** *Denote by $C \in \mathbb{R}^{n \times n}$ the unique solution of the matrix equation $Dg(x_0)^T C + CDg(x_0) = -I$ and define the local Lyapunov function*

$$v(x) = (x - x_0)^T C(x - x_0).$$

*Then, there is a compact set $K$ with a neighborhood $B$ such that $x_0 \in \overset{\circ}{K}$. Moreover, $v'(x) < 0$ holds for all $x \in K \setminus \{x_0\}$ and $K = \{x \in B \mid v(x) \leq R\}$ with $R > 0$.*

We return to Lyapunov functions which have negative orbital derivative for all $x \in A(x_0)\setminus\{x_0\}$. We consider special Lyapunov functions satisfying certain equations for their orbital derivatives. In the first part of Theorem 4.8 $p(x) = \|x - x_0\|_2^2$ is a feasible candidate. For the second part we need

**Definition 4.7 (Non-characteristic hypersurface)** *Let $h \in C^\sigma(\mathbb{R}^n, \mathbb{R})$. The set $\Gamma \subset \mathbb{R}^n$ is called a* non-characteristic hypersurface *if*

- *$\Gamma$ is compact,*
- *$h(x) = 0$ holds for all $x \in \Gamma$,*
- *$h'(x) < 0$ holds for all $x \in \Gamma$, and*
- *for each $x \in A(x_0) \setminus \{x_0\}$ there is a time $\theta(x) \in \mathbb{R}$ such that $S_{\theta(x)}x \in \Gamma$.*

An example for a non-characteristic hypersurface is a level set of the local Lyapunov function, cf. Lemma 4.6.

**Theorem 4.8 ([7])**

1. *Let $p(x) \in C^\sigma(\mathbb{R}^n, \mathbb{R})$ satisfy the following conditions:*

   *(a) $p(x) > 0$ for $x \neq x_0$,*
   *(b) $p(x) = O(\|x - x_0\|_2^\eta)$ with $\eta > 0$ for $x \to x_0$,*
   *(c) for all $\epsilon > 0$, $p$ has a lower positive bound on $\mathbb{R}^n \setminus B_\epsilon(x_0)$.*

   *Then, there exists a Lyapunov function $V_1 \in C^\sigma(A(x_0), \mathbb{R})$ such that*

   $$LV_1(x) = f_1(x) := -p(x) \text{ for all } x \in A(x_0).$$

2. *Let $c > 0$, let $\Gamma$ be a non-characteristic hypersurface, see Definition 4.7, and $F \in C^\sigma(\Gamma, \mathbb{R})$. Then, there is a Lyapunov function $V_2 \in C^\sigma(A(x_0) \setminus \{x_0\}, \mathbb{R})$ such that*

   $$\begin{aligned} LV_2(x) &= f_2(x) := -c \text{ for all } x \in A(x_0) \setminus \{x_0\}, \\ V_2(x) &= F(x) \text{ for all } x \in \Gamma. \end{aligned}$$

## 4.2 Approximating Lyapunov Functions

Theorem 4.8 shows two possibilities to approximate Lyapunov functions. We can use the first part to approximate $V_1$ by solving the problem

$$Ls_1(x) = LV_1(x) = -p(x), \qquad x \in A(x_0).$$

This is an example of an operator problem of type (1) and our theory from Section 3.1 applies.

On the other hand, the second part of Theorem 4.8 implies to solve the boundary value problem

$$\begin{aligned} Ls_2(x) &= f_2(x) = -c, \qquad x \in A(x_0) \setminus \{x_0\}, \\ s_2(x) &= F(x), \qquad x \in \Gamma, \end{aligned}$$

such that we can use our theory from Section 3.2.
However, in both cases the application of our error estimates has now a different character. An error bound of the form

$$|LV(x) - Ls(x)| = |V'(x) - s'(x)| < \epsilon$$

leads to

$$s'(x) \leq V'(x) + \epsilon < 0,$$

provided that $\epsilon$ is sufficiently small. Remember that $V$, as a Lyapunov function satisfies $V'(x) < 0$. Hence, in this case $s$ is itself a Lyapunov function.
However, for the specific choices of Lyapunov functions from Theorem 4.8 we have a problem if $x$ is close to $x_0$. In the first case, $V_1'(x) = f_1(x) = -p(x)$ and $p(x) \to 0$ as $x \to x_0$. Hence, this estimate will not hold near $x_0$ and thus $s_1'$ may be positive near $x_0$. The same problem arises for the approximation $s_2$ of $V_2$, since $V_2$ is not defined in $x_0$. Fortunately, locally, it is easy to determine the basin of attraction by linearization, cf. Lemma 4.6.
Before we can apply the results of this paper to the calculation of Lyapunov functions, we need some information about the level sets of Lyapunov functions. We assume that $g$ is bounded in $A(x_0)$. This can easily be achieved by considering the system $\dot{x} = h(x) := \frac{g(x)}{1+\|g(x)\|^2}$. Note that $\|h(x)\| \leq \frac{1}{2}$. This system has the same equilibria and basins of attraction as the system (25), since $h(x)$ is obtained by multiplication of $g(x)$ by a positive, scalar factor, i.e. the orbits of both systems are the same, but the velocity is different.

**Theorem 4.9 ([7])** *Let $x_0$ be an equilibrium of $\dot{x} = g(x)$, $g \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma \geq 1$ and let the maximal real part of all eigenvalues of $Dg(x_0)$ be negative. Let $g$ be bounded in $A(x_0)$ and let $V = V_i$, $i = 1, 2$ be one of the functions of Theorem 4.8. Then for all $r > 0$ the set $\{x \in A(x_0) \setminus \{x_0\} \mid V(x) \leq r\} \cup \{x_0\}$ is compact. Moreover, there is a $C^\sigma$-diffeomorphism*

$$\phi \in C^\sigma(S^{n-1}, \{x \in A(x_0) \mid V(x) = r\}),$$

*where $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$. For $V_2$ we have $\lim_{x \to x_0} V_2(x) = -\infty$.*

In the second case $V_2$, one first has to link the function $V_2$ to a local Lyapunov function to obtain the above theorem. For details, see [7].
In order to apply the results of Section 3 to approximate the functions $V_1$, $V_2$ of Theorem 4.8, respectively, we have to choose a set $\Omega$ in an appropriate way. For $V_1$ we consider the set $\Omega := \{x \in A(x_0) \mid V_1(x) \leq r\} \setminus \{x_0\}$, which has a smooth

boundary. For $V_2$ we choose $\Omega := \{x \in A(x_0) \setminus \{x_0\} \mid V_2(x) \leq r \text{ and } h(x) \geq 0\}$ where $h$ is the function defining the non-characteristic hypersurface, cf. Definition 4.7, and $r > 0$ is large enough such that $\{x \in A(x_0) \setminus \{x_0\} \mid V_2(x) = r\} \cap \Gamma = \varnothing$. This set $\Omega$ has a smooth boundary as well.

**Theorem 4.10** *Let $k := \lfloor \tau \rfloor > 1+n/2$. Consider the dynamical system defined by the ordinary differential equation $\dot{x} = g(x)$, where $g \in C^k(\mathbb{R}^n, \mathbb{R}^n)$. Let $x_0 \in \mathbb{R}^n$ be an exponentially asymptotically stable equilibrium. Let $g$ be bounded in $A(x_0)$ and denote by $V_1 \in W_2^\tau(A(x_0), \mathbb{R})$, $V_2 \in W_2^\tau(A(x_0) \setminus \{x_0\}, \mathbb{R})$ the Lyapunov functions of Theorem 4.8.*

1. *The reconstruction $s_1$ of the Lyapunov function $V_1$ with respect to the operator $Lu(x) = \langle \nabla u(x), g(x) \rangle$ and a set $X \subseteq \Omega := \{x \in A(x_0) \mid V_1(x) \leq r\} \setminus \{0\}$, $r > 0$, satisfies*

$$\|s_1' - V_1'\|_{L_\infty(\Omega)} = \|s_1' + p\|_{L_\infty(\Omega)} \leq C h_X^{\tau-1-n/2} \|V_1\|_{W_2^\tau(\Omega)}.$$

2. *Let $\Gamma = \{x \in A(x_0) \setminus \{x_0\} \mid h(x) = 0\}$ be a non-characteristic hypersurface and set $\Omega = \{x \in A(x_0) \setminus \{x_0\} \mid V_2(x) \leq r \text{ and } h(x) \geq 0\}$, where $r > 0$ is large enough such that $\{x \in A(x_0) \setminus \{x_0\} \mid V_2(x) = r\} \cap \Gamma = \varnothing$. The reconstruction $s_2$ of $V_2$ with respect to the boundary value problem $Lu(x) = \langle \nabla u(x), g(x) \rangle$, $u(x) = 0 = F(x)$ for $\Gamma$ and the data sites $X_1 \subset \Omega$ and $X_2 \subset \Gamma$ satisfies*

$$\begin{aligned}
\|s_2'(x) - V_2'(x)\|_{L_\infty(\Omega)} &= \|s_2'(x) + c\|_{L_\infty(\Omega)} \\
&\leq C h_{X_1,\Omega}^{\tau-1-n/2} \|V_2\|_{W_2^\tau(\Omega)}, \\
\|s_2(x) - V_2(x)\|_{L_\infty(\Gamma)} = \|s_2(x)\|_{L_\infty(\Gamma)} &\leq C h_{X_2,\Gamma}^{\tau-n/2} \|V_2\|_{W_2^\tau(\Omega)}.
\end{aligned}$$

PROOF: Note that the data sites $x_j$, $1 \leq j \leq N$ are no singular points, i.e. $g(x_j) \neq 0$ or equilibria in this case, since there are no equilibria in $A(x_0) \setminus \{x_0\}$.

1. We apply Theorem 3.5 with $m = 1$. The set $\Omega$ is bounded and has a smooth boundary by Theorem 4.9 and thus satisfies the conditions of Theorem 3.5, cf. [23]. The functions $c_\alpha$ are $g_j \in C^k(\mathbb{R}^n, \mathbb{R})$ and thus in $W_\infty^k(\Omega)$.

2. We apply Theorem 3.10 with $m = 1$. The sets $\Omega$ and $\Gamma \subset \partial\Omega$ are bounded and $\Omega$ has a smooth boundary by Theorem 4.9 (see also [23]). Thus the conditions of Theorem 3.10 are satisfied. The functions $c_\alpha$ are $g_j \in C^k(\mathbb{R}^n, \mathbb{R})$ and thus in $W_\infty^k(\Omega)$.

□

The next proposition describes how the calculation can be achieved for radial basis functions, in particular those from Definition 2.6. We set $\psi_2(0) = 0$ since then $a_{jj} = -\psi_1(0)\|g(x_j)\|^2$, cf. (26). Note that the first term of (26) is at least of order $O(r)$ for $r \to 0$.

**Proposition 4.11 ([7])** *Let $L$ be given by $Lu(x) = \langle \nabla u(x), g(x)\rangle$. Let $\Phi(x) := \psi(\|x\|_2)$ be a sufficiently smooth radial basis function. Define $\psi$ and $\psi_2$ by*

$$\begin{aligned}\psi_1(r) &= \frac{1}{r}\frac{d}{dr}\psi(r) \text{ for } r > 0,\\ \psi_2(r) &= \begin{cases} \frac{1}{r}\frac{d}{dr}\psi_1(r) & \text{for } r > 0,\\ 0 & \text{for } r = 0.\end{cases}\end{aligned}$$

*Then, the matrix elements $a_{jk}$ of the interpolation matrix $A$ in Definition 3.1 are given by*

$$\begin{aligned}a_{jk} &= \psi_2(\|x_j - x_k\|)\langle x_j - x_k, g(x_j)\rangle\langle x_k - x_j, g(x_k)\rangle\\ &\quad -\psi_1(\|x_j - x_k\|)\langle g(x_j), g(x_k)\rangle. \qquad (26)\end{aligned}$$

*The approximant $s$ and its orbital derivative are given by*

$$\begin{aligned}s(x) &= \sum_{k=1}^{N} \alpha_k \langle x_k - x, g(x_k)\rangle \psi_1(\|x - x_k\|),\\ s'(x) &= \sum_{k=1}^{N} \alpha_k \Big[\psi_2(\|x - x_k\|)\langle x - x_k, g(x)\rangle\langle x_k - x, g(x_k)\rangle\\ &\qquad -\psi_1(\|x - x_k\|)\langle f(x), f(x_k)\rangle\Big].\end{aligned}$$

Table 4.2 shows that the necessary functions $\psi_1$ and $\psi_2$ can explicity be computed in case of the Wendland functions and their simple form.

| | $\psi_{3,1}(cr)$ | $\psi_{4,2}(cr)$ |
|---|---|---|
| $\psi(r)$ | $(1-cr)_+^4[4cr+1]$ | $(1-cr)_+^6[35(cr)^2+18cr+3]$ |
| $\psi_1(r)$ | $-20c^2(1-cr)_+^3$ | $-56c^2(1-cr)_+^5[1+5cr]$ |
| $\psi_2(r)$ | $60c^3\frac{1}{r}(1-cr)_+^2$ | $1680c^4(1-cr)_+^4$ |

| | $\psi_{5,3}(cr)$ |
|---|---|
| $\psi(r)$ | $(1-cr)_+^8[32(cr)^3+25(cr)^2+8cr+1]$ |
| $\psi_1(r)$ | $-22c^2(1-cr)_+^7[16(cr)^2+7cr+1]$ |
| $\psi_2(r)$ | $528c^4(1-cr)_+^6[6cr+1]$ |

Table 1: The functions $\psi_1$ and $\psi_2$ for the Wendland functions $\psi_{3,1}(cr)$, $\psi_{4,2}(cr)$ and $\psi_{5,3}(cr)$. Note, that these are the Wendland functions of Definition 2.6 up to a constant.

**Corollary 4.12** *Denote by $k$ the smoothness index of the compactly supported functions from Definition 2.6.*
*Let $k > \frac{1}{2}$ if $n$ is odd or $k > 1$ if $n$ is even. Set $\tau = k + (n+1)/2$ and $\sigma = \lceil \tau \rceil$. Consider the dynamical system defined by the ordinary differential equation $\dot{x} = g(x)$, where $g \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$. Let $x_0 \in \mathbb{R}^n$ be an exponentially asymptotically stable equilibrium. Let $g$ be bounded in $A(x_0)$ and denote by $V_1 \in W_2^\tau(A(x_0), \mathbb{R})$ and $V_2 \in W_2^\tau(A(x_0) \setminus \{x_0\}, \mathbb{R})$ the Lyapunov functions of Theorem 4.8.*

1. *The reconstruction $s_1$ of the Lyapunov function $V_1$ with respect to the operator $Lu(x) = \langle \nabla u(x), g(x) \rangle$ and a set $X \subseteq \Omega := \{x \in A(x_0) \mid V_1(x) \leq r\} \setminus \{x_0\}$, $r > 0$, satisfies*

$$\|s_1' - V_1'\|_{L_\infty(\Omega)} = \|s_1' + p\|_{L_\infty(\Omega)} \leq C h_X^{k-\frac{1}{2}} \|V_1\|_{W_2^{k+(n+1)/2}(\Omega)}. \tag{27}$$

2. *Let $\Gamma = \{x \in A(x_0) \setminus \{x_0\} \mid h(x) = 0\}$ be a non-characteristic hypersurface and set $\Omega = \{x \in A(x_0) \setminus \{x_0\} \mid V_2(x) \leq r$ and $h(x) \geq 0\}$ where $r > 0$ is large enough such that $\{x \in A(x_0) \setminus \{x_0\} \mid V_2(x) = r\} \cap \Gamma = \varnothing$. The reconstruction $s_2$ of $V_2$ with respect to the boundary value problem $Lu(x) = \langle \nabla u(x), g(x) \rangle$, $u(x) = 0 = F(x)$ for $\Gamma$ and the sets of data sites $X_1 \subset \Omega$ and $X_2 \subset \Gamma$ satisfies*

$$\|s_2'(x) - V_2'(x)\|_{L_\infty(\Omega)} \leq C h_{X_1,\Omega}^{k-\frac{1}{2}} \|V_2\|_{W_2^{k+(n+1)/2}(\Omega)}, \tag{28}$$

$$\|s_2(x) - V_2(x)\|_{L_\infty(\Gamma)} \leq C h_{X_2,\Gamma}^{k+\frac{1}{2}} \|V_2\|_{W_2^{k+(n+1)/2}(\Omega)}. \tag{29}$$

PROOF: Apply Corollaries 3.6, 3.11, and 3.12, respectively with $m = 1$. □

The method described in this paper has already been used in [6, 7, 8]. However, the approximation orders derived in those papers were based on Taylor approximation of first order and hence the results in those papers were significantly worse than the results of Corollary 4.12.
The theorems and corollaries of this section ensure that the approximation of the Lyapunov functions $V_1$ and $V_2$ produces functions $s_1$, $s_2$, respectively, with negative orbital derivatives in $\Omega$ if the data sites are dense enough. For the remaining neighborhood of the equilibrium $x_0$ we use a local Lyapunov function, cf. Lemma 4.6. We can combine the approximated function $s$ and the local Lyapunov function $v$ to a new Lyapunov function $\tilde{s}$ such that $\tilde{s}'(x) < 0$ holds for all $x \in \Omega \setminus \{x_0\}$ and such that level sets of $s$ are level sets of $\tilde{s}$.
However, since Theorem 4.5 requires a sublevel set of $s$ within the region where $s'(x) < 0$ we need information about the level sets of the approximants $s$. Here, we make use of the estimate for $s_2$ on $\Gamma$, cf. (29). The following theorem shows that we can cover each compact subset $\tilde{K}$ of the basin of attraction by a sublevel set of $s$ and thus the approximation method finds every compact subset of the basin of attraction provided that the sets $\Omega$ and $\Gamma$ are chosen appropriately and the data sites are dense enough.

**Theorem 4.13 ([7])**

1. *Let $\tilde{K}$ be a compact set with $x_0 \in \overset{\circ}{\tilde{K}} \subset \tilde{K} \subset A(x_0)$. Let $s_1$ be an approximation of $V_1$ as in Corollary 4.12 with $\Omega := \{x \in A(x_0) \mid V_1(x) \leq r\} \setminus \{x_0\}$, where $r > 0$ is large enough and $h_X$ is small enough.*

   *Then there is a $\rho \in \mathbb{R}$ with $\tilde{K} \subset \{x \in \Omega \mid s_1(x) \leq \rho\}$.*

2. *Let $\tilde{K}$ be a compact set with $x_0 \in \overset{\circ}{\tilde{K}} \subset \tilde{K} \subset A(x_0)$. Let $s_2$ be an approximation of $V_2$ as in Corollary 4.12 with $\Omega = \{x \in A(x_0) \setminus \{x_0\} \mid V_2(x) \leq r \text{ and } h(x) \geq 0\}$, where $r > 0$ is large enough and $h_{X_1}$ and $h_{X_2}$ are small enough. Let $U = \{x \in A(x_0) \mid h(x) \leq 0\}$ be a neighborhood of $x_0$.*

   *Then there is a $\rho \in \mathbb{R}$ with $\tilde{K} \subset U \cup \{x \in \Omega \mid s_2(x) \leq \rho\}$.*

The proof of 2. compares level sets of $s_2$ with level sets of $V_2$ using the estimate (29) on $\Gamma$ and (28) along solutions. For 1. we can derive an estimate near $x_0$ since $V_1$ is defined and smooth at $x_0$; then we use the estimate (27) along solutions.

## 4.3 Example

As an example we consider the dynamical system given by

$$\left\{ \begin{array}{rcl} \dot{x} & = & -x - 2y + x^3 \\ \dot{y} & = & -y + \frac{1}{2}x^2 y + x^3 \end{array} \right.$$

and denote the right-hand side by $g(x, y)$. The system has an asymptotically stable equilibrium at $(0, 0)$ with Jacobian

$$Dg(0,0) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}.$$

For a local Lyapunov function, cf. Lemma 4.6, we need the unique solution $C$ of the matrix equation

$$Dg(0,0)^T C + C Dg(0,0) = -I,$$

which is given by

$$C = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

The basin of attraction $A(0,0)$ is bounded by an unstable periodic orbit which we have calculated numerically. We approximate the function $V_1$ satisfying $V_1'(x,y) = -x^2 - y^2$. For the data sites, we use a hexagonal grid of the form

$$\alpha \left[ j\,(1,0)^T + k \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right)^T \right].$$

Then, the mesh norm is $h = \alpha$. Since we have to avoid singular points we must exclude the origin. We use three different grids with parameters $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, and $\alpha_3 = 0.4$ and two different Wendland functions as radial basis functions $\Phi(x) = \psi_{k,l}(c\|x\|_2)$ with $c = 2/3$ and $k = 2, 3$, cf. Figure 1 and 2. We calculate the maximal error on the grid

$$0.1 \left[ j\,(1,0)^T + k \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right)^T + \left( \frac{3}{4}, \frac{\sqrt{3}}{4} \right)^T \right].$$

These grid points are inbetween the grid points of the smallest grid above. By our error analysis the errors $e_{k,\alpha}$ and $e_{k,2\alpha}$ should behave like

$$\frac{e_{k,2\alpha}}{e_{k,\alpha}} \approx \frac{(2\alpha)^{k-1/2}}{(\alpha)^{k-1/2}} = 2^{k-1/2},$$

cf. (27), which is approximately reflected in our numerical results, see Table 2.

| $k$ / $\alpha$ | 0.4 | 0.2 | 0.1 | $e_{0.4}/e_{0.2}$ | $e_{0.2}/e_{0.1}$ | $2^{k-1/2}$ |
|---|---|---|---|---|---|---|
| 2 | 0.8862 | 0.4641 | 0.1814 | 1.9094 | 2.5592 | 2.8284 |
| 3 | 1.1308 | 0.4265 | 0.1041 | 2.6516 | 4.0960 | 5.6569 |

Table 2: The approximation error $e_\alpha = \|s_1'(x) - V_1'(x)\|_2$ for different Wendland functions $\psi_{k+2,k}$ and different grids with mesh norm $\alpha$ for the example discussed in this section. The ratio of the errors $e_\alpha$ is compared to the theoretical bound $2^{k-1/2}$ of Corollary 4.12, (27).

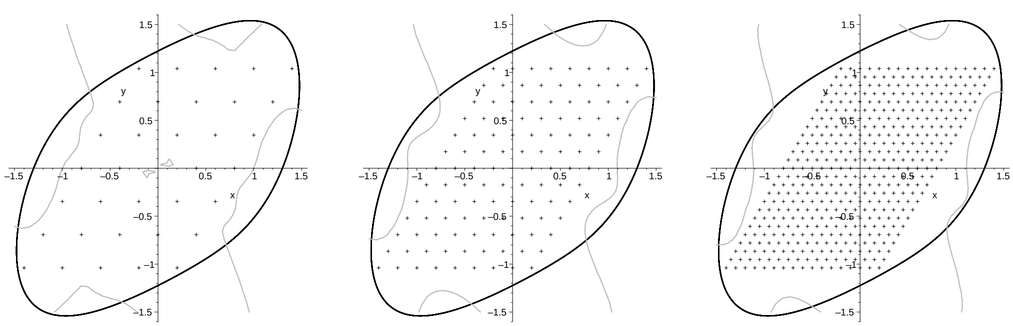

Figure 1: The grid $X_N$ (black $+$), the basin of attraction bounded by the black periodic orbit and the set $\{(x,y) \in \mathbb{R}^2 \mid s'(x,y) = 0\}$ (grey) with the approximation $s$ of the function $V$ where $V'(x,y) = -x^2 - y^2$ with the Wendland function $\psi_{4,2}(2/3\|x\|_2)$ and the grid distance $\alpha$ where left: $\alpha = 0.4$, middle: $\alpha = 0.2$, right: $\alpha = 0.1$.

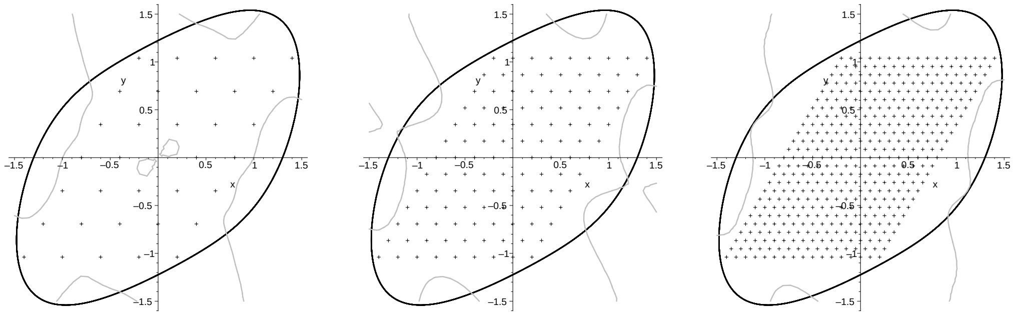


Figure 2: The grid $X_N$ (black $+$), the basin of attraction bounded by the black periodic orbit and the set $\{(x, y) \in \mathbb{R}^2 \mid s'(x, y) = 0\}$ (grey) with the approximation $s$ of the function $V$ where $V'(x, y) = -x^2 - y^2$ with the Wendland function $\psi_{5,3}(2/3\|x\|_2)$ and the grid distance $\alpha$ where left: $\alpha = 0.4$, middle: $\alpha = 0.2$, right: $\alpha = 0.1$.

For the basin of attraction, however, the level sets of $s$ are also important. Even if the set where $s'$ is negative is large, a subset of the basin of attraction is only given by a sublevel set of $s$ within this region. For one example we have calculated such a sublevel set and have compared it to the sublevel set of the local Lyapunov function, see Figure 3. If the function $g$ is bounded in the basin of attraction then one can cover each given compact set in $A(x_0)$ with a sublevel set of $s$ where the data sites are dense enough, see Theorem 4.13.

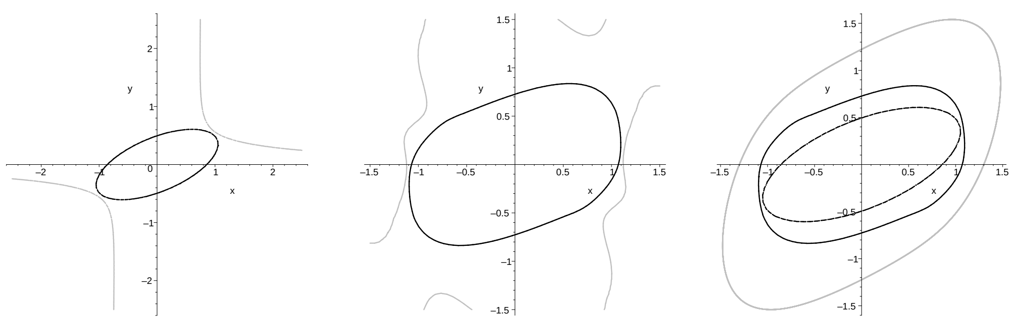


Figure 3: Left: The local Lyapunov function $v(x) = x^T Cx$: level set $v'(x) = 0$ (grey) and a sublevel set $\{x \in \mathbb{R}^2 \mid v(x) \leq 0.37\}$ which is a subset of the basin of attraction. Middle: The calculated Lyapunov function $s$ ($k = 3$, $\alpha = 0.1$): level set $s'(x) = 0$ (grey) and a sublevel set $\{x \in \mathbb{R}^2 \mid s(x) \leq -0.5\}$ which is a subset of the basin of attraction. Right: Comparison of the subsets obtained by the local Lyapunov function $v$ (black small), the calculated Lyapunov function $s$ (black large) and the whole basin of attraction (grey).

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