On the Numerical Solution of 3D Inverse Obstacle Scattering Problems

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Outline

forward scattering problem

Let $K \subset \mathbb{R}^3$: be a compact, smooth, simply connected obstacle and $u_i(x) := \exp(ikx \cdot d)$ an incident wave with wave number $k > 0$ and direction $d \in S^2$.

forward scattering problem: Find the scattered wave u_s such that the total field $u := u_i + u_s$ satisfies:

- the Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^3 \setminus K$$

- the Dirichlet ("sound-soft") boundary condition

$$u = 0 \qquad \text{on } \Gamma := \partial K$$

- the Sommerfeld radiation condition

$$r\left(\frac{\partial u_s}{\partial r} - iku_s\right) \to 0 \qquad r = |x| \to \infty.$$

far field pattern and inverse problem

The Sommerfeld radiation condition implies that u_s has the asymptotic behavior

$$u_s(x) = \frac{e^{i\kappa|x|}}{|x|}\left\{u_\infty\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty.$$

The function $u_\infty : S^2 \to \mathbb{C}$ is called the far field pattern of u_s.

inverse problem: Given u_∞ and u_i, find the obstacle K!

parametrizations

Let Γ_{ref} be a smooth closed reference surface of the same genus as Γ. We parametrize Γ by a C^1-mapping

$$\Psi : \Gamma_{\mathrm{ref}} \to \Gamma$$

which is one-to-one, preserves orientation, and the derivative $D\Psi(x) : T_x \to T_{\Psi(x)}$ is one-to-one for all $x \in \Gamma_{\mathrm{ref}}$.

definition of the forward operator

Let

$$\mathbb{D}(F) := \left\{ \Psi \in H^s(\Gamma_{\mathrm{ref}}; \mathbb{R}^3) : \begin{array}{l} \Psi \text{ one-to-one , orient.-preserving,} \\ \text{and } \det(D\Psi(x)) \neq 0 \text{ for all } x \in \Gamma_{\mathrm{ref}} \end{array} \right\}$$

with $s > 2$ be the set of admissible parametrizations, and let

$$\begin{aligned} F \quad : \quad & \mathbb{D}(F) \to L^2(S^2), \qquad \Psi \mapsto u_\infty \\ & \Psi \to u_\infty \end{aligned}$$

denote the forward solution operator. Then the inverse problem can be formulated as operator equation

$$F(\Psi) = u_\infty.$$

Note that F is not one-to-one since the representation of Γ by Ψ is not unique.

forward operator for star-shaped domains

If K is star-shaped with respect to the origin, there exists a positive function r on S^2 such that

$$\Gamma = \{r(\hat{x})\hat{x} : \hat{x} \in S^2\}.$$

Γ is uniquely represented by r, and we can define the forward operator $F_* : r \mapsto u_\infty$ on the domain

$$\mathbb{D}(F_*) := \{r \in H^s(S^2, \mathbb{R}) : r > 0\}.$$

differentiability of F

Theorem (Hohage 1999): The operator F is analytic even for rough (e.g. fractal) obstacles K.

The derivative $F'[\Psi]\mathbf{V}$ of F at Ψ in direction $\mathbf{V}$ is the far field pattern of a radiating solution u' to the Helmholtz equation. On smooth parts of Γ, u' satisfies the boundary condition

$$u'|_\Gamma = -(\mathbf{V} \cdot \mathbf{n}) \frac{\partial u}{\partial \mathbf{n}}$$

where $\mathbf{n}$ denotes the outward pointing unit normal vector.

see also Simon, Kirsch, Potthast, Hettlich

characterization of the adjoint $F'[\Psi]$

The derivative $F'[\Psi]$ has the factorization $F'[\Psi] = G_\Psi A_\Psi J_s$ where

- $J_s : H^s(\Gamma;\mathbb{R}^3) \hookrightarrow L^2(\Gamma;\mathbb{R}^3)$ is the embedding operator
- $A_\Psi : L^2(\Gamma_{\mathrm{ref}};\mathbb{R}^3) \to L^2(\Gamma;\mathbb{C})$ is defined by

$$(A_\Psi \mathbf{V})(y) := -\frac{\partial u}{\partial \mathbf{n}}(y)\,\mathbf{V}(\Psi^{-1}(y)) \cdot \mathbf{n}(y), \qquad y \in \Gamma.$$

- $G_\Psi : L^2(\Gamma;\mathbb{C}) \to L^2(S^2;\mathbb{C})$ is the operator which maps Dirichlet data $f \in L^2(\Gamma)$ to the far field pattern v_∞ of the radiating solution v to the Helmholtz equation satisfying the Dirichlet condition $v = f$ on Γ.

Hence, $F'[\Psi]^* = J_s^* A_\Psi^* G_\Psi^*$.

characterization of the adjoint $F'[\Psi]$

- Let

$$v_i^g(y) := \frac{1}{4\pi}\int_{S^2} e^{-i\kappa\hat{x}\cdot y} g(\hat{x})\,\mathrm{d}\sigma_{\hat{x}}, \qquad y \in \mathbb{R}^3$$

  denote the Herglotz wave function with kernel $g \in L^2(S^2)$, and let v_g be the total field corresponding to v_i^g as incident field, i.e. $v^g = v_i^g + v_s^g$ where $v = 0$ on Γ and v_s^g is a radiating solution to the Helmholtz equation. Then

$$G_\Psi^* g = \overline{\frac{\partial v^{\overline{g}}}{\partial \mathbf{n}}}.$$

- The adjoint of A_Ψ applied to a function $f \in L^2(\Gamma;\mathbb{C})$ is given by

$$(A_\Psi^* f)(x) = -\Re\left(f(y)\cdot\overline{\frac{\partial u}{\partial \mathbf{n}}(y)}\right)\mathbf{n}(y)\det(D\Psi(x)), \qquad x \in \Gamma_{\mathrm{ref}}$$

  where $y := \Psi(x)$.

Outline

1. operator formulation of the inverse problem
2. numerical solution of the forward problem
3. preconditioned Newton method
4. numerical results

boundary integral operators

Let $E(x,y) = \frac{e^{i\kappa\|x-y\|}}{4\pi\|x-y\|}$ denote the fundamental solution. We define the boundary integral operators

$$\begin{aligned}
(\mathcal{S}\rho)(x) &:= \int_\Gamma E(x,y)\rho(y)\,\mathrm{d}\sigma_y, && x \in \Gamma,\\
(\mathcal{D}'\rho)(x) &:= \int_\Gamma \frac{\partial E(x,y)}{\partial \mathbf{n}(x)}\rho(y)\,\mathrm{d}\sigma_y, && x \in \Gamma,\\
(\mathcal{D}\rho)(x) &:= \int_\Gamma \frac{\partial E(x,y)}{\partial \mathbf{n}(y)}\rho(y)\,\mathrm{d}\sigma_y, && x \in \Gamma.
\end{aligned}$$

evaluation of F

The total field u satisfies

$$u(x) + \int_\Gamma \frac{\partial u(y)}{\partial \mathbf{n}} E(x,y)\,\mathrm{d}\sigma_y = u_i(x), \qquad x \in \mathbb{R}^3 \setminus K.$$

Letting x tend to Γ in this equation and taking the normal derivative, we find that the Neumann-data $\frac{\partial u}{\partial \mathbf{n}}$ of the total field satisfy the integral equation

$$\left(\frac{1}{2}I + \mathcal{D}' - i\eta\mathcal{S}\right)\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial u_i}{\partial \mathbf{n}} - i\eta u_i \quad \text{on } \Gamma$$

for any $\eta > 0$. The far field pattern is given by

$$u_\infty(\hat{x}) = -\frac{1}{4\pi}\int_\Gamma e^{-i\kappa\hat{x}\cdot y}\frac{\partial u}{\partial \mathbf{n}}(y)\,\mathrm{d}\sigma_y, \quad \hat{x} \in \mathbb{S}^2.$$

evalution of $F'[\Psi]$

The ansatz

$$u'(x) = \int_\Gamma \left(\frac{\partial E(x,y)}{\partial \mathbf{n}(y)} - i\eta E(x,y) \right) \rho(y) \, \mathrm{d}\sigma_y \quad x \in \mathbb{R}^3 \setminus \overline{\Omega}.$$

leads to the boundary integral equation

$$\left(\frac{1}{2} I + \mathcal{D} - i\eta S \right) \rho = -(\mathbf{V} \cdot \mathbf{n}) \frac{\partial u}{\partial \mathbf{n}}.$$

Then

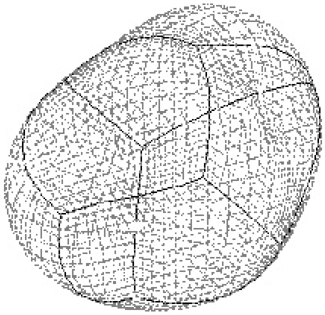
$$(F'[\Psi]\mathbf{V})(\hat{x}) = \frac{1}{4\pi} \int_\Gamma e^{-i\kappa \hat{x} \cdot y} \left(-i\kappa (\mathbf{n}(y) \cdot \hat{x}) - i\eta \right) \rho(y) \, \mathrm{d}\sigma_y, \quad \hat{x} \in \mathbb{S}^2.$$

partitioning of the surface

Let $\square := [0,1]^2$ denote the unit square. We assume that the manifold $\Gamma \subset \mathbb{R}^3$ can be partitioned into a finite number of patches Γ_i

$$\Gamma = \bigcup_{i=1}^{M} \Gamma_i, \qquad \Gamma_i = \gamma_i(\square), \qquad i = 1, 2, \ldots, M,$$

where each $\gamma_i : \square \to \Gamma_i$ defines a diffeomorphism of $\square$ onto Γ_i.



wavelet boundary element method

- Subdivide the unit square into 4^j subsquares and consider the nodal basis $\{\psi_{j,k}\}$ of the space $V_j \subset L^2(\Gamma)$ of piecewise bilinear functions.
- Galerkin approximation of the boundary integral equation $Au = f$: Find $u_j \in V_j$ such that

$$\langle Au_j, v_j\rangle_{L^2(\Gamma)} = \langle f, v_j\rangle_{L^2(\Gamma)} \qquad \text{for all } v_j \in V_j.$$

- The matrix representation w.r.t. the nodal basis is dense (usual BEM). However, the matrix w.r.t. the biorthogonal spline wavelets is quasi-sparse, i.e. it can be approximated by a sparse matrix up to the order of the discretization error.

features, discussion

- The method is 4th order accurate, i.e.
  $\left\| F(\Psi) - F_{h_j}(\Psi) \right\|_{L^2} = \mathcal{O}(h_j^4)$ as $j \to \infty$.
- Compared to multipole or panel clustering methods, the computation of the matrix (which is avoided by those methods) is expensive, but matrix-vector multiplication are then very cheap.
- Hence, wavelet-BEM is particularly attractive in the context of inverse problems where the same integral equation must be solved for many right hand sides in each step of a Newton method.

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ill-posed nonlinear operator equations

We consider a nonlinear ill-posed operator equation

$$F(\Psi) = u_\infty$$

in real Hilbert spaces X and Y. The operator $F : D(F) \to Y$ is assumed to be Fréchet-differentiable on its domain $D(F) \subset X$. We assume that there exists a unique solution $\Psi^\dagger$ to the operator equation for exact data u_∞.

The measured data u_∞^δ are perturbed by noise and satisfy and error bound

$$\|u_\infty^\delta - u_\infty\| \leq \delta$$

for some known noise level $\delta > 0$.

Tikhonov regularized Newton methods

Levenberg-Marquardt algorithm: Compute update $\mathbf{V}_n$ by solving the minimization problem

$$\|F'[\Psi_n^\delta]\mathbf{V} + F(\Psi_n^\delta) - u_\infty^\delta\|_Y^2 + \gamma_n\|\mathbf{V}\|_X^2 = \text{min!} \qquad \mathbf{V} \in X.$$

Iteratively Regularized Gauss-Newton Method (IRGNM): Compute update $\mathbf{V}_n$ by solving the minimization problem

$$\|F'[\Psi_n^\delta]\mathbf{V} + F(\Psi_n^\delta) - u_\infty^\delta\|_Y^2 + \gamma_n\|\mathbf{V} + \Psi_n^\delta - \Psi_0\|_X^2 = \text{min!} \qquad \mathbf{V} \in X.$$

choice of regularization parameters: $\gamma_n = \gamma_0\rho^n$ for some $\rho \in (0, 1)$.

an additional penalty term

Often some parametrizations Ψ of the same obstacle are preferable to others since they lead to better grids. Hence, we introduce an additional penalty term involving an operator $G : \mathbb{D}(F) \to Z$:

$$\begin{aligned} &\|F'[\Psi_n]\mathbf{V} + F(\Psi_n) - u^\delta\|^2 + \alpha_n\|\mathbf{V} + \Psi_n - \Psi_0\|^2 \\ &+\alpha_n\|G'[\Psi_n]\mathbf{V} + G(\Psi_n)\|^2 = \min! \end{aligned}$$

For example,

$$(G(\Psi))(x) := D\Psi(x)^\dagger, \qquad x \in \Gamma_{\mathrm{ref}}.$$

need of a preconditioner

In each step of the IRGNM and the Levenberg-Marquardt algorithm we have to solve a linear operator equation of the form

$$(F'[\Psi_n]^* F'[\Psi_n] + \gamma_n I)\mathbf{V}_n = g_n \tag{1}$$

where the regularization parameter satisfy $\gamma_n \to 0$ as $n \to \infty$. We have

$$\mathrm{cond}(F'[\Psi_n]^* F'[\Psi_n] + \gamma_n I) = \frac{\|F'[\Psi_n]\|^2}{\gamma_n} \to \infty, \qquad n \to \infty.$$

Hence, the number of CG steps to solve (1) will explode as $n \to \infty$.

conjugate gradient method

$$\underbrace{\left(F'[\Psi_m]^* F'[\Psi_m] + \gamma_n I\right)}_{=:A_n} \mathbf{V}_n = g_n, \quad m \le n$$

inner iteration:

$k = 0$; $\mathbf{V}_n^0 = 0$; $r_0 = g_n$;
while $(\|r_k\| > \epsilon\gamma_n\|\mathbf{V}_n^k\|)$
    $k = k + 1$;
    if $(k = 1)$ $p^1 = r^0$;
    else
        $\beta_k = \|r_{k-1}\|^2 / \|r_{k-2}\|^2$;
        $p_k = r_{k-1} + \beta_k p_{k-1}$;
    $\alpha_k = \|r_{k-1}\|^2 / \langle p_k, A_n p_k\rangle$;
    $\mathbf{V}_n^k = \mathbf{V}_n^{k-1} + \alpha_k p_k$;
    $r_k = r_{k-1} - \alpha_k A p_k$;
$\mathbf{V}_n^k \approx \mathbf{V}_n$

Recall that the residuals $r_j = g_n - A_n \mathbf{V}_n^j$ are orthogonal: $\langle r_j, r_l\rangle = 0$ for $j \ne l$

Proposition: After termination of the inner iteration we have

$$\|\mathbf{V}_n^k - \mathbf{V}_n\| \le \frac{\epsilon}{1-\epsilon}\|\mathbf{V}_n\|.$$

Lanczos method

$$A_n R = RB^T B + \mathrm{rem}$$

$$B := \begin{bmatrix} \sqrt{\frac{1}{\alpha_1}} & -\sqrt{\frac{\beta_1}{\alpha_1}} & & \cdots & 0 \\ 0 & \sqrt{\frac{1}{\alpha_2}} & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & -\sqrt{\frac{\beta_{k-2}}{\alpha_{k-2}}} \\ 0 & \cdots & & 0 & \sqrt{\frac{1}{\alpha_{k-1}}} \end{bmatrix}$$

$$R := \left[\frac{r^0}{\| r^0 \|}, \ldots, \frac{r^{k-1}}{\| r^{k-1} \|} \right] \text{ is isometric.}$$

If $\mathrm{rem} = 0$ and $B^T B = VDV^T$ with D diagonal and V orthogonal, then

$$A_n(RV) = (RV)D,$$

i.e. the columns of RV are eigenvectors of A_n, and the diagonal elements of D are eigenvalues.

construction of a preconditioner

Preconditioned equation: $C_n^{-1} A_n C_n^{-1} \tilde{\mathbf{V}}_n = C_n^{-1} g_n$, $\tilde{\mathbf{V}}_n = C_n h_n$. Recall that $A_n := F'[\Psi_m]^* F'[\Psi_m] + \gamma_n I$.

Let $F'[\Psi_m]^* F'[\Psi_m] v_j = \lambda_j v_j$, and assume that the largest K eigenvalues and -vectors are known. Define

$$C_n^{-2} a := a + \sum_{j=1}^{K} \left(\frac{\gamma_n}{\gamma_n + \lambda_j} - 1 \right) \langle a, v_j \rangle \, v_j .$$

Then $C_n^{-1} A_n C_n^{-1} v_k = \gamma_n v_k$ for $k \leq K$ and $C_n^{-1} A_n C_n^{-1} v_k = A_n v_k$ else. Consequently,

$$\sigma(C_n^{-1} A_n C_n^{-1}) = \{\gamma_n\} \cup \{\gamma_n + \lambda_{K+1}, \gamma_n + \lambda_{K+2}, \ldots\}$$

$$\mathrm{cond}(C_n^{-1} A_n C_n^{-1}) = 1 + \frac{\lambda_{K+1}}{\gamma_n};$$

the algorithm

Compute $F(\Psi_0)$;
$n = 0$;
while ($\|F(\Psi_n) - u_\infty^\delta\| \geq \tau\delta$);
    if ($\sqrt{n+1} \in \mathbb{N}$)
        $m = n$;
        Solve $A_n \mathbf{V}_n = g_n$ by CG;
        Compute largest eigenvalues and -vectors of $F'[\Psi_*]^* F'[\Psi_*]$;
    else
        Solve $C_n^{-1} A_n C_n^{-1} (C_n \mathbf{V}_n) = C_n^{-1} g_n$ by CG;
    $\Psi_{n+1} = \Psi_n + \mathbf{V}_n$;
    $n = n + 1$;
    Compute $F(\Psi_n)$;
end!!!

Outline

6 incident waves, wave number k=1

original

reconstruction

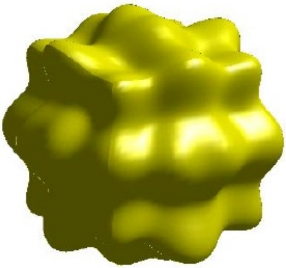
noise level: 5 %

geometry represented by spherical harmonics of degree ≤ 20

6 incident waves, wave number k=2

original

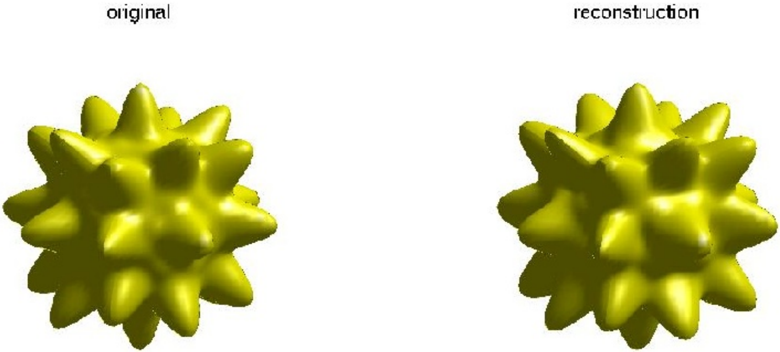
reconstruction



noise level: 5 %

geometry represented by spherical harmonics of degree ≤ 20

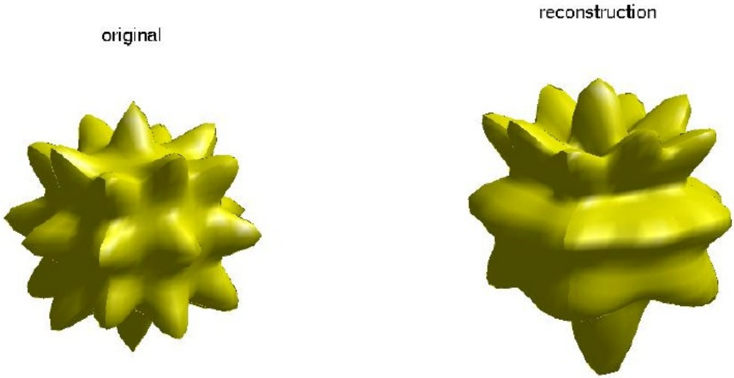
6 incident waves, wave number k=4



noise level: 5 %

geometry represented by spherical harmonics of degree ≤ 20

1 incident wave from top, wave number k=4



noise level: 2 %

geometry represented by spherical harmonics of degree ≤ 20

parameter values:
6 incident waves
wave number k=8
1% noise
spherical harmonics of degree ≤ 20

top: reconstruction
bottom: original