On radiation conditions for rough surface scattering problems

Tilo Arens and Thorsten Hohage

Abstract

It is well known that Sommerfeld’s radiation condition is not a valid characterization of outgoing waves for scattering problems at rough surfaces. Instead, a radiation condition called upward propagating radiation condition (UPRC) is commonly used. Recently a different radiation condition called pole condition has been investigated for scattering problems at bounded obstacles. In this paper we show the equivalence between the UPRC and the pole condition. In doing so, we give a rigorous interpretation of a formula called angular spectrum representation for Dirichlet data in the space of bounded continuous functions.

1 Introduction

In this paper we consider scattering problems in a non-locally perturbed half plane. More specifically, let the scattered field $u$ be a solution to the Helmholtz equation

$$\Delta u + k^2 u = 0$$  \hspace{1cm} (1.1)

in the domain $\Omega := \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_2 > f(x_1)\}$ where $f$ is a smooth and bounded function on $\mathbb{R}$. Throughout the paper, we will suppose that $f \in C^2(\mathbb{R})$ and $\sup_{t \in \mathbb{R}} f(t) < 0$. The scattered field $u$ will be generated by some incident field $u^i$ and some boundary condition is imposed on the surface given by the graph of $f$ coupling the two fields.

The formulation of this problem is not complete without a suitable radiation condition. Physically, such a condition ensures that the scattered field is propagating away from the obstacle while mathematically it ensures
uniqueness of the solution, i.e. well posedness of the problem. The simple example of \( f \) being a constant function and \( u \) a downward propagating plane wave shows that the Sommerfeld radiation condition is not appropriate for such problems: in this case, \( u \) is simply some reflected plane wave and hence satisfies Sommerfeld’s radiation condition
\[
\lim_{r \to \infty} r^{1/2} \left( \frac{\partial u}{\partial r} - i k u \right) = 0, \quad r = |x|,
\]
only in the propagation direction, but no other direction.

A natural approach to obtain a radiation condition, often used in applications, is to consider the field \( u \) only in the upper half plane \( U := \{ x \in \mathbb{R}^2 : x_2 > 0 \} \) and to compute its Fourier transform \((\mathcal{F} u)(\xi, x_2) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\xi x_1} u(x_1, x_2) dx_1\) with respect to the \( x_1 \) variable. This leads to an ordinary differential equation
\[
\frac{\partial^2 \mathcal{F} u}{\partial x_2^2}(\xi, x_2) + (k^2 - \xi^2) \mathcal{F} u(\xi, x_2) = 0, \quad \xi \in \mathbb{R}, \quad (1.2)
\]
with some initial condition, e.g. \((\mathcal{F} u)(\xi, 0) = (\mathcal{F} u_{\text{Dir}})(\xi)\) for \( u(x_1, 0) = u_{\text{Dir}}(x_1), \ x_1 \in \mathbb{R} \). Selecting the outgoing of the two linearly independent solutions, we arrive at \((\mathcal{F} u)(\xi, x_2) = \exp(i x_2 \sqrt{k^2 - \xi^2}) (\mathcal{F} u_{\text{Dir}})(\xi)\), or
\[
u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(ix_1 \xi + ix_2 \sqrt{k^2 - \xi^2}) (\mathcal{F} u_{\text{Dir}})(\xi) \ d\xi. \quad (1.3)
\]
Here, and throughout the paper, the branch cut of the complex square root function is chosen such that \( \text{Im}\sqrt{z} \geq 0, \ z \in \mathbb{C} \).

Equation (1.3) is often referred to as an angular spectrum representation in the literature (see [9]). This representation is also often used as a radiation condition (see [8, 10, 11] and the references contained therein). It shows that \( u \) is a linear superposition of the upward propagating plane waves \( \exp(ix_1 \xi + ix_2 \sqrt{k^2 - \xi^2}) \) for \( |\xi| \leq k \) and evanescent surface waves \( \exp(ix_1 \xi - x_2 \sqrt{x_1^2 - k^2}) \) for \( |\xi| > k \). The formalism can easily be made mathematically rigorous if \( \mathcal{F} u_{\text{Dir}} \in L^2(\mathbb{R}) \). However, again the simple example of plane wave scattering shows that this not necessarily the case. For \( u_{\text{Dir}} \in BC(\mathbb{R}) \), the space of bounded and continuous function on \( \mathbb{R} \), \( \mathcal{F} u_{\text{Dir}} \) is only a distribution in general, and it is not obvious how to interpret the formula (1.3). In the references cited above, the authors remain somewhat sketchy on the question of the exact nature of \( \mathcal{F} u_{\text{Dir}} \) in this case.
A different, and mathematically rigorous, approach is the so-called upward propagating radiation condition (UPRC). It was introduced by Chandler-Wilde in [3] and further analyzed in [6, 7] (see also [1] for an extension to elastic wave scattering problems). The UPRC requires that the scattered field \( u \) satisfy the representation formula

\[
    u(x) = 2 \int_{-\infty}^{\infty} \frac{\partial \Phi(x, (y_1, 0))}{\partial y_2} u_{\text{Dir}}(y_1) \, dy_1, \quad x \in U, \tag{1.4}
\]

where

\[
    \Phi(x, y) := \frac{i}{4} H_0^{(1)}(k |x - y|), \quad x, y \in \mathbb{R}^2, x \neq y,
\]

denotes the fundamental solution of the Helmholtz equation and \( H_0^{(1)} \) the Hankel function of the first kind and of order 0.

The UPRC has been used successfully to establish uniqueness of solution for rough surface scattering problems with a Dirichlet or a Robin type boundary condition on the scattering surface [1,5,7] and to rigorously derive boundary integral equation formulations of these problems [2,18]. It is also straightforward to see that it is equivalent to the angular spectrum representation (1.3) if \( u_{\text{Dir}} \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \). On the other hand, it was shown in [3] that if \( f \) is an \( L \)-periodic function, the UPRC is equivalent to the condition that \( u \) is expansible in a series of upward propagating plane waves and evanescent waves in \( U \) if we additionally know that \( u \) is \( \alpha \)-quasiperiodic for some \( \alpha > 0 \), i.e.

\[
    u(x_1 + L, x_2) = \exp(i\alpha L) u(x_1, x_2), \quad x \in U. \tag{1.5}
\]

That (1.5) holds is a usual and appropriate assumption in the diffraction grating case that \( f \) is \( L \)-periodic and the incident field is a plane wave \( u_i(x_1, x_2) = \exp(i\alpha x_1 + i\beta x_2) \) with \( \alpha^2 + \beta^2 = k^2, \beta < 0 \) (see, e.g., [16]).

A third type of radiation condition recently developed in the context of scattering by bounded obstacles has been called the pole condition and is based on the Laplace transform of the solution in a radial direction. It was first suggested by Schmidt and Deuflhard for time dependent, spatially one-dimensional problems in [17]. Schmidt, Zschiedrich and the second author have shown the equivalence of the pole condition and the Sommerfeld radiation condition for time-harmonic scattering problems with bounded obstacles in \( \mathbb{R}^d \) and radially symmetric potentials [13]. The pole condition leads to new
numerical algorithms which do not rely on the explicit knowledge of a fundamental solution (cf. [12]). A close relationship to the Perfectly Matched Layer (PML) method was pointed out in [14].

In this paper, we want to extend the pole condition method to rough surface scattering problems. We will then explore the connections between the three different types of radiation conditions, clarifying some theoretical questions with regard to the angular spectrum representation. Moreover, we will prove in Theorem 6 that the pole condition is equivalent to the UPRC.

2 The pole condition

To motivate the following definition, we again look at the ordinary differential equation (1.2). The solutions to this equation which are either outgoing or exponentially decreasing are given by \( \exp(i x_2 \sqrt{k^2 - \xi^2}) \), whereas the incoming or exponentially growing solutions are \( \exp(-i x_2 \sqrt{k^2 - \xi^2}) \). Applying the Laplace transform with respect to \( x_2 \) to these functions, we observe that the transforms of the first kind of solutions take the form \( \frac{1}{s - i \sqrt{k^2 - \xi^2}} \) and hence have poles on the positive imaginary axis or the negative real axis, whereas the transforms of the second kind of solutions are \( \frac{1}{s + i \sqrt{k^2 - \xi^2}} \) which have poles on the negative imaginary axis or the positive real axis.

**Definition 1** A function \( u \in L^\infty(\overline{U}) \), \( U := \mathbb{R} \times (0, \infty) \) with

\[
\sup\{|u(x_1, x_2)|(1 + x_2)\beta : x \in \overline{U}\} < \infty \quad \text{for some } \beta \in \mathbb{R}
\]  

(2.1)

satisfies the pole condition if the Laplace transform

\[
(\mathcal{L}u)(x_1, s) := \int_0^\infty e^{-sx_2}u(x_1, x_2) \, dx_2, \quad \Re s > 0, x_1 \in \mathbb{R}
\]  

(2.2)

has an extension to a holomorphic function \( s \mapsto (\mathcal{L}u)(\cdot, s) \) defined on \( D := \{s \in \mathbb{C} : \Im s < 0 \text{ or } \Re s > 0\} \) with values in \( L^\infty(\mathbb{R}) \).

Note that this definition does not involve the Fourier transform with respect to the \( x_1 \)-variable. Also note that condition (2.1) ensures that the Laplace transform (2.2) is well-defined for \( \Re s > 0 \).

We will repeatedly be concerned with Fourier transforms of functions \( \phi \in L^\infty(\mathbb{R}) \). Since \( \phi \) can be interpreted as a tempered distribution, \( \phi \in \mathcal{S}' \),
it follows that $\mathcal{F}\phi \in S'$. Although the following result giving a more precise characterization of $\mathcal{F}\phi$ is well-known (cf. [15, Theorem 7.9.3] for the first part), we reproduce the short proof for the convenience of the reader. We use the Sobolev norms $\|\varphi\|_{H^s} := (\int_{\mathbb{R}^d}(1 + |\xi|^2)^s|\mathcal{F}\varphi(\xi)|^2\,d\xi)^{1/2}$ for $s \in \mathbb{R}$.

**Lemma 2**
1. For all $\sigma > d/2$, $\mathcal{F}(L^\infty(\mathbb{R}^d)) \subset H^{-\sigma}(\mathbb{R}^d)$, and $\|\mathcal{F}\phi\|_{H^{-\sigma}} \leq C\|\phi\|_{L^\infty}$ for all $\phi \in L^\infty(\mathbb{R}^d)$ and some constant $C$ depending only on $\sigma$ and $d$.

2. If $\chi_R$ is the characteristic function of $\{x \in \mathbb{R}^d : |x| \leq R\}$, then

$$\lim_{R \to \infty} \|\mathcal{F}(\chi_R\phi) - \mathcal{F}\phi\|_{H^{-\sigma}} = 0$$

for all $\phi \in L^\infty(\mathbb{R}^d)$ and $\sigma > d/2$.

**Proof.** Let $\phi \in L^\infty(\mathbb{R}^d)$. By the definition of the Fourier transform on $S'$, $\langle \mathcal{F}\phi, \psi \rangle = \langle \phi, \mathcal{F}\psi \rangle$ for all test functions $\psi \in S$. Note that $x \mapsto (1 + |x|^2)^{-\sigma/2}$ belongs to $L^2(\mathbb{R}^d)$ if and only if $\sigma > d/2$. In this case,

$$|\langle \mathcal{F}\phi, \psi \rangle| = \left| \langle (1 + |\cdot|^2)^{-\sigma/2}\phi, (1 + |\cdot|^2)^{\sigma/2}\mathcal{F}\psi \rangle \right| \\ \\ \leq \left( \int_{\mathbb{R}^d} (1 + |x|^2)^{-\sigma} \,dx \right)^{1/2} \|\phi\|_\infty \|\psi\|_{H^s}$$

for all $\psi \in S$. This implies that $\mathcal{F}\phi : S \to \mathbb{C}$ can be extended to a bounded linear functional $\mathcal{F}\phi : H^s(\mathbb{R}^d) \to \mathbb{C}$, i.e. $\mathcal{F}\phi \in H^s(\mathbb{R}^d)' = H^{-s}(\mathbb{R}^d)$.

The second statement follows from the fact that $|\langle \mathcal{F}(\chi_R\phi - \phi), \psi \rangle| \leq \sqrt{c_R} \|\phi\|_\infty \|\psi\|_{H^s}$ and that $c_R := \int_{\{|x| > R\}} (1 + |x|^2)^{-\sigma} \,dx$ tends to 0 as $R \to \infty$. 

**Theorem 3 (Uniqueness)** If $u \in C^2(\overline{U})$ satisfies the pole condition and the Helmholtz equation (1.1) and if

$$u(x_1, 0) = 0 \quad \text{for all } x_1 \in \mathbb{R} \quad (2.3)$$

then there exist $C_1, C_2 \in \mathbb{C}$ such that

$$u(x) = \{C_1 \exp(ikx_1) + C_2 \exp(-ikx_1)\} x_2 \quad (2.4)$$

for $x \in \overline{U}$. If $u$ additionally satisfies (2.1) with $\beta > -1$, then $u = 0$. 

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Proof. Due to (2.3), $u$ can be extended to a function $u \in C^2(\mathbb{R})$ by $u(x_1, -x_2) := -u(x_1, x_2)$, and the extended function satisfies the Helmholtz equation everywhere. By interior elliptic regularity estimates, the growth condition (2.1) for $u$ implies the same growth conditions for all partial derivatives $\frac{\partial^\alpha u}{\partial x^\alpha}$ (here $\alpha$ denotes a multi-index) of $u$:

$$\sup \left\{ \left| \frac{\partial^\alpha u}{\partial x^\alpha}(x_1, x_2) \right| (1 + x_2)^\beta : x \in \mathcal{U} \right\} < \infty \quad (2.5)$$

As a consequence, the Laplace transform of partial derivatives of $u$ is well defined, and $\frac{\partial^2}{\partial x_1^2}(\mathcal{L}u) = \mathcal{L}\frac{\partial^\alpha u}{\partial x^\alpha}$. Integrating by parts twice and using (2.3) we obtain that $(\mathcal{L}\frac{\partial^\alpha u}{\partial x^\alpha})(s, x_1) = s^2(\mathcal{L}u)(x_1, s) - u_{Neu}(x_1)$ with the Neumann data $u_{Neu}(x_1) := \frac{\partial u}{\partial x_2}(x_1, 0)$, $x_1 \in \mathbb{R}$. Hence, the Helmholtz equation (1.1) implies

$$\frac{\partial^2}{\partial x_1^2}(\mathcal{L}u)(x_1, s) + s^2(\mathcal{L}u)(x_1, s) + k^2(\mathcal{L}u)(x_1, s) = u_{Neu}(x_1)$$

for $x_1 \in \mathbb{R}$ and Re $s > 0$. Taking the Fourier transform with respect to $x_1$ gives

$$\langle (\mathcal{F}\mathcal{L}u)(\cdot, s), (s^2 + k^2 - \cdot^2)\psi \rangle = \langle \mathcal{F}u_{Neu}, \psi \rangle \quad \text{for all } \psi \in S.$$
Note that \((s^2 + k^2 - \lambda^2)\psi \in S\) for all \(s \in \mathbb{C}\). By virtue of (2.1), (2.5), and Lemma 2, we have \(\langle \mathcal{F}L \psi, s \rangle, \mathcal{F}u_{\text{Neu}} \in H^{-\sigma}(\mathbb{R})\) for \(\sigma > \frac{1}{2}\) and \(\text{Re } s > 0\). Since the mapping \(\phi \mapsto \frac{\phi}{s^2 + k^2 - \lambda^2}\) is bounded and boundedly invertible on \(H^\sigma(\mathbb{R})\) for all \(s \in \mathbb{C} \setminus (\mathbb{R} \cup [-ik, ik])\), it follows that

\[
\langle \mathcal{F}L \psi, s \rangle = \mathcal{F}u_{\text{Neu}}, \quad \text{for all } \phi \in H^\sigma(\mathbb{R})
\]  

(2.6)

and \(s \in \{z \in \mathbb{C} : \text{Re } z > 0, \text{Im } z \neq 0\}\). The right-hand side of this equation is a holomorphic function on \(\mathbb{C} \setminus (\mathbb{R} \cup [-ik, ik])\) since by Taylor expansion the mapping \(\mathbb{C} \setminus (\mathbb{R} \cup [-ik, ik]) \to H^\sigma(\mathbb{R})\), \(s \mapsto \frac{\phi}{s^2 + k^2 - \lambda^2}\) is holomorphic (cf. Lemma 7 for an analogous, but slightly more complicated argument). As the left-hand side of eq. (2.6) is a holomorphic function in \(s\) on \(D\) by virtue of the pole condition, it follows that the right-hand side has a holomorphic extension to \(D\). Therefore, by Cauchy’s Integral Theorem

\[
\int_\gamma \langle \mathcal{F}u_{\text{Neu}}, \frac{\phi}{s^2 + k^2 - \lambda^2} \rangle \, ds = 0, \quad \phi \in H^\sigma(\mathbb{R})
\]  

(2.7)

for all closed paths in \(D\). We approximate the distribution \(\mathcal{F}u_{\text{Neu}}\) by the smooth functions \(\mathcal{F}(\chi_R u_{\text{Neu}})\). Moreover, we choose some closed interval \([a, b] \subset \mathbb{R}\) not containing the points \(k\) and \(-k\) and consider test functions \(\phi \in H^\sigma(\mathbb{R})\) with \(\text{supp } \phi \subset [a, b]\). Finally, we choose \(\gamma\) as indicated in Figure 1. Interchanging the order of integration and using the residue theorem gives

\[
\int_\gamma \langle \mathcal{F}(\chi_R u_{\text{Neu}}), \frac{\phi}{s^2 + k^2 - \lambda^2} \rangle \, ds
\]

\[
= \int_a^b \langle \mathcal{F}(\chi_R u_{\text{Neu}})(\xi), \int_\gamma \frac{\phi(\xi)}{s^2 + k^2 - \xi^2} \, ds \, d\xi
\]

\[
= \int_a^b \langle \mathcal{F}(\chi_R u_{\text{Neu}})(\xi), \frac{\phi(\xi)}{-2i\sqrt{k^2 - \xi^2}} \, d\xi.
\]  

(2.8)

We now consider the limit \(R \to \infty\) in this equation. Due to our assumption on \(\text{supp } \phi\), the mapping \(s \mapsto \frac{\phi}{s^2 + k^2 - \lambda^2}\) is well-defined and continuous (even holomorphic) from \(\mathbb{C} \setminus \{\pm i\sqrt{k^2 - \xi^2} : \xi \in [a, b]\}\) to \(H^\sigma(\mathbb{R})\). Using Lemma 2 and \(u_{\text{Neu}} \in L^\infty\), it follows that for all \(s \in \gamma\) the integrand on the left-hand side of eq. (2.8) converges to \(\langle \mathcal{F}u_{\text{Neu}}, \frac{\phi}{s^2 + k^2 - \lambda^2} \rangle\). Since \(\gamma\) is compact, \(\text{sup} \,_{s \in \gamma} \|\frac{\phi}{s^2 + k^2 - \lambda^2}\|_{H^\sigma} < \infty\), and hence \(\|\langle \mathcal{F}u_{\text{Neu}}, \frac{\phi}{s^2 + k^2 - \lambda^2} \rangle\| < C\|u_{\text{Neu}}\|_{\infty}\) for all
Since the mapping \( \phi \mapsto \frac{\phi}{-2\sqrt{k^2 - \cdot^2}} \) is an isomorphism of \( H_0^s([a,b]) \) onto itself, the restriction of the distribution \( \mathcal{F}u_{\text{Neu}} \) to \([a,b]\) vanishes. As \([a,b]\) was an arbitrary interval not containing \( \pm k \), we have shown that \( \text{supp}(\mathcal{F}u_{\text{Neu}}) \subset \{-k,k\} \). Therefore, \( \mathcal{F}u_{\text{Neu}} \) must be a finite sum of derivatives of delta-distributions at \( \pm k \) (cf. [15, Theorem 2.3.4]). As \( \mathcal{F}u_{\text{Neu}} \in H^{-1}(\mathbb{R}) \), we get

\[
\mathcal{F}u_{\text{Neu}} = C_1\delta_k + C_2\delta_{-k}
\]

for some constants \( C_1, C_2 \in \mathbb{C} \), and hence \( u_{\text{Neu}}(x_1) = C_1e^{ikx_1} + C_2e^{-ikx_1} \). Since by the Cauchy-Kowalesvskaia Theorem \( u \) is uniquely determined by its Cauchy data \( u(\cdot,0) \) and \( u_{\text{Neu}} \), this implies (2.4).

The last statement follows from the fact that the right hand side of (2.4) satisfies (2.1) with \( \beta > -1 \) if and only if \( C_1 = C_2 = 0 \). \( \blacksquare \)

3 The angular-spectrum representation formula

We start from the upward propagating radiation condition (1.4). Since the integral on the right hand side exists as an improper integral for \( u_{\text{Dir}} \in BC(\mathbb{R}) \), eq. (1.4) can be written equivalently as

\[
u(x_1,h) = \lim_{R \to \infty} \left( \chi_R u_{\text{Dir}}, \frac{K_h(\cdot - x_1)}{L^2} \right)
\]

with the convolution kernel

\[
K_h(x_1) := \frac{ikh}{2} \frac{H_1^{(1)}(k\sqrt{x_1^2 + h^2})}{\sqrt{x_1^2 + h^2}}, \quad h > 0.
\]
Here \( (f, g)_{L^2} := \int_{\mathbb{R}} f \overline{g} \, dx \) denotes the usual inner product of \( L^2(\mathbb{R}) \). By the unitarity of the Fourier transform we obtain
\[
u(x_1, h) = \lim_{R \to \infty} \left( \mathcal{F}(\chi_R u_{\text{Dir}}), \mathcal{F}K_h(\cdot - x_1) \right)_{L^2}.
\] (3.1)

From tables of Fourier transforms we find
\[
(\mathcal{F}K_h)(\xi) = \frac{1}{\sqrt{2\pi}} \exp \left( i h \sqrt{k^2 - \xi^2} \right),
\] i.e. \( (\mathcal{F}K_h(x_1 - \cdot))(\xi) = \frac{1}{\sqrt{2\pi}} \exp \left( i x_1 \xi + i h \sqrt{k^2 - \xi^2} \right) \). Hence, (3.1) coincides with (1.3) for \( u_{\text{Dir}} \in L^2(\mathbb{R}) \). We know from Lemma 2 that \( \mathcal{F}(\chi_R u_{\text{Dir}}) \) converges to \( \mathcal{F}u_{\text{Dir}} \) in \( H^{-\sigma}(\mathbb{R}) \) as \( R \to \infty \) for \( \sigma > \frac{1}{2} \). Therefore, if we can show that \( \mathcal{F}K_h \in H^{\sigma}(\mathbb{R}) \), then we can go to the limit \( R \to \infty \) in (3.1) after replacing the inner product \( (\cdot, \cdot)_{L^2} \) by the dual pairing \( \langle \cdot, \cdot \rangle \) of \( H^{-\sigma} \) and \( H^{\sigma} \).

**Lemma 4** The functions \( \mathcal{F}K_h \) belong to the Sobolev space \( H^{\sigma}(\mathbb{R}) \) for \( \sigma < 1 \), and there exists a constant \( C > 0 \) depending only on \( \sigma \) and \( k \) such that \( \| \mathcal{F}K_h \|_{H^{\sigma}} \leq C(h^{-1/2} + h) \) for all \( h > 0 \).

**Proof.** The idea of the proof is to subtract from \( \mathcal{F}K_h \) a function with the same singularity and explicitly known Fourier transform. Hence, we consider the function
\[
f(\xi) := \begin{cases} \sqrt{\xi} e^{-\xi}, & \xi \geq 0, \\ 0, & \xi < 0 \end{cases}
\]
with Fourier transform
\[
(\mathcal{F}f)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(1+i\xi)x} \sqrt{\xi} \, d\xi = \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{2\pi}} (1 + ix)^{-3/2}.
\]
It satisfies
\[
\|f\|_{H^{\sigma}}^2 = \int_0^\infty (1 + |x|^2)^{\sigma} |(\mathcal{F}f)(x)|^2 \, dx < \infty
\]
for \( \sigma < 1 \). An elementary computation shows that the function
\[
\varphi_h(\xi) := \sqrt{2\pi}(\mathcal{F}K_h)(\xi) + \sqrt{2kh}f(\xi - k) - i\sqrt{2kh}f(k - \xi) - i\sqrt{2kh}f(\xi + k) + \sqrt{2kh}f(k + \xi)
\]
belongs to the Hölder space $C^{1,1/2}(\mathbb{R})$. Moreover, both $\varphi_h(\xi)$ and $\varphi'_h(\xi)$ decay exponentially as $|\xi| \to \infty$, and there exists a (generic) constant $C > 0$ such that $\|\varphi_h\|_{L^2} \leq C(h^{-1/2} + 1)$ and $\|\varphi'_h\|_{L^2} \leq C(h + 1)$. Hence,

$$\sqrt{2\pi}\|\mathcal{F}K_h\|_{H^\sigma} \leq \|\varphi_h\|_{L^2} + \|\varphi'_h\|_{L^2} + 4\sqrt{2kh}\|f\|_{H^\sigma} \leq C(h^{-1/2} + h).$$

Let us summarize the results of this section:

**Proposition 5** If $u$ satisfies the UPRC (1.4) with $u_{\text{Dir}} \in BC(\mathbb{R})$, then

$$u(x) = \frac{1}{\sqrt{2\pi}} \langle \mathcal{F}u_{\text{Dir}}, \exp(iz_1 \cdot +iz_2\sqrt{k^2 - z^2}) \rangle$$

for $x \in \mathbb{R} \times (0, \infty)$

where $\langle \cdot, \cdot \rangle$ denotes the bilinear dual pairing of $H^{-\sigma}$ and $H^\sigma$ with $\frac{1}{2} < \sigma < 1$.

4 Equivalence of the pole condition and the upward propagating radiation condition

**Theorem 6** Let $u \in C^2(\overline{U})$ satisfy (1.1) and (2.1) with $\beta = -\frac{1}{2}$. Then $u$ satisfies the upward propagating radiation condition (1.4) if and only if it satisfies the pole condition.

**Proof.** We first assume that $u$ satisfies the UPRC (1.4). Then the angular-spectrum representation formula of Proposition 5 holds true. Together with Lemma 2 we obtain

$$(Lu)(x_1, s) = \int_0^\infty e^{-sh} \lim_{R \to \infty} \int_{-\infty}^\infty (\mathcal{F} \chi_R u_{\text{Dir}})(\xi)
\times \exp \left( ix_1 \xi + ih\sqrt{k^2 - \xi^2} \right) d\xi dh.$$ 

By the Lebesgue’s Dominated Convergence Theorem, we may write the limit outside of the first integral since the integrands converge pointwise, and by virtue of Lemma 2 and 4 they are pointwise bounded by the integrable function $h \mapsto Ce^{-sh}(h^{-1/2} + h)\|u_{\text{Dir}}\|_\infty$ for $\Re s > 0$. Then we interchange the
order of integration, which is justified by Fubini’s Theorem. This gives

\[(Lu)(x_1, s) = \lim_{\mathcal{H} \to \infty} \int_{-\infty}^{\infty} (\mathcal{F} \chi_{\mathcal{H}} u_{\text{Dir}})(\xi) \frac{\exp(ix_1 \xi)}{\sqrt{2\pi}} \int_0^\infty e^{-h(s-i\sqrt{k^2-\xi^2})} dh \, d\xi \]

\[= \frac{1}{\sqrt{2\pi}} \left( \mathcal{F} u_{\text{Dir}}(\cdot - x_1), \frac{1}{s - i\sqrt{k^2 - \xi^2}} \right) \]

for Re \(s > 0\). By Lemma 2, there exists a constant \(C_\sigma > 0\) for all \(\sigma > \frac{1}{2}\) such that \(\|\mathcal{F} u_{\text{Dir}}(\cdot - x_1)\|_{H^{-\sigma}} \leq C_\sigma \|u_{\text{Dir}}\|_\infty\) for all \(x_1 \in \mathbb{R}\). It follows from Lemma 7 below that the mapping \(D \to L^\infty(\mathbb{R}), s \mapsto (Lu)(\cdot, s)\) defined by the right hand side of the last equation is holomorphic for \(\frac{1}{2} < \sigma < 1\).

Assume vice versa that \(u\) satisfies the pole condition, the boundedness condition (2.1) with \(\beta = -\frac{1}{2}\), and the Helmholtz equation (1.1). Define

\[\tilde{u}(x) := 2 \int_{\mathbb{R} \times \{0\}} \frac{\partial\Phi(x, y)}{\partial n(y)} u(y) \, dy \quad \text{for } x \in \mathbb{R} \times (0, \infty).\]

Then \(\tilde{u}\) satisfies (2.1) with \(\beta = -\frac{1}{2}\), the Helmholtz equation (1.1), and the boundary condition \(\lim_{x_2 \to 0} \tilde{u}(x_2) = u(x_1, 0)\) (cf. [4, Theorem 3.2]). By the first part of the proof, \(\tilde{u}\) also satisfies the pole condition. Therefore, applying Theorem 3 to \(\tilde{u} - u\) shows that \(u = \tilde{u}\), i.e. that \(u\) satisfies (1.4).

**Lemma 7** For \(\sigma < 1\), the mapping \(D \to H^\sigma(\mathbb{R}), s \mapsto \frac{1}{s - i\sqrt{k^2 - \xi^2}}\) is holomorphic.

**Proof.** For every \(\xi \in \mathbb{R}, \ s \in D, \ \text{and } n \in \mathbb{N}\) we have

\[\frac{\partial^n}{\partial s^n} \left( \frac{1}{s - i\sqrt{k^2 - \xi^2}} \right) = \frac{(-1)^n n!}{(s - i\sqrt{k^2 - \xi^2})^{n+1}}.\]

It can be shown as in the proof of Lemma 4 that \((s - i\sqrt{k^2 - \xi^2})^{-n-1} \in H^\sigma(\mathbb{R})\) for all \(s \in D\) and \(n \in \mathbb{N}\) and that the mapping \(s \mapsto (s - i\sqrt{k^2 - \xi^2})^{-n-1}\) is continuous from \(D\) to \(H^\sigma(\mathbb{R})\). Therefore, the integrand in the Taylor formula

\[\frac{1}{s + \Delta s - i\sqrt{k^2 - \xi^2}} - \frac{1}{s - i\sqrt{k^2 - \xi^2}} + \frac{1}{(s - i\sqrt{k^2 - \xi^2})^2} \Delta s\]

\[= (\Delta s)^2 \int_0^1 (1-t) \frac{1}{(s + t\Delta s - i\sqrt{k^2 - \xi^2})^3} \, dt\]

is uniformly bounded for \(|\Delta s| < \frac{1}{2} \text{dist}(s, \partial D)\) as a function of \(\xi\) with respect to the norm \(\| \cdot \|_{H^\sigma}\). This implies the assertion.

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References


