Nonlocal impedance conditions in direct and inverse obstacle scattering

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Dedicated to the memory of Armin Lechleiter

Abstract

We discuss the use of nonlocal impedance conditions within the use of boundary integral equations for the solution of direct and inverse obstacle scattering problems for penetrable obstacles with constant index of refraction. In the first part, for the classical transmission problem we present an approach that leads to a two-by-two system of nonlinear integral equations in the spirit of the method initiated by Kress and Rundell in 2005 rather than the three-by-three system arising from the traditional boundary integral equation approach to the transmission conditions. In the second part we survey on recent work of Cakoni and Kress from 2017 on the use of boundary integral equations for the characterization and numerical computation of transmission eigenvalues. In particular, we modify and simplify the analysis by the use of a nonlocal rather than a local impedance condition as in the 2017 paper.

1 Introduction

Scattering from a sound-soft or a sound-hard obstacle provides the two basic classical scattering problems for time-harmonic acoustic waves. Assume an impenetrable obstacle described by a bounded domain $D \in \mathbb{R}^3$ with a connected C^2 smooth boundary ∂D and an incident field given by the plane wave

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$U^i(x,t) = \text{Re } e^{i(k\,x\cdot d - \omega t)}$ where $k = \omega/c$ is the wave number, ω the frequency, c the speed of sound, and

$$d \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$$

the direction of propagation. Then, setting $u^i(x) := e^{ik\,x\cdot d}$, the two basic problems are to find the total field $u = u^i + u^s \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \bar{D})$ as a solution to the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{1.1}$$

such that the scattered wave u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \quad r = |x|, \tag{1.2}$$

uniformly for all directions. Solutions to the Helmholtz equation satisfying the Sommerfeld radiation condition are called radiating solutions. The boundary conditions are given by the homogeneous Dirichlet condition

$$u = 0 \quad \text{on } \partial D \tag{1.3}$$

in the sound-soft case and the homogeneous Neumann condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \tag{1.4}$$

in the sound-hard case, with ν the unit normal to ∂D directed into $\mathbb{R}^3 \setminus D$.

After renaming the unknown functions the two scattering problems are special cases of the following exterior Dirichlet and Neumann boundary value problems: Find a radiating solution $v \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \bar{D})$ to the Helmholtz equation satisfying

$$v = f \quad \text{on } \partial D \tag{1.5}$$

or

$$\frac{\partial v}{\partial \nu} = g \quad \text{on } \partial D \tag{1.6}$$

with given $f \in H^{\frac{1}{2}}(\partial D)$ for the Dirichlet case and $g \in H^{-\frac{1}{2}}(\partial D)$ for the Neumann case, respectively.

In addition to these two standard boundary conditions, so called *impedance boundary conditions* were introduced mainly to model scattering problems for penetrable obstacles approximately by scattering problems for impenetrable obstacles. The classical impedance condition, also known as Leontovich condition, is given by

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \quad \text{on } \partial D \tag{1.7}$$

where $\lambda \in C(\partial D)$ is a given complex valued function with non-negative real part. The generalized impedance boundary condition is described by

$$\frac{\partial u}{\partial \nu} + i\,(\lambda u - \operatorname{Div} \mu \operatorname{Grad} u) = 0 \quad \text{on } \partial D, \tag{1.8}$$

where Grad and Div denote the surface gradient and surface divergence on ∂D and $\mu \in C^1(\partial D)$ and $\lambda \in C(\partial D)$ are given complex valued functions with non-negative real parts. As compared with the Leontovich condition, the wider class of impedance conditions (1.8) can provide more accurate models, for example, for imperfectly conducting or coated obstacles.

The above impedance conditions (1.7) and (1.8) are local conditions whereas in this survey we want to discuss *nonlocal impedance conditions* of the form

$$A\,u + B\,\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D \tag{1.9}$$

for solutions u to the Helmholtz equation defined either in $\mathbb{R}^3 \setminus \bar{D}$ or in D. Here, at least one of the two operators A and B will contain integral operators, or more general pseudo differential operators defined in Sobolev trace spaces on ∂D. These nonlocal impedance conditions have no immediate physical interpretation and only serve as an analytic tool for the investigation of mathematical problems related to direct and inverse obstacle scattering.

The plan of the paper is as follows. We begin in Section 2 by indicating that the classical suggestion by Panich [24] to overcome the non-uniqueness issue of the single-layer potential approach for solving the exterior Neumann problem (1.6) can be interpreted as an application of a nonlocal impedance condition for the interior domain D. In the following Section 3 we illustrate the use of the same type of interior impedance problem to formulate a solution method for the scattering problem from a penetrable scatterer D with constant refractive index by one single boundary integral equation for one unknown instead of the classical approach with two equations for two unknowns. Complemented by a far field equation, in the spirit of the method proposed by Kress and Rundell [17], this new integral equation creates a system of two nonlinear equations for solving the inverse scattering problem to determine the shape of the scatterer from a known far field pattern. A couple of numerical examples is presented as proof of concept for this method. This is then followed in Section 4 by a brief description of the same idea for scattering from an extended source with constant density. The final Section 5 reviews recent work of Cakoni and Kress [5] employing boundary integral equations and using impedance conditions for the characterization and numerical computation for transmission eigenvalues for penetrable obstacles with constant refractive index. This research was motivated by the work of Cossonnière and Haddar [9] who presented a two-by-two system of integral equations based on Green's representation theorem for the interior transmission problem whereas in [5] again a single boundary integral equation was derived. In the present, we eliminate some negligence within the presentation in [5] and simplify the analysis by the use of a nonlocal rather than a local impedance condition. Although, except the numerical examples, our presentation will be in $\mathbb{R}^3$, we note that all our results remain valid in $\mathbb{R}^2$ with some slight modifications.

2 An early occurrence in history

Following the notation of [8, Section 3.1], in terms of the fundamental solution

$$\Phi_k(x,y) := \frac{1}{4\pi}\,\frac{e^{ik|x-y|}}{|x-y|}\,, \quad x \neq y,$$

to the Helmholtz equation in $\mathbb{R}^3$ we introduce the classical boundary integral operators given by the single- and double-layer operators

$$(S_k\varphi)(x) := 2\int_{\partial D} \Phi_k(x,y)\varphi(y)\,ds(y) \tag{2.1}$$

and

$$(K_k\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\,\varphi(y)\,ds(y) \tag{2.2}$$

and the corresponding normal derivative operators

$$(K_k'\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu(x)}\,\varphi(y)\,ds(y) \tag{2.3}$$

and

$$(T_k\varphi)(x) := 2\,\frac{\partial}{\partial \nu(x)}\,\int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\,\varphi(y)\,ds(y) \tag{2.4}$$

for $x \in \partial D$. For convenience of the reader we present the basic properties of acoustic single- and double-layer potentials in the contemporary Sobolev space setting. The single-layer potential

$$u(x) = \int_{\partial D} \Phi_k(x,y)\varphi(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D, \tag{2.5}$$

with density $\varphi \in H^{-\frac{1}{2}}(\partial D)$ defines a bounded operator from $H^{-\frac{1}{2}}(\partial D)$ into both $H^1(D)$ and $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ with boundary traces given by the jump relations

$$u_\pm = \frac{1}{2}\, S\varphi \quad \text{and} \quad \frac{\partial u_\pm}{\partial \nu} = \frac{1}{2}\, K'\varphi \mp \frac{1}{2}\, \varphi. \tag{2.6}$$

Here, by subscripts $+$ and $-$ we distinguish the traces taken in $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ and $H^1(D)$, respectively. The double-layer potential

$$u(x) = \int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D, \tag{2.7}$$

with density $\varphi \in H^{\frac{1}{2}}(\partial D)$ defines a bounded operator from $H^{\frac{1}{2}}(\partial D)$ into both $H^1(D)$ and $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ with boundary traces given by the jump relations

$$u_\pm = \frac{1}{2}\, K\varphi \pm \frac{1}{2}\, \varphi \quad \text{and} \quad \frac{\partial u_\pm}{\partial \nu} = \frac{1}{2}\, T'\varphi. \tag{2.8}$$

In analogy to Laplace's equation, the classical solution approach to the exterior Dirichlet problem (1.5) via boundary integral equations is to seek the solution u in the form of a double-layer potential (2.7) in $\mathbb{R}^3 \setminus \bar{D}$ The Dirichlet condition (1.5) is satisfied provided the density φ solves the integral equation of the second kind $\varphi + K_k\varphi = 2f$. It can be shown that this equation is uniquely solvable both in $C(\partial D)$, assuming $f \in C(\partial D)$, and in $H^{\frac{1}{2}}(\partial D)$ provided k^2 is not a Neumann eigenvalue for the negative Laplacian in the interior domain D. This failure of the double-layer approach at the interior eigenvalues can be remedied by replacing (2.7) through a combined double- and single-layer potential of the form

$$u(x) = \int_{\partial D} \left\{ \frac{\partial \Phi_k(x,y)}{\partial \nu(y)} - i\Phi_k(x,y) \right\} \varphi(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}. \tag{2.9}$$

For the details of the analysis we refer the reader to [8, Section 3.2].

Analogously, the approach to the exterior Neumann problem (1.6) via a single-layer approach fails when k^2 is a Dirichlet eigenvalue for the negative

Laplacian in D. However, here the attempt to remedy this via the combined potential approach faces analytical and numerical challenges since the resulting integral equation

$$i\varphi - iK_k'\varphi + T_k\varphi = 2g$$

contains the hypersingular operator T_k for the normal derivative of the double-layer potential. To obtain an integral equation that can be handled by using only the Riesz theory for compact operators, in 1965 Panich [24] suggested to include a smoothing operator in (2.9) and replace it by

$$u(x) = \int_{\partial D} \left\{ \Phi_k(x,y)\,\varphi(y) + i\,\frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\,(S_0^2\varphi)(y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \quad (2.10)$$

where S_0 denotes the single-layer operator (2.1) in the potential theoretic limit case $k = 0$. For the resulting integral equation

$$\varphi - K_k'\varphi - iT_kS_0^2\varphi = -2g \qquad (2.11)$$

the Riesz theory is applicable both in $C(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$ since $T_kS_0^2$ is compact. We note that in the 1960s when Panich made the above suggestion the boundary integral equations where discussed only in the classical function spaces of continuous and Hölder continuous functions and not yet in the contemporary Sobolev trace spaces. Injectivity for the operator on the left hand side of (2.11) follows from the fact that, for a solution φ of the homogeneous form of the equation (2.11), with the aid of the potential theoretic jump relations the function u defined by (2.10) in the interior domain D can be seen to satisfy the nonlocal impedance condition

$$u + iS_0^2\,\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D. \qquad (2.12)$$

By Green's integral theorem and the self-adjointness of S_0, from (2.12) we have that

$$\int_D \left\{ |\operatorname{grad} u|^2 - k^2|u|^2 \right\} dx = i \int_{\partial D} \left| S_0 \frac{\partial u}{\partial \nu} \right|^2 ds.$$

The positive definiteness of S_0 now implies that $\partial_\nu u = 0$ on ∂D whence $u = 0$ in D follows and consequently $\varphi = 0$ by the jump relations. For details again we refer to [8, Section 3.2].

We conclude this section by noting that in (2.12) we may replace S_0 by any positive definite pseudo differential operator of order -1. For example, using diagonal operators in terms of an orthogonal system (such as spherical harmonics) can create computational advantages as compared with the full matrix operator obtained by discretizing S_0.

3 Scattering by a penetrable obstacle

Modeling scattering by a penetrable obstacle D with constant density ρ_D and speed of sound c_D differing from the density ρ and speed of sound c in the surrounding background medium $\mathbb{R}^3 \setminus \bar{D}$ leads to a transmission problem. Here, in addition to the superposition $u = u^i + u^s$ of the incoming wave u^i and the scattered wave u^s in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the Helmholtz equation with wave number $k = \omega/c$, we also have a transmitted wave v in D satisfying the Helmholtz equation with wave number $k_D = \omega/c_D \neq k$. The continuity of the pressure and of the normal velocity across the interface ∂D leads to the *transmission conditions*

$$u = v, \quad \frac{1}{\rho}\frac{\partial u}{\partial \nu} = \frac{1}{\rho_D}\frac{\partial v}{\partial \nu} \quad \text{on } \partial D. \tag{3.1}$$

For the sake of simplicity, we only consider the case where $\rho_D = \rho$. The extension of the following analysis to the case $\rho_D \neq \rho$ is straightforward. We also want to allow absorption, i.e., complex wave numbers k_D with non-negative real and imaginary part. For an incident field $u^i = 0$, by Green's integral theorem we find that

$$\operatorname{Im} \int_{\partial D} u^s \frac{\partial \bar{u}^s}{\partial \nu}\, ds = \operatorname{Im} \int_{\partial D} v \frac{\partial \bar{v}}{\partial \nu}\, ds = 2 \operatorname{Re} k_D \operatorname{Im} k_D \int_D |v|^2\, dx \geq 0.$$

By Theorem 2.13 in [8], as consequence of Rellich's lemma, it follows that $u^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$, that is, we have uniqueness for the solution.

Usually this transmission problem is reduced to a two-by-two system of boundary integral equations over the interface ∂D for a pair of unknowns, see among others [7, 16]. This can be done either by the direct method combining the Calderón projectors for the domains D and $\mathbb{R}^3 \setminus \bar{D}$ or by a potential approach in the indirect method. For a survey on methods for solving the transmission problem using only a single integral equation over ∂D we refer to [15]. As an addition to the selection of available single integral equations for the transmission problem, in the present paper we will reduce the transmission problem to a scattering problem in $\mathbb{R}^3 \setminus \bar{D}$ with a nonlocal impedance boundary condition in terms of the Dirichlet-to-Neumann operator for the domain D which then can be solved via one integral equation.

We assume that k_D^2 is not a Dirichlet eigenvalue for $-\Delta$ in the domain D. Then the Dirichlet-to-Neumann operator

$$A_{k_D} : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$$

is well defined by the mapping taking $f \in H^{\frac{1}{2}}(\partial D)$ into the normal derivative $A_{k_D} f := \partial_\nu v$ of the unique solution $v \in H^1(D)$ of $\Delta v + k_D^2 v = 0$ satisfying the Dirichlet condition $v = f$ on ∂D. From the single-layer approach for the interior Dirichlet problem we note the representation

$$A_{k_D} = (I + K'_{k_D}) S_{k_D}^{-1}. \tag{3.2}$$

Then, for $\rho_D = \rho$, the transmission problem (3.1) can be seen to be equivalent to the scattering problem for $u = u^i + u^s$ in $\mathbb{R}^3 \setminus \bar{D}$ with the nonlocal impedance condition

$$\frac{\partial u^s}{\partial \nu} - A_{k_D} u^s = -\frac{\partial u^i}{\partial \nu} + A_{k_D} u^i \quad \text{on } \partial D. \tag{3.3}$$

Once we have determined the scattered wave u^s in $\mathbb{R}^3 \setminus \bar{D}$ from (3.3), the transmitted wave v in D can be obtained via Green's representation theorem from its Cauchy data $v = u$ and $\partial_\nu v = A_{k_D} v = A_{k_D} u = \partial_\nu u$ on ∂D.

The single-layer potential

$$u^s(x) = \int_{\partial D} \Phi_k(x, y)\, \varphi(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{3.4}$$

satisfies the boundary condition (3.3) provided the density $\varphi \in H^{-\frac{1}{2}}(\partial D)$ solves the equation

$$-\varphi + K'_k \varphi - A_{k_D} S_k \varphi = -2\, \frac{\partial u^i}{\partial \nu} + 2 A_{k_D} u^i. \tag{3.5}$$

From the uniqueness for the solution of the transmission problem and consequently also for the solution of its equivalent reformulation (3.3) one can deduce that for a solution φ of the homogeneous form of (3.5) the corresponding potential (3.4) vanishes $u^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Taking the boundary trace it follows that $S_k\varphi = 0$. If we assume that in addition to k_D^2 also k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D we have injectivity of S_k and therefore $\varphi = 0$. Therefore the operator $-I + K_k' - A_{k_D} S_k$ is injective.

With the aid of (3.2) we rewrite

$$-I + K_k' - A_{k_D} S_k = -I + K_k' - I - K_{k_D}' - A_{k_D}(S_k - S_{k_D}).$$

Since $S_k - S_{k_D} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is compact (see [8, Lemma 5.37]) and the Dirichlet to Neumann operator is bounded from $H^{\frac{1}{2}}(\partial D)$ into $H^{-\frac{1}{2}}(\partial D)$ in addition to $K_k', K_{k_D}' : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ also $A_{k_D}(S_k - S_{k_D}) : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is compact. Thus, finally, the Riesz theory applies to equation (3.5) and we can summarize in the following theorem.

Theorem 3.1 *Under the assumption that both k^2 and k_D^2 are not Dirichlet eigenvalues for the negative Laplacian in D the equation (3.5) is uniquely solvable.*

To avoid the restriction on k_D, instead of using the Dirichlet-to-Neumann operator, we propose to work with the Robin-to-Neumann operator

$$R_{k_D} : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$$

defined by the mapping taking $f \in H^{\frac{1}{2}}(\partial D)$ into the normal derivative $R_{k_D} f = \partial_\nu v$ of the unique solution $v \in H^1(D)$ of $\Delta v + k_D^2 v = 0$ satisfying the Robin condition

$$v + i\,\frac{\partial v}{\partial \nu} = f \quad \text{on } \partial D. \tag{3.6}$$

Uniqueness for the solution follows analogously to that for (2.12) by applying Green's integral theorem to v and $\bar{v}$ and taking the imaginary part. From the single-layer approach for the solution of (3.6) we observe that

$$R_{k_D} = (I + K_{k_D}')[S_{k_D} + i(I + K_{k_D}')]^{-1}. \tag{3.7}$$

The corresponding nonlocal impedance condition now becomes

$$\frac{\partial u^s}{\partial \nu} - R_{k_D}\left[u^s + i\,\frac{\partial u^s}{\partial \nu}\right] = -\frac{\partial u^i}{\partial \nu} + R_{k_D}\left[u^i + i\,\frac{\partial u^i}{\partial \nu}\right] \quad \text{on } \partial D. \tag{3.8}$$

For all wave numbers k and k_D we are allowing, this impedance problem can be dealt with via a uniquely solvable integral equation as derived from the combined single- and double-layer approach (2.9). We omit working out the details.

The Sommerfeld radiation condition is equivalent to the asymptotic behavior

$$u^s(x) = \frac{e^{ik\,|x|}}{|x|}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \tag{3.9}$$

uniformly for all directions $\hat{x} = x/|x|$ where the function u_∞ defined on the unit sphere $\mathbb{S}^2$ is known as the far field pattern of u^s. One of the main inverse obstacle scattering problems is to determine the shape of the scatterer D from the far field pattern u_∞ on $\mathbb{S}^2$ for one or several incident plane waves.

Some fifteen years ago in [17] a class of Newton type iterations for obstacle scattering problems was initiated which starts from a boundary integral equation approach for the solution of the forward scattering problem. Together with a corresponding representation of the far field pattern a system of two nonlinear integral equations for the unknown boundary ∂D and a density function on the boundary as a sort of slip variable can be derived. For the above approach to the transmission problem this equation has the form

$$S_\infty \varphi = u_\infty \tag{3.10}$$

with the known far field pattern u_∞ and the operator $S_\infty : H^{-\frac{1}{2}}(\partial D) \to L^2(\mathbb{S}^2)$ given by

$$(S_\infty \varphi)(\hat{x}) := \frac{1}{4\pi}\int_{\partial D} e^{-ik\,\hat{x}\cdot y}\varphi(y)\,ds(y), \quad \hat{x} \in \mathbb{S}^2, \tag{3.11}$$

describing the far field pattern of the single-layer potential (3.4). Then the basis of the inverse method under consideration can be cast into the following theorem.

Theorem 3.2 *For a given incident field u^i and a given far field pattern u_∞, assume that the boundary ∂D and the density φ satisfy the system* (3.5) *and* (3.10)*. Then ∂D solves the inverse scattering problem.*

These two equations are nonlinear with respect to the boundary and ill-posed. Following [17] we suggest to solve them by simultaneous linearization with respect to both unknowns φ and ∂D and iteration, i.e., by Newton iterations that require regularization because of the ill-posedness. We note that the derivatives of the operators in (3.5) and (3.10) can be written down explicitly in terms of boundary integral operators which offers computational advantages. For an overview and survey on this idea we refer to Section 5.4 in [8] and to [10].

To conclude this section, we proceed with a couple of numerical examples in two dimensions as proof of concept. For a brief description we assume that the boundary curve ∂D is given by a regular 2π periodic parameterization

$$\partial D = \{z(t) : 0 \le t \le 2\pi\}$$

and we redefine the operators $S, K' : H^{-\frac{1}{2}}[0, 2\pi] \times C^2[0, 2\pi] \to H^{\frac{1}{2}}[0, 2\pi]$ by

$$S(\psi, z)(t) := \frac{i}{2} \int_0^{2\pi} H_0^{(1)}(k|z(t) - z(\tau)|)\psi(\tau)\, d\tau$$

and

$$K'(\psi, z)(t) := \int_0^{2\pi} \frac{ik}{2} \, \frac{[z'(t)]^{\perp} \cdot [z(\tau) - z(t)]}{|z(t) - z(\tau)|} \, H_1^{(1)}(k|z(t) - z(\tau)|)\, \psi(\tau) d\tau$$

for $t \in [0, 2\pi]$ as well as $S_\infty : H^{-\frac{1}{2}}[0, 2\pi] \times C^2[0, 2\pi] \to L^2(\mathbb{S}^1)$ given by

$$S_\infty(\psi, z)(\hat{x}) := \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_0^{2\pi} e^{-ik\,\hat{x}\cdot z(\tau)} \psi(\tau)\, d\tau, \quad \hat{x} \in \mathbb{S}^1,$$

in their parameterized form. Here, $H_0^{(1)}$ and $H_1^{(1)}$ are the Hankel functions of order zero and one and of the first kind and we have indicated the dependence of the operators on ∂D with parameterization z and we used the fundamental solution to the Helmholtz equation $\Phi(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|)$ in $\mathbb{R}^2$. For any vector $a = (a_1, a_2)$ we denote $a^\perp := (a_2, -a_1)$, that is, $a^\perp$ is obtained by rotating a clockwise by 90 degrees.

Now, setting $\psi := |z'|\, \varphi \circ z$ and multiplying the first equation by $|z'|$, the parameterized form of the system (3.5) and (3.10) becomes

$$\psi - K_k'(\psi, z) + (I + K_{k_D}')S_{k_D}^{-1} S_k(\psi, z) = g(z) \tag{3.12}$$

and

$$S_\infty(\psi, z) = u_\infty \tag{3.13}$$

where

$$g(z) := 2[z']^{\perp} \cdot \operatorname{grad} u^i \circ z - 2(I + K'_{k_D}) S_{k_D}^{-1}(u^i \circ z, z). \tag{3.14}$$

This system needs to be solved for ψ and z and we do this via linearization. Given an approximation for (ψ, z) we solve the linearized equations

$$\begin{aligned} &\eta - K'_k(\eta, z) + (I + K'_{k_D}) S_{k_D}^{-1} S_k(\eta, z) \\ &-d_z K'_k(\psi, z; \zeta) + d_z[(I + K'_{k_D}) S_{k_D}^{-1} S_k](\psi, z; \zeta) - d_z g(z; \zeta) \\ &\quad = g(z) - \psi + K'_k(\psi, z) - (I + K'_{k_D}) S_{k_D}^{-1} S_k(\psi, z) \end{aligned} \tag{3.15}$$

and

$$S_\infty(\eta, z) + d_z S_\infty(\psi, z; \zeta) = u_\infty - S_\infty(\psi, z) \tag{3.16}$$

for (η, ζ) and update (ψ, z) into $(\psi+\eta, z+\zeta)$. These equations contain Fréchet derivatives of S_k, K'_k and S_∞ with respect to z acting as linear operators on ζ. These are obtained by differentiating the kernels with respect to z and for an explicit presentation we refer the reader, for example, to [19] where this inverse scattering method is applied to the inverse scattering problem with the local Leontovich impedance condition (1.7). Basic differentiation rules imply

$$d_z[(I+K'_{k_D}) S_{k_D}^{-1}](\psi, z; \zeta) = d_z K'_k(S_{k_D}^{-1}\psi, z; \zeta) - (I+K'_{k_D}) S_{k_D}^{-1} d_z S_{k_D}(S_{k_D}^{-1}\psi, z; \zeta)$$

and, for plane wave incidence $u^i(x) = e^{ik\,x\cdot d}$, we have

$$\begin{aligned} d_z g(z; \zeta) \;\; = \;\; & 2u^i \circ z \left\{ ik[\zeta']^{\perp} \cdot d - k^2 d \cdot [z']^{\perp}\, d \cdot \zeta \right\} \\ & -2iku^i \circ z (I + K'_{k_D}) S_{k_D}^{-1}(u^i \circ z, z)\, d \cdot \zeta \\ & -2 d_z[(I + K'_{k_D}) S_{k_D}^{-1}](u^i \circ z, z; \zeta). \end{aligned}$$

For finitely many incident fields $u_1^i, \dots, u_M^i$ and corresponding far field patterns $u_{\infty,1}, \dots, u_{\infty,M}$, one iteration step of our algorithmis is as follows: Given an approximation z for the parameterization of the boundary ∂D and $\psi_1, \dots, \psi_m$ for the densities, after defining g_m analogously to (3.14) in terms of u_m^i, we solve the linearized system

$$\begin{aligned}
&\eta_m - K_k'(\eta_m, z) + (I + K_{k_D}')S_{k_D}^{-1}S_k(\eta_m, z) \\
&-d_z K_k'(\psi_m, z;\zeta) + d_z[(I + K_{k_D}')S_{k_D}^{-1}S_k](\psi_m, z;\zeta) - d_z g_m(z;\zeta) \\
&\quad = g_m(z) - \psi + K_k'(\psi_m, z) - (I + K_{k_D}')S_{k_D}^{-1}S_k(\psi_m, z)
\end{aligned} \tag{3.17}$$

and

$$S_\infty(\eta_m, z) + d_z S_\infty(\psi_m, z;\zeta) = u_{\infty,m} - S_\infty(\psi_m, z) \tag{3.18}$$

where $m = 1, \dots, M$. We solve the system (3.17)–(3.18) of $2M$ equations for the $M+1$ unknowns η_m and ζ and update ψ_m into $\psi_m + \eta_m$ and z into $z+\zeta$. For the initial step of the iteration only an initial guess for the shape z is required. The initial densities ψ_m then can be obtained by solving (3.12).

For the numerical implementation we need to discretize the boundary integral operators S_k and K_k' and their Fréchet derivatives. All four operators have weakly singular kernels with logarithmic singularities. For their numerical approximation by weighted trigonometric interpolation quadratures with spectral convergence we refer to [8].

In our numerical examples we use parameterizations of the form

$$z(t) = r(t)(\cos t, \sin t), \quad 0 \le t \le 2\pi, \tag{3.19}$$

with a non-negative function r representing the radial distance of ∂D from the origin and approximate r by trigonometric polynomials of degree J. We collocate the two equations (3.12) and (3.13), each at $2n$ equidistant collocation points, the first equation at the points $t_j = j\pi/n$, $j = 1, \dots, 2n$, and the second equation at the points $(\cos t_j, \sin t_j) \in \mathbb{S}^1$. The resulting linear system for the $2J+1$ Fourier coefficients and the $2n$ (or $2nM$ if we use M incident waves) nodal values of the density function ψ is solved in the least squares sense, penalized via Tikhonov regularization. As experienced in the application of regularized Newton iterations for related problems, it is advantageous to use an H^p Sobolev penalty term for the shape rather than an L^2 penalty in the Tikhonov regularization for some small $p \in \mathbb{N}$. For the density function just L^2 regularization suffices. Thus, in our Tikhonov functional we added the penalty term $\alpha_\zeta \|\zeta\|_{H^p}^2 + \alpha_\eta \|\eta\|_{L^2}^2$ with regularization parameters α_ζ and α_η to the square of the L_2 residual for the linear system (3.15) and (3.16).

As pointed out already, the following two numerical examples are intended as proof of concept and not as presentation of an already fully developed method. In particular, the regularization parameters and the number of iterations were chosen by trial and error instead of, for example, a discrepancy principle. To avoid committing an inverse crime the synthetic far field data were obtained by solving the two-by-two system of integral equations presented in [16].

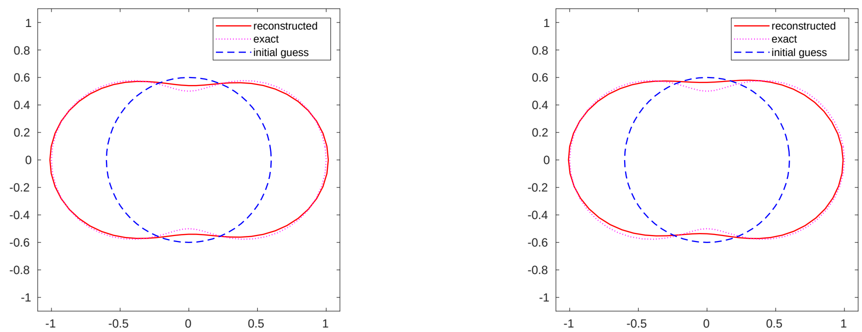

Figure 3.1: Reconstruction of the peanut (3.20) for exact data (left) and 5% noise (right).

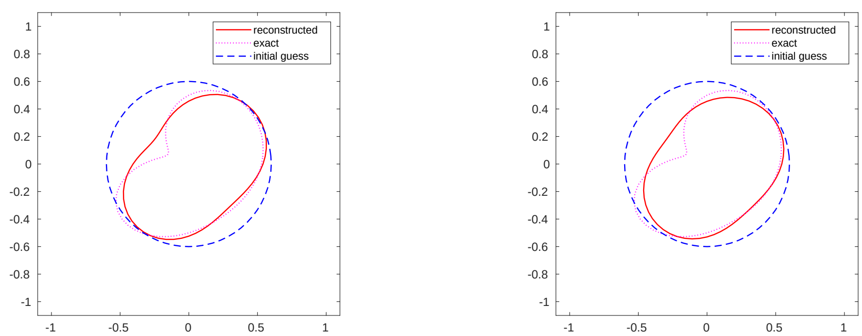

Figure 3.2: Reconstruction of the apple (3.21) for exact data (left) and 5% noise (right).

As boundary curves we considered a peanut-shaped obstacle with parametric representation

$$z(t) = \sqrt{\cos^2 t + 0.25 \sin^2 t}\,(\cos t, \sin t) \tag{3.20}$$

and an apple-shaped obstacle with parametric representation

$$z(t) = \frac{0.5 + 0.4\cos t + 0.1 \sin 2t}{1 + 0.7\cos t}\,(\cos t, \sin t). \tag{3.21}$$

The wave numbers were $k = 1$ and $k_D = 2+3i$ for both cases. The number of quadrature points was $2n = 64$. The degree of the trigonometric polynomials was chosen as $J = 6$ for exact data and $J = 4$ for perturbed data. The regularization parameter for an H^2 regularization for the shape was 0.1×0.9^m for the m-th iteration step for exact data and 0.9^m for perturbed data. The L^2 regularization parameter for the density was 10^{-10} for exact data and 10^{-7} for perturbed data. The iteration was stopped after 20 iterations for exact data and 12 iterations for perturbed data, For the perturbed data, random noise was added point wise with relative error in the L^2 norm. The iterations were started with an initial guess given by a circle of radius 0.6 centered at the origin. For the peanut we used one incident wave with $d = (1, 0)$ and for the apple two incident waves with $d = (\pm 1, 0)$.

In the figures the exact ∂D is given as dotted (magenta), the reconstruction as full (red) and the initial guess as dashed (blue) curve. They compare well with the reconstructions in [1] which were also obtained by the Kress-Rundell approach, but based on a two-by-two system for the transmission problem and with two incident waves for the peanut and four incident waves for the apple.

4 A source problem

We proceed by considering scattering from an extended source described by an L^2 function F with support in $\bar{D}$. Given a plane wave $u^i(x) = e^{ik\,x\cdot d}$ as incident field, the problem is to find the total field $u = u^i + u^s \in H^1_{\rm loc}(\mathbb{R}^3 \setminus \bar{D})$ as a solution to the inhomogeneous Helmholtz equation

$$\Delta u + k^2 u = -F \quad \text{in } \mathbb{R}^3 \tag{4.1}$$

such that the scattered wave u^s satisfies the Sommerfeld radiation condition. Uniqueness of the solution follows from the observation that the difference of two solutions is an entire solution to the Helmholtz equation satisfying the radiation condition. Following similar ideas as in the previous section, we present a solution method by transforming this problem equivalently to a scattering problem in $\mathbb{R}^3 \setminus \bar{D}$ with a nonlocal impedance condition on ∂D.

The volume potential

$$u_0(x) := \int_D \Phi(x, y) F(y)\, dy, \quad x \in \mathbb{R}^3 \tag{4.2}$$

defines a solution of (4.1) and belongs to $H^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ (see[8]). Therefore the solution of the scattering problem also belongs to $H^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ (see[8]) and consequently both the trace and the normal derivative trace on both sides of ∂D coincide.

We assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in the domain D and make use of the Dirichlet-to-Neumann operator $A_k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ for D from the previous section, now with the wave number k. For a solution u of (4.1) the function $u - u_0$ in D satisfies the Helmholtz equation and

$$\frac{\partial(u - u_0)}{\partial \nu} = A_k(u - u_0)|_{\partial D} \quad \text{on } \partial D.$$

Therefore, in view of the continuity of the trace and the normal derivative across ∂D, the scattering problem from a source can equivalently be separated into two parts. First we solve solve the scattering problem for u in $\mathbb{R}^3 \setminus \bar{D}$ with the nonlocal impedance condition

$$\frac{\partial u}{\partial \nu} - A_k u = \frac{\partial u_0}{\partial \nu} - A_k u_0 \quad \text{on } \partial D. \tag{4.3}$$

Then, knowing u on ∂D we solve for a solution to the Helmholtz equation v in D with Dirichlet values $v = u - u_0$ on ∂D and set $u := u_0 + v$ in D. Again we seek the scattered wave u^s in the form of the single-layer potential (3.4) and the boundary condition

$$\frac{\partial u^s}{\partial \nu} - A u^s = \frac{\partial u_0}{\partial \nu} - \frac{\partial u^i}{\partial \nu} - A(u_0 - u^i) \quad \text{on } \partial D$$

leads to an integral equation analogous to (3.5). As in the previous section the restriction on k^2 not a Dirichlet eigenvalue for D can be avoided by the use of a Robin-to-Neumann operator for D.

We note that instead of u_0 given by the volume potential (4.2) one can use any special solution to (4.1). In particular, when $F = 1$ is constant in D a simple alternative is given by

$$u_0(x) := -\frac{1}{k^2}\,, \quad x \in D.$$

In this particular case, the analogue of the two integral equations of the previous section can be employed for the solution of the inverse problem to determine D from the far field pattern u_∞ of the scattered wave. For the limiting case of a related problem for the Laplace case this method has been successfully implemented in two dimension by Kress and Rundell [18].

5 Transmission eigenvalues

Deviating for a couple of paragraphs from the theme of obstacle scattering, we consider the case of an inhomogeneous medium with space dependent refractive index n. We assume that n is real valued and nonnegative and that the contrast $m := 1 - n$ has support given by our obstacle domain $\bar{D}$ and is continuous in $\bar{D}$. Then, for an incident plane wave $u^i(x) = e^{ik\,x\cdot d}$, the simplest inhomogeneous medium scattering problem is to find the total field $u \in H^1_{\rm loc}(\mathbb{R}^3)$ such that $u = u^i + u^s$ fulfills

$$\Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^3 \tag{5.1}$$

and u^s satisfies the Sommerfeld radiation condition (1.2).

A complex number k is called a transmission eigenvalue if there exist nontrivial functions $v, w \in L^2_\Delta(D) := \{u \in L^2(D) : \Delta u \in L^2(D)\}$ such that $w - v \in H^2(D)$ and

$$\Delta v + k^2 v = 0, \quad \Delta w + k^2 n w = 0 \quad \text{in } D \tag{5.2}$$

and

$$v = w, \quad \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \quad \text{on } \partial D. \tag{5.3}$$

Here, the differential equations in (5.2) have to be understood in the distributional sense. In view of the the transmission condition (5.3) the space $H^2(D)$ of functions u with vanishing trace $u|_{\partial D}$ and normal trace $\partial_\nu u|_{\partial D}$ is the natural solution space for the difference $v - w$. Then from

$$\Delta v - \Delta w = k^2(w - v) - k^2 m w$$

we observe that we must demand $v, w \in L^2_\Delta(D)$. We endow $L^2_\Delta(D)$ with the norm

$$\|u\|^2_{L^2_\Delta(D)} := \|u\|^2_{L^2(D)} + \|\Delta u\|^2_{L^2(D)}.$$

This transmission eigenvalue problem was introduced by Kirsch [11] in 1986 in connection with the completeness of the far field patterns $\{u_\infty(\cdot, d) : d \in \mathbb{S}^2\}$ in $L^2(\mathbb{S}^2)$ for scattering of plane waves from an inhomogeneous medium. In particular, this set of far field patterns is not complete if k is a transmission eigenvalue.

The transmission eigenvalues can be seen as an extension of the idea of resonant frequencies for impenetrable obstacles to the case of penetrable media and are related to non-scattering frequencies. If k is a real transmission eigenvalue and the corresponding eigenfunction v can be extended outside D as an entire solution to the Helmholtz equation, then if the extended field is used as incident field the corresponding scattered wave is identically zero, i.e., this field does not scatter at the wave number k. Therefore it is not surprising that certain methods for the solution of the inverse scattering problem for inhomogeneous media such as the linear sampling method and the factorization method fail when k is a transmission eigenvalue. This inconvenient property explains why for a long time transmission eigenvalues were viewed as something to avoid and therefore left aside. It lasted 20 years after their introduction before Päivärinta and Sylvester [23] proved the existence of real transmission eigenvalues for the general case.

Together with his co-workers Armin Lechleiter [13, 21, 22], to whom this survey is dedicated, has given a promising characterization of the smallest real transmission eigenvalue k_0 in terms of the behavior of the eigenvalues of the far field operator F in the vicinity of their accumulation point zero when k approaches k_0. The far field operator is just the integral operator with the far field pattern $u_\infty(\hat{x}, d)$ (for observation directions $\hat{x}$ and incident directions d) as kernel and is related to the scattering operator. This interesting relation between transmission eigenvalues and the eigenvalues of the far field operator is known as inside-outside duality for the far field operator and, together with monotonicity properties of transmission eigenvalues, might open up possibilities for designing a target signature for scatterers.

Here, following work of Cossonnière and Haddar [9] and Cakoni and Kress [5], we want to illustrate how boundary integral equations can be used to characterize and numerically compute transmission eigenvalues in the case were n is constant in D. We need to adjust the spaces in which we must investigate the boundary integral operators since we have to search for the eigenfunctions v, w in $L^2_\Delta(D)$. This implies that their trace and their normal derivative on the boundary belong to $H^{-\frac{1}{2}}(\partial D)$ and $H^{-\frac{3}{2}}(\partial D)$, respectively, and are defined by duality. Motivated by Green's integral theorem

$$\int_{\partial D} u\,\frac{\partial w}{\partial \nu}\,ds = \int_D (u\Delta w - w\Delta u)\,dx$$

for $u, w \in H^2(D)$ with $w = 0$ on ∂D, for $\tau \in H^{\frac{1}{2}}(\partial D)$ we define the duality pairing

$$\langle u, \tau\rangle_{H^{-\frac{1}{2}}(\partial D), H^{\frac{1}{2}}(\partial D)} := \int_D (u\Delta w - w\Delta u)\,dx \tag{5.4}$$

where $w \in H^2(D)$ is biharmonic in D such that $w = 0$ and $\partial_\nu w = \tau$ on ∂D. However, we note that he right hand side of (5.4) has the same value for all $w \in H^2(D)$ with boundary traces $w = 0$ and $\partial_\nu w = \tau$ on ∂D. The well-posedness of the biharmonic Dirichlet problem (see, among others, [3]) implies

$$\|w\|_{H^2(D)} \le c\|\tau\|_{H^{\frac{1}{2}}(\partial D)}$$

with some positive constant c independent of τ. Consequently

$$\left|\langle u, \tau\rangle_{H^{-\frac{1}{2}}(\partial D), H^{\frac{1}{2}}(\partial D)}\right| \le C\|u\|_{L^2_\Delta(D)}\|\tau\|_{H^{\frac{1}{2}}(D)} \tag{5.5}$$

for all $\tau \in H^{\frac{1}{2}}(\partial D)$ and some positive constant C. Thus for each $u \in H^2(D)$ by (5.4) we have defined a bounded linear functional γu with

$$\|\gamma u\|_{H^{-\frac{1}{2}}(\partial D)} \le C\|u\|_{L^2_\Delta(D)}, \tag{5.6}$$

that means $\gamma : H^2(D) \to H^{-\frac{1}{2}}(\partial D)$ is a bounded operator. By denseness we can extend γ as a bounded operator $\gamma : L^2_\Delta(D) \to H^{-\frac{1}{2}}(\partial D)$.

Analogously, for $u \in L^2_\Delta(D)$ the normal trace $\partial_\nu u \in H^{-\frac{3}{2}}(\partial D)$ is defined by duality using the identity

$$\left\langle \frac{\partial u}{\partial \nu}, \tau\right\rangle_{H^{-\frac{3}{2}}(\partial D), H^{\frac{3}{2}}(\partial D)} := -\int_D (u\Delta w - w\Delta u)\,dx \tag{5.7}$$

where $w \in H^2(D)$ is such that $w = \tau$ and $\partial_\nu w = 0$ on ∂D. Then we also have

$$\left\|\frac{\partial u}{\partial \nu}\right\|_{H^{-\frac{3}{2}}(\partial D)} \le C\|u\|_{L^2_\Delta(D)} \tag{5.8}$$

for some positive constant C.

For convenience we introduce the wave number for the refracting case by

$$k_n := \sqrt{n}\, k.$$

As in the direct boundary integral equation approach to the classical transmission problem of Section 3, Cossonnière and Haddar combined the Calderón projectors for the wave numbers k and k_n in the domain D. From Green's representation theorem and the potential theoretic jump relations it follows that for a solution $v \in L^2_\Delta(D)$ to $\Delta v + k^2 v = 0$ we have that

$$\begin{pmatrix} v \\ \partial_\nu v \end{pmatrix} = \begin{pmatrix} -K_k & S_k \\ -T_k & K'_k \end{pmatrix} \begin{pmatrix} v \\ \partial_\nu v \end{pmatrix}. \tag{5.9}$$

The validity of this Calderón equation for $v \in L^2_\Delta(D)$ can be justified from the classical spaces by denseness arguments. We now combine (5.9) with the corresponding equation for w to observe that if k is a transmission eigenvalue then the operator A_k defined by

$$A_k := \begin{pmatrix} -K_k + K_{k_n} & S_k - S_{k_n} \\ -T_k + T_{k_n} & K'_k - K'_{k_n} \end{pmatrix}$$

has a nontrivial kernel in $H^{-\frac{1}{2}}(\partial D) \times H^{-\frac{3}{2}}(\partial D)$, that is, the transmission eigenvalue k is an eigenvalue of A_k considered as a nonlinear and analytic operator in k.

The main result of Cossonnière and Haddar [9] for this two-by-two system now can be stated as the following theorem.

Theorem 5.1 *The operator*

$$(|k|^2 - |k_n|^2)A(i|k|) : H^{-\frac{1}{2}}(\partial D) \times H^{-\frac{3}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D) \times H^{\frac{3}{2}}(\partial D)$$

is coercive and the operator

$$A(k) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} A(i|k|) : H^{-\frac{1}{2}}(\partial D) \times H^{-\frac{3}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D) \times H^{\frac{3}{2}}(\partial D)$$

is compact.

The main tools for its proof are Green's integral theorem for the coercitivity and an elaborate analysis of the mapping properties of the difference of the boundary integral operators for k and k_n. In particular, this result implies that the analytic Fredholm theory can be used to deduce the discreteness of transmission eigenvalues, a result that had been established some 30 years ago by Colton, Kirsch and Päivärinta [6] and Rynne and Sleeman [25] by different concepts.

The above boundary integral equation formulation of Cossonnière and Haddar was used by Kleefeld [14] for actual computations of transmission eigenvalues with the aid of the attractive new algorithm for solving nonlinear eigenvalue problems for large sized matrices that are analytic with respect to the eigenvalue parameter as proposed by Beyn [2]. We note that exterior transmission eigenvalues in $\mathbb{R}^3 \setminus \bar{D}$ are also eigenvalues of A_k which has to be taken into account in the numerical implementation in an appropriate way. Before Kleefeld's work, in the literature, the majority of numerical methods were based on finite element methods applied after a transformation of the transmission eigenvalue problem to a generalized eigenvalue problem for a fourth order partial differential equation.

As for the classical transmission problem as discussed in Section 3, one can also try to base the transmission eigenvalue problem on only one integral equation via the Dirichlet-to-Neumann operator for the domain D. This idea was pursued by Cakoni and Kress [5]. Assuming that k and k_n are not Dirichlet eigenvalues of $-\Delta$ in the domain D, we have that k is a transmission eigenvalue if and only if the kernel of the operator $A_k - A_{k_n}$ given by the difference of the Dirichlet-to-Neumann operators for the wave numbers k and k_n is nontrivial. To avoid the annoying restriction on the wave numbers instead of using the Dirichlet-to-Neumann operator, Cakoni and Kress [5] originally worked with the Robin-to-Neumann operator defined by the Leontovich impedance condition (1.7). However, later it was discovered that the related analysis can be simplified by replacing (1.7) by a nonlocal impedance condition analogous to (2.12). For this, taking into account that we require $L^2_\Delta(D)$ solutions in the analysis of the transmission eigenvalues, we redefine the Robin-to-Neumann operator by

$$R_k : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{3}{2}}(\partial D)$$

as the mapping taking $f \in H^{-\frac{1}{2}}(\partial D)$ into the normal derivative $R_k f = \partial_\nu u$ of the unique solution $u \in L^2_\Delta(D)$ of $\Delta u + k^2 u = 0$ satisfying the nonlocal impedance boundary condition

$$u + iP^3 \frac{\partial u}{\partial \nu} = f \quad \text{on } \partial D. \tag{5.10}$$

Here P is a positive definite pseudo-differential operator of order -1.

The first task is to establish the well-posedness of this boundary value problem in the required space $L^2_\Delta(D)$. One of our main tools in the subsequent analysis is the following regularity result for the Poisson equation. To avoid tedious adjustments of the regularity assumptions on ∂D related to the order of the respective Sobolev trace spaces for each of the following statements, for the remainder of our paper we assume that ∂D is C^∞ smooth.

Lemma 5.2 *Let $F \in H^m(D)$ and $g \in H^{m+\frac{3}{2}}(\partial D)$. Then the unique solution $v \in L^2_\Delta(D)$ of $\Delta v = F$ in D and $v = g$ on ∂D belongs to $H^{m+2}(D)$ and the mapping taking (F, g) into v is bounded from $H^m(D) \times H^{m+\frac{3}{2}}(\partial D)$ into $H^{m+2}(D)$ for $m = 0, 1, \dots$*

Proof. By Theorem 1.3 in [26, p. 305]) the unique solution $v \in H^1_0(D)$ of $\Delta v = F$ for $F \in H^m(D)$ belongs to $H^{m+2}(D)$ and the mapping taking F into v is bounded from $H^m(D)$ into $H^{m+2}(D)$ for $m = 0, 1, \dots$.

First we show that this result can be extended to solutions $v \in L^2_\Delta(D)$ that vanish on ∂D in the sense of the $H^{-\frac{1}{2}}(\partial D)$ trace. For this we observe from the definition (5.4) that for any harmonic function $v \in L^2(D)$ with vanishing trace on the boundary ∂D we have that $\int_D v\Delta w dx = 0$ for all $w \in H^2(D)$ with $w = 0$ on ∂D. Inserting the solution $w \in H^1_0(D)$ of $\Delta w = \overline{v}$ which by the above automatically belongs to $H^2(D)$ yields $v = 0$ in D. For a solution $v \in L^2_\Delta(D)$ of $\Delta v = F$ for $F \in L^2(D)$ with vanishing $H^{-\frac{1}{2}}(\partial D)$ trace on ∂D we denote by $\tilde{v}$ the solution of $\Delta \tilde{v} = F$ in $H^1_0(D)$ and apply the just established uniqueness result for the difference $v - \tilde{v}$ to obtain that $v = \tilde{v} \in H^1_0(D)$.

The statement of the lemma now follows from the observation that the unique solution $w \in H^1(D)$ of the Laplace equation $\Delta w = 0$ satisfying $w = g$ on ∂D for $g \in H^{m+\frac{3}{2}}(\partial D)$ is in $H^{m+2}(D)$ and that the mapping taking g into w is bounded from $H^{m+\frac{3}{2}}(\partial D)$ into $H^{m+2}(D)$ as can be observed from the single-layer boundary integral equation approach and the mapping properties of the the single-layer potential. $\square$

Now we establish uniqueness for a solution $u \in L^2_\Delta(D)$ to (5.10). From $f = 0$ and the assumptions on P we observe that u has boundary trace in $H^{\frac{3}{2}}(\partial D)$. Hence by Lemma 5.2 we have $u \in H^2(D)$ and uniqueness for all k with positive real and nonnegative imaginary part now follows analogously to that for (2.12).

In order to represent the Robin-to-Neumann operator we introduce the single-layer potential $\mathcal{S}_k$ defined by

$$(\mathcal{S}_k\psi)(x) := 2\int_{\partial D} \psi(y)\Phi_k(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D.$$

In particular, it is known that the operator $\mathcal{S}_k : H^{-\frac{3}{2}}(\partial D) \to L^2(D)$, its restriction $S_k : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ and the normal derivative operator $K'_k : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{3}{2}}(\partial D)$ all are bounded (see [12, 20]). Further, the jump relations for the single-layer potential can be extended to the case of $H^{-\frac{3}{2}}(\partial D)$ densities. and, if k is not a Dirichlet eigenvalue of $-\Delta$ for D then $S_k : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is an isomorphism.

The single-layer potential with density $\varphi \in H^{-\frac{3}{2}}(\partial D)$ satisfies (5.10) provided φ satisfies the equation

$$A_k\varphi = f \tag{5.11}$$

where

$$A_k := S_k + iP^3(I + K'_k). \tag{5.12}$$

From uniqueness both for the interior problem (5.10) in D and for the exterior Dirichlet problem together with the jump relations for the single-layer potential it can be deduced that A_k has a trivial kernel in $H^{-\frac{3}{2}}(\partial D)$ for all k with positive real part and nonnegative imaginary part. After picking a wave number k_0 such that k_0^2 is not a Dirichlet eigenvalue for $-\Delta$ in D we write $A_k = S_{k_0} + B_k$ where $B_k := S_k - S_{k_0} + iP^3(I + K'_k)$. Then $S_{k_0} : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is an isomorphism and $B_k : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is compact since the difference $S_k - S_{k_0}$ is bounded from $H^{-\frac{3}{2}}(\partial D)$ into $H^{\frac{1}{2}}(\partial D)$ (see [8, Lemma 5.37]) and $P^3(I + K'_k)$ is bounded from $H^{-\frac{3}{2}}(\partial D)$ into $H^{\frac{3}{2}}(\partial D)$ because of our assumption on P. Therefore, by the Riesz theory $A_k : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is an isomorphism and we can write

$$R_k = (I + K'_k)A_k^{-1}. \tag{5.13}$$

Hence, we finally can state that k is a transmission eigenvalue if and only if the operator

$$M(k) := R_k - R_{k_n} \tag{5.14}$$

has a nontrivial kernel in $H^{-\frac{1}{2}}(\partial D)$.

We note that if for positive κ we define

$$\widetilde{A}_\kappa := S_{i\kappa} + P^3(I + K'_{i\kappa}) \tag{5.15}$$

then analogous to the above it can shown that $\widetilde{A}_\kappa : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is an isomorphism and $\widetilde{R}_\kappa = (I + K'_{i\kappa})\tilde{A}_\kappa^{-1}$ represents the Robin-to-Neumann operator for $\Delta u - \kappa^2 u = 0$ in D with boundary condition

$$u + P^3 \frac{\partial u}{\partial \nu} = f \quad \text{on } \partial D \tag{5.16}$$

for a given $f \in H^{-\frac{1}{2}}(\partial D)$. Setting $\kappa_n := \kappa\sqrt{n}$, we also have that $i\kappa$ is a transmission eigenvalue if and only if the operator

$$\widetilde{M}(\kappa) := \widetilde{R}_\kappa - \widetilde{R}_{\kappa_n} \tag{5.17}$$

has a nontrivial kernel in $H^{-\frac{1}{2}}(\partial D)$.

We now want to show that

$$M(k) = (I + K'_k)A_k^{-1} - (I + K'_{k_n})A_{k_n}^{-1} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$$

is a Fredholm operator of index zero. Our analysis is based on the properties of the difference of the single-layer potentials

$$u := \mathcal{S}_k A_k^{-1}\varphi - \mathcal{S}_{k_n} A_{k_n}^{-1}\varphi \tag{5.18}$$

for $\varphi \in H^{-\frac{1}{2}}(\partial D)$ for which we expect $M(k) = \partial_\nu u$. We collect these properties in the following lemma.

Lemma 5.3 *For $\varphi \in H^{-\frac{1}{2}}(\partial D)$ define u by (5.18). Then $u \in H^2(D)$ and $\Delta u \in L^2_\Delta(D)$ with the mappings $\varphi \mapsto u$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$ and $\varphi \mapsto \Delta u$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $L^2_\Delta(D)$ such that*

$$\|\Delta u\|_{L^2_\Delta(D)} \leq \widetilde{C}\|u\|_{L^2_\Delta(D)} \tag{5.19}$$

for some positive constant $\widetilde{C}$ independent of φ. Further we have the differential equation

$$(\Delta + k^2)(\Delta + k_n^2)u = 0 \quad \textit{in } D \tag{5.20}$$

and the boundary conditions

$$u + iP^3 \frac{\partial u}{\partial \nu} = 0 \quad \textit{and} \quad \Delta u + iP^3 \frac{\partial \Delta u}{\partial \nu} = (k_n^2 - k^2)\varphi \quad \textit{on } \partial D. \tag{5.21}$$

Proof. From

$$\Delta u = -k^2 \mathcal{S}_k A_k^{-1}\varphi + k_n^2 \mathcal{S}_{k_n} A_{k_n}^{-1}\varphi \tag{5.22}$$

we observe that u is in $L^2_\Delta(D)$ with the mapping $\varphi \to u$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $L^2_\Delta(D)$. The differential equation (5.20) is obvious and via $\Delta\Delta u = -(k^2+k_n^2)\Delta u - k^2 k_n^2 u$ it implies that Δu in $L^2_\Delta(D)$ with the estimate (5.19) and that the mapping $\varphi \to \Delta u$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $L^2_\Delta(D)$. Furthermore we have

$$[\mathcal{S}_k A_k^{-1}\varphi]|_{\partial D} + iP^3 \frac{\partial}{\partial \nu} \mathcal{S}_k A_k^{-1}\varphi = [S_k + iP^3(I + K_k')]A_k^{-1}\varphi = \varphi. \tag{5.23}$$

Subtracting from this the corresponding equation for the wave number k_n the first of the boundary conditions (5.21) follows. Multiplying (5.23) by k^2 and subtracting the corresponding equation for the wave number k_n in view of (5.22) we obtain the second of the boundary conditions (5.21).

Since $\varphi \mapsto u$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $L^2_\Delta(D)$ we have that $\varphi \mapsto \partial_\nu u$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ to $H^{-\frac{3}{2}}(\partial D)$ and our assumption on the operator P ensures that the mapping $\varphi \to P^3\partial_\nu u$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{3}{2}}(\partial D)$. Now, the first boundary condition in (5.21) and Lemma 5.2 for $m = 0$ imply that $\varphi \mapsto u$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$. □

Analogously, for $\kappa > 0$ the statement of this lemma carries over to

$$v := \mathcal{S}_{i\kappa}\widetilde{A}_\kappa^{-1}\varphi - \mathcal{S}_{i\kappa_n}\widetilde{A}_{\kappa_n}^{-1}\varphi \tag{5.24}$$

with (5.20) and (5.21) replaced by

$$(\Delta - \kappa^2)(\Delta - \kappa_n^2)v = 0 \quad \text{in } D \tag{5.25}$$

and

$$v + P^3 \frac{\partial v}{\partial \nu} = 0 \quad \text{and} \quad \Delta v + P^3 \frac{\partial \Delta v}{\partial \nu} = (\kappa^2 - \kappa_n^2)\varphi \quad \text{on } \partial D. \tag{5.26}$$

Corollary 5.4 *The linear operators* $M(k), \widetilde{M}(\kappa) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ *are bounded.*

Proof. The statement is obtained by taking the normal traces of the mappings $\varphi \mapsto u$ and $\varphi \mapsto v$. $\square$

Theorem 5.5 *Let* $\kappa > 0$*. Then*

$$(\kappa^2 - \kappa_n^2)\widetilde{M}(\kappa) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$$

is coercive.

Proof. For $v \in H^2(D)$ with $\Delta v \in L^2_\Delta(D)$, using Green's integral theorem, we can transform

$$\begin{aligned}
&\int_D \bar{v}(\Delta - \kappa^2)(\Delta - \kappa_n^2)v\,dx \\
&- \int_D \left[|\Delta v|^2 + (\kappa^2 + \kappa_n^2)|\operatorname{grad} v|^2 + \kappa^2\kappa_n^2|v|^2\right] dx \\
= &\int_D (\bar{v}\Delta\Delta v - \Delta\bar{v}\Delta v)\,dx - (\kappa^2 + \kappa_n^2)\int_D (\bar{v}\Delta v + |\operatorname{grad} v|^2)\,dx \\
= &\int_{\partial D}\left(\bar{v}\,\frac{\partial \Delta v}{\partial \nu} - \Delta v\,\frac{\partial \bar{v}}{\partial \nu}\right) ds - (\kappa^2 + \kappa_n^2)\int_{\partial D} \bar{v}\,\frac{\partial v}{\partial \nu}\,ds
\end{aligned} \tag{5.27}$$

where the integrals over ∂D containing Δv and $\partial_\nu \Delta v$ are to be understood in the sense of the dualities (5.4) and (5.7). In this expression the second domain integral is equivalent to the $\|\cdot\|_{H^2}$ norm as can be seen with the aid of Green's representation formula, that is,

$$\int_D \left[|\Delta v|^2 + (\kappa^2 + \kappa_n^2)|\operatorname{grad} v|^2 + \kappa^2\kappa_n^2|v|^2\right] dx \geq c\|v\|^2_{H^2(D)} \tag{5.28}$$

for all $v \in H^2(D)$ and some constant $c > 0$.

Now, for $\varphi \in H^{-\frac{1}{2}}(\partial D)$ we consider v as defined by (5.24). From the boundary conditions (5.26) and the self adjointness of P we observe that

$$\int_{\partial D}\left(\bar{v}\,\frac{\partial \Delta v}{\partial \nu} - \Delta v\,\frac{\partial \bar{v}}{\partial \nu}\right) ds = -(\kappa^2 - \kappa_n^2)\int_{\partial D} \varphi\,\frac{\partial \bar{v}}{\partial \nu}\,ds \tag{5.29}$$

Then (5.25), (5.27), and (5.29) imply

$$\int_D \left[|\Delta v|^2 + (\kappa^2+\kappa_n^2)|\operatorname{grad} v|^2 + \kappa^2\kappa_n^2|v|^2\right] dx$$

$$= \ (\kappa^2-\kappa_n^2)\int_{\partial D}\varphi\,\frac{\partial\bar v}{\partial\nu}\,ds - (\kappa^2+\kappa_n^2)\int_{\partial D} P^{\frac{3}{2}}\frac{\partial v}{\partial\nu}\,P^{\frac{3}{2}}\frac{\partial\bar v}{\partial\nu}\,ds.$$

Inserting $\partial_\nu v = \widetilde{M}(\kappa)\varphi$ and using the positive definiteness of P and the estimate (5.28) we obtain

$$(\kappa^2-\kappa_n^2)\int_{\partial D}\varphi\overline{\widetilde{M}(\kappa)\varphi}\,ds \ge \tilde c\,\|v\|^2_{H^2(D)} \tag{5.30}$$

for some constant $\tilde c > 0$. From the boundary condition (5.26), the boundedness of P, the trace estimates (5.6) and (5.8), applied to Δu, and the estimate (5.19) we can conclude that

$$\|\varphi\|^2_{H^{-\frac{1}{2}}(\partial D)} \le C\|\Delta v\|_{L^2_\Delta(D)} \le C\widetilde{C}\|v\|_{L^2_\Delta(D)} \le c\|v\|^2_{H^2(D)}$$

for some positive constant c. Inserting this estimate into (5.30) finishes the proof. □

Theorem 5.6 *The operator*

$$M(k) + \frac{k^2-k_n^2}{|k|^2-|k_n|^2}\widetilde{M}(|k|) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$$

is compact.

Proof. For $\varphi \in H^{-\frac{1}{2}}(\partial D)$ we consider u defined by (5.18) and v defined by (5.24) for $\kappa = |k|$ and let

$$U := u + \frac{k^2-k_n^2}{|k|^2-|k_n|^2}\,v. \tag{5.31}$$

From (5.21) and (5.26) we can deduce the boundary conditions

$$U = -P^3\,\frac{\partial U}{\partial\nu} + (1-i)P^3\,\frac{\partial u}{\partial\nu} \tag{5.32}$$

and

$$\Delta U = -P^3 \, \frac{\partial \Delta U}{\partial \nu} + (1-i) P^3 \, \frac{\partial \Delta u}{\partial \nu} \tag{5.33}$$

on ∂D. (We note that the coefficient in the definition of U in (5.31) is chosen such that we obtain (5.33).) Since the mappings $\varphi \mapsto U$ and $\varphi \mapsto u$ are bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$, in view of our assumption on P, the right hand side g_1 of (5.32) is in $H^{\frac{7}{2}}(\partial D)$ with the mapping $\varphi \mapsto g_1$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{7}{2}}(\partial D)$. Because the mappings $\varphi \mapsto U$ and $\varphi \mapsto u$ also are bounded from $H^{-\frac{1}{2}}(\partial D)$ into $L^2_\Delta(D)$ The right hand side g_2 of (5.33) is in $H^{\frac{3}{2}}(\partial D)$ with the mapping $\varphi \mapsto g_2$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{3}{2}}(\partial D)$.

Furthermore, it is straightforward to check that

$$\Delta\Delta U = F(u, v)$$

where

$$F(u,v) := -k^2 k_n^2 u - (k^2 + k_n^2)\Delta u$$
$$-\frac{k^2 - k_n^2}{|k|^2 - |k_n|^2}\left[|k|^2 |k_n|^2 v - (|k|^2 + |k_n|^2)\Delta v\right]$$

belongs to $L^2(D)$ with the mapping $\varphi \to F$ bounded from $H^{-\frac{1}{2}}(\partial D)$ to $L^2(D)$.

Now, we can use Lemma 5.2 again. Applying it first for ΔU we obtain that $\Delta U \in H^2(D)$ with the mapping $\varphi \mapsto \Delta U$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$. Applying the lemma then for U shows that $U \in H^4(D)$ with the mapping $\varphi \mapsto U$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^4(D)$. Therefore, the mapping $\varphi \mapsto \partial_\nu U$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{5}{2}}(\partial D)$. Now, in view of

$$\frac{\partial U}{\partial \nu} = M(k) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2}\widetilde{M}(|k|)$$

the statement of the theorem follows from the compact embedding of $H^{\frac{5}{2}}(\partial D)$ into $H^{\frac{1}{2}}(\partial D)$. □

Noting that $M(k)$ is analytic in k since the kernels of S_k and K'_k are analytic in k, now Theorems 5.5 and 5.6 imply the following final result. From this, in particular, we can again reestablish the discreteness of the set of transmission eigenvalues for the special case of a constant refractive index and the finite multiplicity of the transmission eigenvalues.

Theorem 5.7 *$M(k) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is a Fredholm operator with index zero and analytic in $\{k \in \mathbb{C} : \operatorname{Re} k > 0$ and $\operatorname{Im} k \geq 0\}$.*

Cakoni and Kress [5] also used their boundary integral equation formulation for actual computations of transmission eigenvalues with the aid Beyn's [2] algorithm mentioned above. Comparing the computational costs for Beyn's algorithm as applied to Cossonnière and Haddar's two-by-two system it can be shown that the approach presented here reduces the costs by about 50 percent. For details of the implementation and numerical results we refer to [5] and for a very recent extension of this approach to the Maxwell equations including numerical results for transmission eigenvalues in three dimensions we refer the reader to [4].

References

[1] Altundag, A. and Kress, R.; An iterative method for a two-dimensional inverse scattering problem for a dielectric. Jour. on Inverse and Ill-Posed Problem **20**, 575–590 (2012) .

[2] Beyn, W.-J.: An integral method for solving nonlinear eigenvalue problems. Linear Algebra Appl. **436**, 3839–3863 (2012).

[3] Cakoni, F., Hsiao, G. C., and Wendland, W. L.: On the boundary integral equations method for a mixed boundary value problem of the biharmonic equation. Complex Var. Theory Appl. **50**, 681–696 (2005).

[4] Cakoni, F., Ivanyshyn, O., Kress, R., and Le Louër, F. : A boundary integral equation for the transmission eigenvalue problem for Maxwell's equation. Math. Meth. Appl. Math. **41**, 1316–1330 (2018).

[5] Cakoni, F. and Kress, R.: A boundary integral equation method for the transmission eigenvalue problem. Applicable Analysis **96**, 23–38 (2017).

[6] Colton, D., Kirsch, A., and Päivärinta, L.: Far field patterns for acoustic waves in an inhomogeneous medium. SIAM J. Math. Anal. **20**, 1472–1483 (1989).

[7] Colton, D. and Kress, R.: *Integral Equation Methods in Scattering Theory.* Classics in Applied Mathematics Vol. 72, SIAM, Philadelphia, 2013.

[8] Colton, D. and Kress, R.: *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd. ed. Springer, New York, 2013.

[9] Cossonnière, A. and Haddar, H.: Surface integral formulation of the interior transmission problem. J. Integral Equations Applications **25**, 341–376 (2013).

[10] Ivanyshyn, O., Kress, R., and Serranho, P.: Huygens' principle and iterative methods in inverse obstacle scattering. Adv. Comput. Math. **33**, 413–429 (2010).

[11] Kirsch, A.: The denseness of the far field patterns for the transmission problem. IMA J. Appl. Math. **37**, 213–225 (1986).

[12] Kirsch, A.: Surface gradient and continuity properties of some integral operators in classical scattering theory. Math. Meth. Appl. Sci. **11**, 789–804 (1989).

[13] Kirsch, A. and Lechleiter, A.: The inside-outside duality for scattering problems by inhomogeneous media. Inverse Problems **29**, 104011 (2013).

[14] Kleefeld, A.: A numerical method to compute interior transmission eigenvalues. Inverse Problems, **29**, 104012 (2013).

[15] Kleinman, R. and Martin, P.: On single integral equations for the transmission problem of acoustics. SIAM J. Appl. Math. **48**, 307–325 (1988).

[16] Kress, R. and Roach, G.: Transmission problems for the Helmholtz equation Jour. Math. Phys. **19** 1433–1437 (1978).

[17] Kress, R. and Rundell, W.: Nonlinear integral equations and the iterative solution for an inverse boundary value problem. Inverse Problems **21**, 1207–1223 (2005).

[18] Kress, R. and Rundell, W.: A nonlinear integral equation and an iterative algorithm for an inverse source problem. Jour. Integral Equations and Appl. **26**, 179–198 (2015).

[19] Kress, R. and Rundell, W.: Inverse scattering for shape and impedance revisited. Jour. Integral Equations and Appl. **30**, 293–311 (2018).

[20] McLean, W.: *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.

[21] Lechleiter, A. and Peters, S.: The inside-outside duality for inverse scattering problems with near field data. Inverse Problems **31**, 085004 (2015).

[22] Lechleiter, A. and Rennoch, M.: Inside-outside duality and the determination of electromagnetic interior transmission eigenvalues. SIAM J. Math. Anal. **47**, 684–705 (2015).

[23] Päivärinta, L. and Sylvester, J.: Transmission eigenvalues. SIAM J. Math. Anal. **40**, 738–758 (2008).

[24] Panich, O.I.: On the question of the solvability of the exterior boundary-value problems for the wave equation and Maxwell's equations. Usp. Mat. Nauk **20A**, 221–226 (1965) (in Russian).

[25] Rynne, B.P. and Sleeman, B.D.: The interior transmission problem and inverse scattering from inhomogeneous media. SIAM J. Math. Anal. **22**, 1755–1762 (1991).

[26] Taylor, M.E.: *Partial Differential Equations.* Springer, New York, 1996.