Some Old and Some New Results in Inverse Obstacle Scattering

Rainer Kress∗

Abstract

We will survey on uniqueness, that is, identifiability and on reconstruction issues for inverse obstacle scattering for time-harmonic acoustic and electromagnetic waves. In the first part we begin by presenting two classical uniqueness proofs and after that proceed with two recent uniqueness results for inverse obstacle scattering subject to a generalized impedance boundary condition. Then we proceed with an iterative reconstruction algorithm via non-linear boundary integral equations for the case of the generalized impedance boundary condition. In the final part we present new integral equation formulations for transmission eigenvalues that play an important role through their connections with the linear sampling method and the factorization method for inverse scattering problems for penetrable objects.

1 Uniqueness in inverse obstacle scattering

Scattering theory is concerned with the effects that obstacles and inhomogeneities have on the propagation of waves and in particular time-harmonic waves. For simplicity, we focus our attention on acoustic waves and only give passing references to electromagnetic waves. Throughout the paper we will consider scattering objects within a homogeneous background that are described by a bounded domain $D \subset \mathbb{R}^m$ for $m = 2, 3$ with a connected $C^2$ smooth boundary $\partial D$ and can be either impenetrable or penetrable. We note

∗Institut für Numerische und Angewandte Mathematik, Universität Göttingen, Göttingen, Germany (kress@math.uni-goettingen.de)
that the smoothness assumption, in principle, can be weakened and Lipschitz boundaries can also be allowed.

Given as incident field a plane wave \( u^i(x) = e^{ikx \cdot d} \) propagating in the direction \( d \in S^{m-1} := \{ x \in \mathbb{R}^m : |x| = 1 \} \), the simplest obstacle scattering problem is to find the total field \( u \in H^1_{\text{loc}}(\mathbb{R}^m \setminus \bar{D}) \) as superposition \( u = u^i + u^s \) of the incident field and the scattered field \( u^s \) such that the Helmholtz equation

\[
\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^m \setminus \bar{D}
\]  

with positive wave number \( k \) and the boundary condition

\[
u = 0 \quad \text{on} \quad \partial D
\]

are satisfied together with the Sommerfeld radiation condition

\[
\lim_{r \to \infty} r^{\frac{m-1}{2}} \left( \frac{\partial u^s}{\partial r} - iku^s \right) = 0, \quad r = |x|,
\]

uniformly for all directions. The homogeneous Dirichlet boundary condition (1.2) corresponds to a sound-soft obstacle. Boundary conditions other than (1.2) to be considered are the homogeneous Neumann or sound-hard boundary condition or the impedance boundary condition, also known as Leontovich boundary condition,

\[
\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \quad \text{on} \quad \partial D
\]

where \( \nu \) is the unit outward normal to \( \partial D \) and \( \lambda \) is a given continuous complex valued function with non-negative real part. In addition to plane waves other incident fields can be considered.

The radiation condition (1.3) was introduced by Sommerfeld in 1912 to characterize an outward energy flux. It is equivalent to the asymptotic behavior

\[
u^s(x) = \frac{e^{ik|x|}}{|x|^{\frac{m-1}{2}}} \left[ u_\infty(\hat{x}) + O\left( \frac{1}{|x|} \right) \right], \quad |x| \to \infty,
\]

uniformly for all directions \( \hat{x} = x/|x| \) and where \( u_\infty \) is defined on \( S^{m-1} \) and is called the far field pattern of \( u^s \). Solutions to the Helmholtz equation satisfying (1.3) are called radiating. For plane wave incidence we will indicate the dependence of the far field pattern on the incident direction \( d \) by writing \( u_\infty(\hat{x}, d) \).
Uniqueness of a solution to the obstacle scattering problem is a consequence of the following fundamental lemma which is due to Rellich (1943) and Vekua (1943) and is known as Rellich’s lemma. This lemma later on also plays an essential role in connection with uniqueness for the inverse scattering problems. For a proof we refer to [16]. Existence of a solution was first established by Vekua, Weyl and Müller in the 1950s by a boundary integral equation approach.

**Lemma 1.1** Any radiating solution $u^s \in H^1_{\text{loc}}(\mathbb{R}^m \setminus \bar{D})$ to the Helmholtz equation with far field pattern $u_\infty = 0$ vanishes identically in $\mathbb{R}^m \setminus \bar{D}$.

Given the incident field $u^i(x) = e^{ikx \cdot d}$, the basic inverse obstacle scattering problem is to determine $D$ from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for all observation directions $\hat{x} \in S^{m-1}$ and one or a few incident directions $d \in S^{m-1}$ and a fixed wave number $k$. This inverse problem serves as a model problem for analyzing inverse scattering techniques in nondestructive evaluation such as radar, sonar, ultrasound imaging, seismic imaging etc. However, we note that in practical applications the inverse scattering problem will never occur in the above idealized form. In particular, the far field pattern or some other measured quantity of the scattered wave will be available only for observation directions within a limited aperture either in the near or in the far field region.

We begin by noting that the inverse obstacle scattering problem is nonlinear in the sense that the scattered wave depends nonlinearly on the scatterer $D$. More importantly, it is ill-posed since the determination of $D$ does not depend continuously on the far field pattern in any reasonable norm.

We illustrate the nonlinearity and ill-posedness of the inverse obstacle scattering problem by looking at a simple example. For this we consider as incident field the entire solution $v^i$ to the Helmholtz equation given by

$$v^i(x) = \frac{\sin k|x|}{|x|}, \quad x \in \mathbb{R}^3. \quad (1.5)$$

Because of

$$\frac{\sin k|x|}{|x|} = \frac{k}{4\pi} \int_{S^2} e^{ikx \cdot d} ds(d), \quad x \in \mathbb{R}^3,$$

the field $v^i$ is a superposition of plane waves. For $D$ a sound-soft ball of radius $R$ centered at the origin the scattered wave is given by

$$v^s(x) = \frac{-\sin kR}{e^{ikR} |x|} e^{ik|x|}, \quad |x| \geq R. \quad (1.6)$$
This leads to the total wave
\[ v(x) = \frac{1}{|x|e^{ikR}} \sin k(|x| - R), \quad |x| \geq R, \quad (1.7) \]
and the constant far field pattern
\[ v_\infty(\hat{x}) = -\frac{\sin kR}{e^{ikR}}, \quad \hat{x} \in S^2. \quad (1.8) \]
Therefore, assuming the a priori information that the scatterer is a ball centered at the origin, (1.8) provides a non-linear equation for determining the radius \( R \).

Concerning the ill-posedness we consider a perturbed far field pattern
\[ v^\delta(\hat{x}) = -\frac{\sin kR}{e^{ikR}} + \delta Y_n(\hat{x}) \]
with some \( \delta \in \mathbb{R} \) and a spherical harmonic \( Y_n \) of degree \( n \). Then, in view of the asymptotic behavior of the spherical Hankel functions for large argument, the corresponding total field is given in terms of an outgoing spherical wave function
\[ v^\delta(x) = \frac{\sin k(|x| - R)}{e^{ikR} |x|} + \delta k i^{n+1} h_n^{(1)}(k|x|)Y_n \left( \frac{x}{|x|} \right) \]
with the spherical Hankel function \( h_n^{(1)} \) of order \( n \) and of the first kind (see in [16, Theorem 2.16]). This implies
\[ v^\delta(x) = \delta k i^{n+1} h_n^{(1)}(kR)Y_n \left( \frac{x}{|x|} \right), \quad |x| = R, \]
and consequently, by the asymptotics of the spherical Hankel functions for larger order, it follows that
\[ |v^\delta(x)| \approx \delta k \left( \frac{2n}{ekR} \right)^n Y_n \left( \frac{x}{|x|} \right), \quad |x| = R. \]
This illustrates that small changes in the data \( v_\infty \) can cause large errors in the solution of the inverse problem, or a solution even may not exist anymore since \( v^\delta \) may fail to have a closed surface as zero level surface.

From a functional analytic point of view the ill-posedness is a consequence of the compactness property of the mapping \( \partial D \mapsto u_\infty \) (see [16, Theorem 5.7]).

The following classical uniqueness result is due to Schiffer.
Theorem 1.2 Assume that $D_1$ and $D_2$ are two sound-soft scatterers such that their far field patterns coincide for an infinite number of incident plane waves with distinct directions and one fixed wave number. Then $D_1 = D_2$.

Proof. Assume that $D_1 \neq D_2$. By Rellich’s lemma for each incident wave $u^i$ the scattered waves $u^s_1$ and $u^s_2$ for the obstacles $D_1$ and $D_2$ coincide in the unbounded component $G$ of the complement of $D_1 \cup D_2$. Without loss of generality, one can assume that $D^* := (\mathbb{R}^m \setminus G) \setminus \bar{D}_2$ is nonempty. Then $u^s_2$ is defined in $D^*$, and the total wave $u = u^i + u^s_2$ satisfies the Helmholtz equation in $D^*$ and the homogeneous boundary condition $u = 0$ on $\partial D^*$. Hence, $u$ is a Dirichlet eigenfunction of $-\Delta$ in the domain $D^*$ with eigenvalue $k^2$. The proof is now completed by showing that the total fields for distinct incident plane waves are linearly independent, since this contradicts the fact that for a fixed eigenvalue the Dirichlet eigenspace of $-\Delta$ in $H^1_0(D^*)$ has finite dimension. □

Schiffer’s uniqueness result was obtained around 1960 and appeared as a private communication in the monograph by Lax and Philipps [33]. This is notable since nowadays in a time of permanent evaluation and competition for grants nobody would want to give away such a valuable result as a private communication. Noting that the proof presented in [33] contains a slight technical fault since the fact that the complement of $D_1 \cup D_2$ might be disconnected was overlooked, it is comforting to observe that even eminent authors can have errors in their books.

Using the strong monotonicity property of the Dirichlet eigenvalues of $-\Delta$, extending Schiffer’s ideas in 1983 Colton and Sleeman [17] showed that a sound-soft scatterer is uniquely determined by the far field pattern for one incident wave under the a priori assumption that it is contained in a ball of radius $R$ such that $kR < c_{m,0}$. Here, $c_{2,0}$ and $c_{3,0} = \pi$ are the smallest zeros of the Bessel function $J_0$ and the spherical Bessel function $j_0$, respectively, representing the smallest eigenvalue for the unit ball which is a simple eigenvalue. Hence, exploiting the fact that the wave functions are complex valued with linearly independent real and imaginary parts, in 2005 Gintides [21] improved this bound to $kR < c_{m,1}$ in terms of the smallest positive zeros $c_{2,1}$ and $c_{3,1} = 4.49\ldots$ of the Bessel function $J_1$ and the spherical Bessel function $j_1$, respectively. For other than the Dirichlet boundary condition there is no analogue to the results in [17, 21] since there is no monotonicity property for the eigenvalues of $-\Delta$ for other boundary conditions.
Although there is widespread belief that the far field pattern for one single incident direction and one single wave number determines the scatterer without any additional a priori information, establishing this result still remains a challenging open problem. To illustrate the difficulty of a proof, we consider scattering of the entire solution $v^i$ given by (1.5) from a sound-soft ball $D$ of radius $R$ centered at the origin. Then from (1.7) we observe that the total field $v$ vanishes on the spheres with radius $R_n := R + n\pi/k$ centered at the origin for all integers $n$ for which $R_n > 0$. This indicates that proving uniqueness for the inverse obstacle scattering problem with one single incident plane wave needs to incorporate special features of the incident field.

Starting in 2003 in a series of papers by Alessandrini, Cheng, Liu, Rondi and Yamamoto [1, 13, 34, 35] it was established that one incident plane wave is sufficient to uniquely determine a sound-soft polyhedron. Assuming that there exist two polyhedral scatterers producing the same far field pattern for one incident plane wave, the main idea of their proofs is to use the reflection principle to construct a zero field line extending to infinity. However, in view of the fact that the scattered wave tends to zero uniformly at infinity, this contradicts the property that the incident plane wave has modulus one everywhere. These results for the polyhedron have analogs for other boundary conditions and also for electromagnetic waves.

The finiteness of the dimension of the eigenspaces for eigenvalues of $-\Delta$ for the Neumann or impedance boundary condition requires the boundary of the intersection $D^*$ from the proof of Theorem 1.2 to be sufficiently smooth which, in general, is not the case. Therefore, there does not exist an immediate extension of Schiffer’s approach to other boundary conditions.

Assuming that two different scatterers have the same far field patterns for all incident directions, in 1990 Isakov [23] obtained a contradiction by considering a sequence of solutions with a singularity moving towards a boundary point of one scatterer that is not contained in the other scatterer. He used weak solutions and the proofs are technically involved. During a hike in the Dolomites, on a long downhill walk in 1993 Kirsch and Kress [29] realized that these proofs can be simplified by using classical solutions rather than weak solutions and by obtaining the contradiction by considering point-wise limits of the singular solutions rather than limits of $L^2$ norms. For boundary conditions of the form $Bu = 0$ on $\partial D$, where $Bu = u$ for a sound-soft scatterer and $Bu = \partial u/\partial \nu + ik\lambda u$ for the impedance boundary condition one can state the following theorem. For its proof and for later use throughout
the remainder of the paper we introduce the notation

\[\Phi_k(x,y) := \begin{cases} 
\frac{1}{4\pi} e^{ik|x-y|}, & m = 3, \\
\frac{i}{4} H_0^{(1)}(k|x-y|), & m = 2,
\end{cases}\]

for the fundamental solution of the Helmholtz equation, where \(H_0^{(1)}\) denotes the Hankel function of order zero of the first kind.

**Theorem 1.3** Let two scatterers \(D_1\) and \(D_2\) with boundary conditions \(B_1\) and \(B_2\) have the same far field patterns for an infinite number of incident plane waves with distinct directions and one fixed wave number. Then \(D_1 = D_2\) and \(B_1 = B_2\).

**Proof.** In addition to scattering of plane waves, we also consider scattering of point sources \(\Phi_k(\cdot,z)\) with source location \(z\) in \(\mathbb{R}^m\setminus \bar{D}\). We will make use of the mixed reciprocity relation

\[w^s_\infty(-d,z) = \gamma_m u^s(z,d), \quad z \in \mathbb{R}^m \setminus \bar{D}, \ d \in S^2,\]

which, for scattering of a point source located in \(z\), connects the far field pattern \(w^s_\infty\) of the scattered wave in observation direction \(-d\) with the scattered wave \(u^s\) for plane wave incidence in direction \(d\) evaluated at \(z\) and where

\[\gamma_2 = \frac{e^{i\pi}}{\sqrt{8\pi k}} \quad \text{and} \quad \gamma_3 = \frac{1}{4\pi}.\]

(see [16, Theorem 3.16]). Using twice Rellich’s lemma and (1.10) from the assumption of the theorem one can deduce that \(w^s_1(x,z) = w^s_2(x,z)\) for all \(x,z \in G\). Here, we assume again that \(D_1 \neq D_2\) and that \(G\) is defined as in the proof of Theorem 1.2 and \(w_1\) and \(w_2\) are the scattered waves for point source incidence for the obstacles \(D_1\) and \(D_2\), respectively. Now a contradiction can be obtained choosing \(x \in \partial G\) such that \(x \in \partial D_1\) and \(x \not\in \partial D_2\) and a sequence \(z_n \in G\) such that \(z_n \to x\) as \(n \to \infty\). Hence \(D_1 = D_2\) and then \(B_1 = B_2\) follows from \(u_1 = u_2\). \(\square\)

The idea of the proof for Theorem 1.3 has been applied to a number of other boundary conditions such as for example a generalized impedance
boundary condition by Bourgeois, Chaulet and Haddar [4] and other differential equations such as the Maxwell equations for electromagnetic waves. The generalized impedance boundary condition will be the subject of the next section.

2 Generalized impedance boundary condition

Given the plane wave \( u^i(x) = e^{ikx} \) as incident field, the obstacle scattering problem with the generalized impedance boundary condition (GIBC) consists in finding the total field \( u \in H^2_{\text{loc}}(\mathbb{R}^m \setminus \bar{D}) \) as superposition \( u = u^i + u^s \) of the incident field and the scattered field \( u^s \) such that \( u \) satisfies the Helmholtz equation (1.1) and the boundary condition

\[
\frac{\partial u}{\partial \nu} + ik(\lambda u - \text{Div} \mu \text{Grad} u) = 0 \quad \text{on } \partial D \tag{2.1}
\]

together with the Sommerfeld radiation condition (1.3). Here, Grad and Div denote the surface gradient and surface divergence on \( \partial D \) and \( \mu \in C^2(\partial D) \) and \( \lambda \in C^1(\partial D) \) are given complex valued functions with non-negative real parts. In the two dimensional case both Grad and Div correspond to the tangential derivative \( d/ds \). Recall that the unit normal vector \( \nu \) to \( \partial D \) is directed towards the exterior of \( D \). The boundary condition (2.1) requires \( u \in H^2_{\text{loc}}(\mathbb{R}^m \setminus \bar{D}) \) and, in view of \( u|_{\partial D} \in H^{3/2}(\partial D) \) has to be understood in the weak sense, that is,

\[
\int_{\partial D} \left( \eta \frac{\partial u}{\partial \nu} + ik\lambda \eta u + k\mu \text{Grad} \eta \cdot \text{Grad} u \right) ds = 0 \tag{2.2}
\]

for all \( \eta \in H^{3/2}(\partial D) \).

We note that the classical Leontovich boundary condition (1.4) is contained in (2.1) as the special case where \( \mu = 0 \). As compared with the Leontovich condition, the wider class of impedance conditions (2.1) provides more accurate models, for example, for imperfectly conducting obstacles (see [20, 22, 39]). For further interpretation of the generalized impedance boundary condition we refer to [3, 4, 5] where the direct and the inverse scattering problem are analyzed by variational methods. Here, we will base our analysis on boundary integral equations.
2.1 The direct problem

Extending the analysis in [32] from the two dimensional to the three dimensional case we briefly sketch an existence analysis via boundary integral equations.

**Theorem 2.1** Any solution \( u \in H^2_{\text{loc}}(\mathbb{R}^m \setminus \bar{D}) \) to (1.1) and (2.1) satisfying the Sommerfeld radiation condition vanishes identically.

**Proof.** Inserting \( \eta = \bar{u}|_{\partial D} \) in the weak form (2.2) of the boundary condition we obtain that

\[
\int_{\partial D} \bar{u} \frac{\partial u}{\partial \nu} \, ds = -ik \int_{\partial D} \{ \lambda |u|^2 + \mu |\text{Grad} \, u|^2 \} \, ds.
\]

Hence in view of our assumption \( \text{Re} \lambda \geq 0 \) and \( \text{Re} \mu \geq 0 \) we can conclude that

\[
\text{Im} \int_{\partial D} \bar{u} \frac{\partial u}{\partial \nu} \, ds \leq 0
\]

and from this and the radiation condition the statement of the theorem follows from Theorem 2.13 in [16]. \( \square \)

**Corollary 2.2** The obstacle scattering problem with generalized impedance boundary condition has at most one solution.

For the existence analysis, following [16, Section 3.1] we introduce the classical boundary integral operators in scattering theory given by the single- and double-layer operators

\[
(S_k \varphi)(x) := 2 \int_{\partial D} \Phi_k(x,y) \varphi(y) \, ds(y)
\]  

(2.3)

and

\[
(K_k \varphi)(x) := 2 \int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu(y)} \varphi(y) \, ds(y)
\]  

(2.4)

and the corresponding normal derivative operators

\[
(K'_k \varphi)(x) := 2 \int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu(x)} \varphi(y) \, ds(y)
\]  

(2.5)
and
\[(T_k \varphi)(x) := 2 \frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi_k(x, y)}{\partial \nu(y)} \varphi(y) \, ds(y)\] (2.6)
for \(x \in \partial D\). We note that for \(\partial D \in C^{4,\alpha}\) the operators \(S_k : H^{\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)\), and \(K'_k : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)\) are bounded (see [26, 36]). (The subscript \(k\) for the operators will be needed in the next section.)

We seek the solution in the form of a single-layer potential for the scattered wave
\[u^s(x) = \int_{\partial D} \Phi_k(x, y) \varphi(y) \, ds(y), \quad x \in \mathbb{R}^m \setminus \overline{D},\] (2.7)
with density \(\varphi \in H^{\frac{1}{2}}(\partial D)\) and note that the regularity \(\varphi \in H^{\frac{1}{2}}(\partial D)\) guarantees that \(u \in H^2_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D})\) (see [36]). From the jump relations for single-layer potentials (see [16, Theorem 3.1]) we observe that the boundary condition (2.1) is satisfied provided \(\varphi\) solves the integro-differential equation
\[\varphi - K'_k \varphi - ik (\lambda - \text{Div} \mu \text{Grad}) S_k \varphi = g\] (2.8)
where we set
\[g := 2 \frac{\partial u^i}{\partial \nu} \bigg|_{\partial D} + 2ik (\lambda - \text{Div} \mu \text{Grad}) u^i|_{\partial D}\] (2.9)
in terms of the incident wave \(u^i\). After defining a bounded linear operator \(A_k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)\) by
\[A_k \varphi := \varphi - K'_k \varphi - ik (\lambda - \text{Div} \mu \text{Grad}) S_k \varphi\] (2.10)
we summarize the above into the following theorem.

**Theorem 2.3** The single-layer potential (2.7) solves the scattering problem (1.1), (2.1) and (1.3) provided the density \(\varphi\) satisfies the equation
\[A_k \varphi = g.\] (2.11)

**Lemma 2.4** The modified Laplace–Beltrami operator given by
\[L \varphi := - \text{Div Grad} \varphi + \varphi\] (2.12)
is an isomorphism from \(H^{\frac{3}{2}}(\partial D)\) onto \(H^{-\frac{1}{2}}(\partial D)\).
Proof. In view of the Gauss surface divergence theorem the surface divergence of a vector field \( w \in L^2(\partial D) \) is given by the duality pairing
\[
(Div w, \psi) = -(w, Grad \psi), \quad \psi \in H^1(\partial D).
\]
This in turn implies
\[
(L\varphi, \psi) = (Grad \varphi, Grad \psi) + (\varphi, \psi)
\]
for \( \varphi, \psi \in H^1(\partial D) \) and consequently
\[
\|L\varphi\|_{H^{-1}(\partial D)} = \sup_{\psi \in H^1(\partial D)} |(L\varphi, \psi)| \leq C_1 \|\varphi\|_{H^1(\partial D)} \quad (2.13)
\]
and
\[
|(L\varphi, \varphi)| \geq C_2 \|\varphi\|_{H^1(\partial D)}^2 \quad (2.14)
\]
for all \( \varphi \in H^1(\partial D) \) and some positive constants \( C_1 \) and \( C_2 \). From (2.13) we have that \( L : H^1(\partial D) \to H^{-1}(\partial D) \) is bounded and from (2.14) we can conclude that it is injective and has closed range. Assuming that it is not surjective implies the existence of some \( \chi \neq 0 \) in the dual space \( (H^{-1}(\partial D))^* = H^1(\partial D) \) that vanishes on \( L(H^1(\partial D)) \), that is,
\[
(L\varphi, \chi) = 0
\]
for all \( \varphi \in H^1(\partial D) \). Choosing \( \varphi = \chi \) yields \( (L\chi, \chi) = 0 \) and from (2.14) we obtain the contradiction \( \chi = 0 \). Hence \( L : H^1(\partial D) \to H^{-1}(\partial D) \) is bijective and consequently by Banach’s open mapping theorem it is an isomorphism.

Clearly the operator \( L : H^2(\partial D) \to L^2(\partial D) \) is bounded and proceeding as in the proof of Theorem 1.3 in [41] using elliptic regularity analysis it can be shown that its inverse is also bounded (see also Lemma 3.2 below). Now the statement of the lemma follows by Sobolev space interpolation.

Lemma 2.5 The operator
\[
A_k - ik\mu LS_k : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)
\]
is compact.

Proof. The boundedness of the operators \( S_k : H^{1/2}(\partial D) \to H^{3/2}(\partial D) \) and \( K'_k : H^{3/2}(\partial D) \to H^1(\partial D) \) mentioned above and our assumption \( \lambda \in C^1(\partial D) \)
imply that all terms in the sum (2.10) defining the operator $A_k$ are bounded from $H^\frac{1}{2}(\partial D)$ into $H^\frac{1}{2}(\partial D)$ except the term
$$\varphi \mapsto ik\text{ Div }\mu \text{ Grad }S_k \varphi.$$ Therefore, after splitting
$$\text{Div }\mu \text{ Grad }S_k \varphi = \mu \text{ Div Grad }S_k \varphi + \text{Grad }\mu \cdot \text{Grad }S_k \varphi$$
we observe that the operator $A_k - ik\mu LS_k : H^\frac{1}{2}(\partial D) \to H^\frac{1}{2}(\partial D)$ is bounded since we assumed $\mu \in C^2(\partial D)$. Hence the statement of the lemma follows from the compact embedding of $H^\frac{1}{2}(\partial D)$ into $H^{-\frac{1}{2}}(\partial D)$.

**Theorem 2.6** Assume that $|\mu| > 0$ and that $k^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $D$. Then for each $g \in H^{-\frac{1}{2}}(\partial D)$ the equation (2.11) has a unique solution $\varphi \in H^\frac{1}{2}(\partial D)$ and this solution depends continuously on $g$.

**Proof.** Since under our assumption on $k$ the operator $S_k : H^\frac{1}{2}(\partial D) \to H^\frac{3}{2}(\partial D)$ is an isomorphism, by Lemma 2.4 and our assumptions on $\mu$ the operator $ik\mu LS_k : H^\frac{1}{2}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ also is an isomorphism. Therefore, in view of Lemma 2.5, by the Riesz theory it suffices to show that the operator $A_k$ is injective. Assume that $\varphi \in H^\frac{1}{2}(\partial D)$ satisfies $A_k \varphi = 0$. Then, by Theorem 2.3 the single-layer potential $u$ defined by (2.7) solves the scattering problem for the incident wave $u^i = 0$. Hence, by the uniqueness Theorem 2.1 we have $u = 0$ in $\mathbb{R}^m \setminus \overline{D}$. Taking the boundary trace of $u$ it follows that $S_k \varphi = 0$ and consequently $\varphi = 0$. □

To remedy the failure of the single-layer potential approach at the interior Dirichlet eigenvalues, as in the case of the classical impedance condition, we modify it into the form of a combined single- and double-layer potential for the scattered wave
$$u^s(x) = \int_{\partial D} \left\{ \Phi_k(x, y) + i \frac{\partial \Phi_k(x, y)}{\partial \nu(y)} \right\} \varphi(y) ds(y), \quad x \in \mathbb{R}^m \setminus \overline{D}, \quad (2.15)$$
with density $\varphi \in H^\frac{3}{2}(\partial D)$. The boundary condition (2.1) is satisfied provided $\varphi$ solves the integro-differential equation
$$\varphi - K_k \varphi - iT_k \varphi - ik (\lambda - \text{Div }\mu \text{ Grad}) (S_k \varphi + i \varphi + iK_k \varphi) = g \quad (2.16)$$
with $g$ given by (2.9). Then with the same ideas as applied in the analysis of the integro-differential equation (2.8) the following existence result can be established. For the two-dimensional case we refer to [32].
Theorem 2.7 Under the assumption $|\mu| > 0$, the direct scattering problem with generalized impedance boundary condition has a unique solution.

For the numerical solution in two dimensions collocation methods based on numerical quadratures using trigonometric polynomial approximations are the most efficient methods for solving boundary integral equations for scattering problems in planar domains with smooth boundaries (see [16, Section 3.5]). Here, additionally an approximation is required for the operator $\varphi \mapsto d/ds \mu d\varphi/ds$ as the new feature in the integro-differential equations for the generalized impedance boundary condition. For this, we recommend trigonometric differentiation. It can be shown that this approach leads to spectral convergence for infinitely smooth boundaries and impedance coefficients. Details on this, including numerical examples, are presented in [32].

In three dimension for smooth boundaries that are homoemorphic to the unit sphere numerical methods with spectral convergence for the boundary integral equations for scattering problems can be obtained via approximations by spherical harmonics by means of a hyperinterpolation operator on the unit sphere (see [16, Section 3.6]). This operator, in principle, can also be employed to approximate the surface gradient and the surface divergence. However, a numerical implementation of this idea at the time of this writing has not yet been done.

2.2 The inverse problem

The most general inverse scattering problem is the inverse shape and impedance problem to determine $\partial D$, $\mu$ and $\lambda$ from a knowledge of a number of far field patterns $u_\infty$ of solutions $u$ to (1.1), (2.1) and (1.3). Here we will be only concerned with two less general cases, namely the inverse shape problem and the inverse impedance problem. The inverse shape problem consists in determining $\partial D$ knowing the impedance coefficients $\mu$ and $\lambda$. With the roles reversed, the inverse impedance problem requires to determine the impedance functions $\mu$ and $\lambda$ for a known shape $\partial D$.

We briefly discuss the uniqueness issue and begin with the inverse impedance problem. In two dimensions, Cakoni and Kress [11] have shown that for a given shape $\partial D$ three far field patterns corresponding to the scattering of three plane waves with different incident directions uniquely determine the impedance functions $\mu$ and $\lambda$. For two cylindrical wave functions as incident fields in [32] a counter example is given where different impedance
coefficients lead to the same two far field patterns. The uniqueness proof is sort of constructive and can be employed for an algorithm for the solution of the inverse impedance problem. For details and numerical reconstructions we refer to [32].

The following uniqueness result for the full inverse shape and impedance problem (in two and three dimensions) was obtained by Bourgeois, Chaulet and Haddar [4] by using the method presented in Theorem 1.3.

**Theorem 2.8** Both the shape and the impedance functions of a scattering obstacle with generalized impedance boundary condition are uniquely determined by the far field patterns for an infinite number of incident plane waves with different incident directions and one fixed wave number.

We now outline an iterative algorithm for approximately solving the inverse shape problem which extends the method proposed by Johansson and Sleeman [24] for sound-soft or perfectly conducting obstacles. For this, we introduce the operator

\[ S_\infty : H^\frac{1}{2}(\partial D) \to L^2(S^{m-1}) \]

by

\[ (S_\infty \varphi)(\hat{x}) := \gamma_m \int_{\partial D} e^{-ik\hat{x} \cdot y} \varphi(y) \, ds(y), \quad \hat{x} \in S^{m-1}, \]  

where \( \gamma_m \) is given by (1.11). Then, in view of the asymptotic for the Hankel functions, the far field pattern for the solution to the scattering problem (1.1), (2.1) and (1.3) is given by

\[ u_\infty = S_\infty \varphi \]  

in terms of the solution \( \varphi \) to (2.8). Hence we can state the following theorem as theoretical basis of the inverse algorithm.

**Theorem 2.9** For a given incident field \( u^i \) and a given far field pattern \( u_\infty \), assume that \( \partial D \) and the density \( \varphi \) satisfy the system

\[ \varphi - K'_k \varphi - ik (\lambda - \text{Div} \mu \, \text{Grad}) S_k \varphi = g \]  

and

\[ S_\infty \varphi = u_\infty \]  

where \( g \) is given in terms of the incident field by (2.9). Then \( \partial D \) solves the inverse shape problem. (Note that the operators \( S_k, K'_k \) and \( S_\infty \) and the right hand side \( g \) depend on \( \partial D \).)
The operator $S_\infty$ is compact with exponentially decreasing singular values and therefore the linear equation (2.20) is severely ill-posed reflecting the ill-posedness of the inverse shape problem. We denote this equation as the data equation. Note that the system (2.19)–(2.20) is linear with respect to the density $\varphi$ and nonlinear with respect to the boundary $\partial D$. This opens up a variety of approaches to solve (2.19)–(2.20) by linearization and iteration. Here, we are going to proceed as follows. Given an approximation for the unknown $\partial D$ we solve the equation (2.19) that we denote as the field equation for the unknown density $\varphi$, that is, we solve the forward problem for the approximate boundary. Then, keeping $\varphi$ fixed we linearize the data equation (2.20) with respect to the boundary to update the approximation.

To describe this in more detail, for simplicity, we assume $\partial D$ to be star-like with respect to the origin, i.e., $\partial D$ is represented in the parametric form

$$\partial D := \{ r(z) : z \in S^{m-1} \} \quad (2.21)$$

with a positive function $r \in C^2(S^{m-1})$. Then, indicating its dependence on the boundary $\partial D$, the parametrized form of the operator $S_\infty$ is given by

$$(\tilde{S}_\infty(\psi, r))(\hat{x}) = \gamma_m \int_{S^{m-1}} e^{-ikr(y) \hat{x} \cdot \hat{y}} J_r(y) \psi(y) \, ds(y), \quad \hat{x} \in S^{m-1}. \quad (2.22)$$

Here $J_r$ is the Jacobian of the mapping (2.21) given by $J_r = \sqrt{r^2 + |dr/ds|^2}$ if $m = 2$ and $J_r = r \sqrt{r^2 + |\text{Grad} \, r|^2}$ if $m = 3$. For notational convenience we introduce the mapping $p$ taking the scalar function $r$ onto the vector function $(p(r))(z) := r(z)$ for $z \in S^{m-1}$. Then the parameterized form of (2.20) is given by

$$((\tilde{S}_\infty(\psi, r))(\hat{x}) = u_{\infty} \quad (2.23)$$

where $\psi = \varphi \circ p(r)$. Its linearization with respect to $r$ in direction $q$ becomes

$$(\tilde{S}_\infty(\psi, r) + \tilde{S}'_\infty(\psi, r; q) = u_{\infty} \quad (2.24)$$

and is an ill-posed linear equation for the perturbation $q$ to obtain the update $r + q$. Here, the Fréchet derivative $\tilde{S}'_\infty$ of the operator $\tilde{S}_\infty$ with respect to the boundary $r$ in the direction $q$ is given by

$$\tilde{S}'_\infty(\psi, r; q)(\hat{x}) := \gamma_m \int_{S^{m-1}} e^{-ikr(y) \hat{x} \cdot \hat{y}} \left[ -ikq(y) \hat{x} \cdot \hat{y} J_r(y) + (J'_r q)(y) \right] \psi(y) \, ds(y)$$
for \( \hat{x} \in S^{m-1} \) where \( J'_r q \) denotes the Fréchet derivative of \( J_r \) in direction \( q \). We have

\[
J'_r q = \frac{rq + r'q'}{\sqrt{r^2 + (dr/ds)^2}}
\]

if \( m = 2 \) and

\[
J'_r q = q \sqrt{r^2 + |\text{Grad } r|^2} + r \frac{rq + \text{Grad } r \cdot \text{Grad } q}{\sqrt{r^2 + |\text{Grad } r|^2}}
\]

if \( m = 3 \).

Now, given an approximation for \( \partial D \) with parameterization \( r \), each iteration step of the proposed inverse algorithm consists of two parts.

1. We solve the parameterized well-posed field equation \((2.19)\) for \( \psi \). In two dimensions, this can be done through the numerical method described at the end of the previous subsection.

2. Then we solve the ill-posed linearized equation \((2.24)\) for \( q \) and obtain an updated approximation for \( \partial D \) with the parameterization \( r + q \). Since the kernels of the integral operators in \((2.24)\) are smooth, for its numerical approximation the composite trapezoidal rule in two dimensions or the Gauss trapezoidal rule in three dimensions can be employed. Because of the ill-posedness, the solution of \((2.24)\) requires stabilization, for example, by Tikhonov regularization.

This algorithm has a straightforward extension for the case of more than one incident wave. Assume that \( u_1^i, \ldots, u_N^i \) are \( N \) incident waves with different incident directions and \( u_{\infty,1}, \ldots, u_{\infty,N} \) the corresponding far field patterns for scattering from \( \partial D \). Given an approximation \( r \) for the boundary we first solve the field equations \((2.19)\) for the \( N \) different incident fields to obtain \( N \) densities \( \psi_1, \ldots, \psi_N \). Then we solve the linearized equations

\[
\tilde{S}_\infty(\psi_n, r) + \tilde{S}_\infty'(\psi_n, r; q) = u_{\infty,n}, \quad n = 1, \ldots, N,
\]

for the update \( r + q \) by interpreting them as one ill-posed equation with an operator from \( L^2(S^{m-1}) \) into \( (L^2(S^{m-1}))^N \) and applying Tikhonov regularization.

For more details on the numerical implementation and numerical examples in two dimensions we refer to [32]. Numerical examples in three dimensions are not available for the time being. Further research is required for
the solution of the full inverse problem by simultaneous linearization of both equations (2.19) and (2.20) with respect to the shape $\partial D$, the impedance functions $\lambda$ and $\mu$ and the density $\varphi$ analogous to [9].

3 Transmission eigenvalues

Roughly speaking, for the solution of inverse scattering problems one can distinguish between two main groups of methods, namely iterative methods and sampling methods. Iterative methods reformulate the inverse problem as a nonlinear ill-posed operator equation and solve it by iteration schemes such as regularized Newton methods, Landweber iterations or conjugate gradient methods. Sampling methods develop criteria in terms of the behavior of appropriately chosen ill-posed linear integral equations that decide on whether a point lies inside or outside the scatterer. In the previous section we met the approach by Johansson and Sleeman as an example for an iteration method and two of the prominent examples for a sampling method are the linear sampling method and the factorization method. They are based on the far field operator

$$F : L^2(S^{m-1}) \rightarrow L^2(S^{m-1})$$

defined by

$$Fg(\hat{x}) := \int_{S^{m-1}} u_{\infty}(\hat{x}, d) g(d) \, ds(d), \quad \hat{x} \in S^{m-1},$$

that is, the integral operator with the far field pattern as the kernel. Further for the far field $w_{\infty}(\cdot, z)$ of the point source $\gamma^{-1} m \Phi(\cdot, z)$ located at $z$ (where $\gamma_m$ is given by (1.11)) we note that

$$w_{\infty}(\hat{x}, z) = e^{-ik\hat{x}z}, \quad \hat{x} \in S^{m-1}, \quad z \in \mathbb{R}^m.$$  

Now, the following theorem due to Kirsch [27] provides a short and concise description of the factorization method. Its proof relies on deep functional analytic tools, together with a factorization of the far field operator that explains the name of the method.

**Theorem 3.1** Assume that $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ for $D$. Then the equation

$$(F^*F)^{1/4} g(\cdot, z) = w_{\infty}(\cdot, z),$$

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where \( F^* \) is the adjoint operator of \( F \), is solvable in \( L^2(S^{m-1}) \) if and only if \( z \in D \).

Picard’s theorem (see [16, Theorem 4.8]) on the solution of equations of the first kind with compact operators can be employed for the numerical implementation of this criterion.

The linear sampling method introduced by Colton and Kirsch [14] is based on the far field equation

\[ Fg(\cdot, z) = w_\infty(\cdot, z) \]

and decides on the behavior of its Tikhonov solution whether \( z \) belongs to the scatterer \( D \). We refrain from the concise formulation since it is more involved as compared with Theorem 3.1 and only note that the linear sampling method also requires that \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) for \( D \).

An important feature of the factorization method and the linear sampling method is that they both work independently on the nature of the scatterer and also for scattering from inhomogeneous media.

Deviating for a couple of paragraphs from the theme of obstacle scattering, we consider the case of an isotropic medium with refractive index \( n \). We assume that \( n \) is real valued and nonnegative and that the contrast \( m := 1 - n \) has support given by our obstacle domain \( D \) and is continuous in \( \overline{D} \). Then, for an incident plane wave \( u^i(x) = e^{ikx \cdot d} \), the simplest inhomogeneous medium scattering problem is to find the total field \( u \in H^1_{\text{loc}}(\mathbb{R}^m) \) such that \( u = u^i + u^s \) satisfies

\[ \Delta u + k^2 n u = 0 \quad \text{in} \ \mathbb{R}^m \] (3.1)

and \( u^s \) satisfies the Sommerfeld radiation condition (1.3).

As shown by Kirsch [28], Theorem 3.1 remains valid for the inhomogeneous medium scattering problem, and also the linear sampling method has been extended to this case. Only the assumption on the wave number has to be modified. For the medium problem \( k \) is required not to be an interior transmission eigenvalue. A complex number \( k \) is called a transmission eigenvalue if there exist nontrivial functions \( v, w \in L^2(D) \) with \( \Delta v, \Delta w \in L^2(D) \) and \( w - v \in H^2_0(D) \) such that

\[ \Delta v + k^2 v = 0, \quad \Delta w + k^2 nw = 0 \quad \text{in} \ D \] (3.2)
and
\[ v = w, \quad \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \quad \text{on } \partial D \] (3.3)

In view of the transmission condition (3.3) the space \( H^2(D) \) of functions \( u \) with vanishing trace \( u|_{\partial D} \) and normal trace \( \partial_\nu u|_{\partial D} \) is the natural solution space for the difference \( v - w \). Then from
\[ \Delta v - \Delta w = k^2(w - v) - k^2mw \]

we observe that we must demand \( v, w \in L^2(D) \) and \( \Delta v, \Delta w \in L^2(D) \).

This eigenvalue problem was first introduced by Kirsch [25] in 1986 in connection with the denseness and injectivity of the far field operator. The transmission eigenvalues can be seen as the extension of the idea of resonant frequencies for impenetrable obstacles to the case of penetrable media and related to non-scattering frequencies. As show in [8], if \( k \) is a real transmission eigenvalue and \( v \) can be extended outside \( D \) as a solution to the Helmholtz equation, then the extended field does not scatter at this wave number \( k \). The transmission eigenvalue problem is a non-linear and non-selfadjoint eigenvalue problem that is not covered by the standard theory of eigenvalue problems for elliptic equations. With respect to the factorization method and the linear sampling method, for a long time transmission eigenvalues were viewed as something to avoid and only in 2008 Päivärinta and Sylvester [37] proved the existence of real transmission eigenvalues. Discreteness of the set of transmission eigenvalues was shown much earlier by Colton, Kirsch and Päivärinta [15] and Rynne and Sleeman [40]. More recently it has been indicated that monotonicity properties of transmission eigenvalues in terms of the refractive index [6, 7] might open the possibility to use transmission eigenvalues as target signature for inverse media.

Here, following recent work of Cakoni and Kress [12], we want to illustrate how boundary integral equations can be used to characterize and compute transmission eigenvalues in the case where \( n \) is constant in \( D \). The main idea is to derive an integral equation from a characterization of the transmission eigenvalues in terms of the Robin-to-Neumann operator as defined by
\[ N_k : f \mapsto \frac{\partial u}{\partial \nu} \quad (3.4) \]

where \( u \in H^1(D) \) is the unique solution to
\[ \Delta u + k^2u = 0 \quad \text{in } D \quad (3.5) \]
satisfying the nonlocal impedance boundary condition

\[ u - i\eta P^3 \frac{\partial u}{\partial \nu} = f \quad \text{on } \partial D \]  

for \( f \in H^{1/2}(\partial D) \). Here \( \eta \) is a positive constant and \( P \) is a positive definite pseudo-differential operator of order \(-1\). For example, we may choose \( P = S_0 \) where \( S_0 \) is the single-layer boundary integral operator (2.3) for the Laplace case \( k = 0 \) which needs to be modified in the two-dimensional case as in Theorem 7.41 in [31]. Our approach differs slightly from that in [12] through the use of the nonlocal impedance boundary condition rather than the classical Leontovich impedance condition (1.4). Using the smoothing operator \( P \) slightly simplifies the analysis.

For any solution of (3.5) and (3.6) for \( f = 0 \) from Green’s integral theorem we have that

\[
\int_D \left[ |\nabla u|^2 - k^2 |u|^2 \right] \, dx = i\eta \int_{\partial D} \left| P^3 \frac{\partial u}{\partial \nu} \right|^2 \, ds
\]

which implies uniqueness of the solution for all \( k \) with \( \text{Re } k > 0 \) and \( \text{Im } k \geq 0 \). Existence of a solution can be shown analogous to Theorem 2.6. The single-layer potential with density \( \varphi \in H^{-1/2}(\partial D) \) solves (3.5) and (3.6) provided \( \varphi \) satisfies the equation

\[ A_k \varphi = f \]  

where we redefined \( A_k := S_k - i\eta P^3(I + K'_k) \).

From uniqueness both for the interior impedance problem (3.5) and (3.6) in \( D \) and for the exterior Dirichlet problem in \( \mathbb{R}^m \setminus \bar{D} \) together with the jump relations for the single-layer potential it can be checked that \( A_k \) has a trivial kernel in \( H^{-1/2}(\partial D) \) for all \( k \) with positive real part and nonnegative imaginary part. After picking a wave number \( k_0 \) such that \( k_0^2 \) is not a Dirichlet eigenvalue for \(-\Delta \) in \( D \) we write \( A_k = S_{k_0} + B_k \) where

\[ B_k := S_k - S_{k_0} - i\eta P^3(I + K'_k). \]

Then \( S_{k_0} : H^{-1/2}(\partial D) \to H^{1/2}(\partial D) \) is an isomorphism and \( B_k : H^{-1/2}(\partial D) \to H^{1/2}(\partial D) \) is compact since the difference \( S_k - S_{k_0} \) is bounded from \( H^{-1/2}(\partial D) \) into \( H^{1/2}(\partial D) \) (see [16, Lemma 5.37]) and \( P^3(I + K'_k) \) is bounded from \( H^{-1/2}(\partial D) \)
into $H^{\frac{3}{2}}(\partial D)$ because of our assumption on $P$. Therefore, by the Riesz theory $A_k : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ is an isomorphism and we can write

$$N_k = (I + K'_k)A^{-1}_k. \quad (3.9)$$

Now, setting

$$k_n := k\sqrt{n},$$

we have that $k$ is a transmission eigenvalue if and only if the kernel of the operator

$$M(k; \eta) := N_k - N_{k_n} \quad (3.10)$$

in non-trivial.

We need to adjust the spaces in which we have to investigate the kernel of $M(k; \eta)$ since we must search for the eigenfunctions $v, w$ in $L^2(D)$. This implies that their trace and their normal derivative on the boundary belong to $H^{-\frac{1}{2}}(\partial D)$ and $H^{-\frac{3}{2}}(\partial D)$, respectively. Indeed if $u \in L^2_\Delta(D) := \{u \in L^2(D) : \Delta u \in L^2(D)\}$ then its trace $u \in H^{-\frac{1}{2}}(\partial D)$ is defined by duality using the identity

$$\langle u, \tau \rangle_{H^{-\frac{1}{2}}(\partial D), H^{\frac{3}{2}}(\partial D)} = \int_D (u\Delta w - w\Delta u) \, dx$$

where $w \in H^2(D)$ is such that $w = 0$ and $\partial_{\nu} w = \tau$ on $\partial D$. Similarly, the trace of $\partial_{\nu} u \in H^{-\frac{3}{2}}(\partial D)$ is defined by duality using the identity

$$\left\langle \frac{\partial u}{\partial \nu}, \tau \right\rangle_{H^{-\frac{3}{2}}(\partial D), H^{\frac{3}{2}}(\partial D)} = -\int_D (u\Delta w - w\Delta u) \, dx$$

where $w \in H^2(D)$ is such that $w = \tau$ and $\partial_{\nu} w = 0$ on $\partial D$.

Therefore, when we represent $v$ and $w$ by single-layer potentials we must work with densities in $H^{-\frac{3}{2}}(\partial D)$. For convenience we introduce

$$(S_k \varphi)(x) := 2 \int_{\partial D} \varphi(y)\Phi_k(x, y) \, ds(y), \quad x \in D.$$ 

Obviously, $S_k \varphi$ satisfies the Helmholtz equation, hence we can conclude that $S_k : H^{-\frac{3}{2}}(\partial D) \to L^2_\Delta(D)$ is bounded. Further, by a duality argument it is possible to extend the jump relations for single-layer potentials across $\partial D$ to the case of densities in $H^{-\frac{3}{2}}(\partial D)$. The standard theory of single-layer potentials implies that both operators $S_{k_0} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$
\[ S_{k_0} : H^{\frac{3}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D) \] are isomorphisms under our assumption on \( k_0 \) not to be a Dirichlet eigenvalue. From this, again by duality it follows that \( S_{k_0} : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{3}{2}}(\partial D) \) is an isomorphism. Consequently, from the above we have that \( A_k \) also is an isomorphism from \( H^{-\frac{3}{2}}(\partial D) \) onto \( H^{-\frac{3}{2}}(\partial D) \).

We note that the above statements remain valid in the case when \( k = i \) and \( \eta = i \) because of the uniqueness for the Robin problem \( \Delta u - u = 0 \) in \( D \) with \( u + P^3 \partial \nu u = 0 \) on \( \partial D \).

To analyze the kernel of \( M(k) \) we now want to show that

\[ M(k; \eta) = (I + K_k')A_k^{-1} - (I + K_k')^2A_k^{-1} : H^{-\frac{3}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D) \]

is a Fredholm operator of index zero and to this end we begin with a regularity result.

**Lemma 3.2** Let \( F \in H^m(D) \) and \( g \in H^{m+\frac{5}{2}}(\partial D) \). Then the unique solution \( v \in L^2(\partial D) \) of \( \Delta v = F \) in \( D \) and \( v = g \) on \( \partial D \) belongs to \( H^{m+2}(D) \) and the mapping taking \( (F, g) \) into \( v \) is bounded from \( H^m(D) \times H^{m+\frac{5}{2}}(\partial D) \) into \( H^{m+2}(D) \) for \( m = 0, 1, \ldots \)

**Proof.** We make use of a regularity theorem on the Poisson equation which guarantees that the unique solution \( v \in H^1_0(D) \) of \( \Delta v = F \) for \( F \in H^m(D) \) belongs to \( H^{m+2}(D) \) and that the linear mapping taking \( F \) into \( v \) is bounded from \( H^m(D) \) into \( H^{m+2}(D) \) for \( m = 0, 1, \ldots \) (see Theorem 1.3 in [41, p. 305]).

First we show that this property can be extended to solutions \( v \in L^2(D) \) that vanish on \( \partial D \) in the sense of the \( H^{-\frac{3}{2}}(\partial D) \) trace. For this we observe from the definition of the \( H^{-\frac{3}{2}}(\partial D) \) trace that for any harmonic function \( v \in L^2(D) \) vanishing on the boundary \( \partial D \) we have that \( \int_D v \Delta w \, dx = 0 \) for all \( w \in H^2(D) \) with \( w = 0 \) on \( \partial D \). Inserting the solution \( w \in H^1_0(D) \) of \( \Delta w = \bar{v} \) which automatically belongs to \( H^2(D) \) by the above theorem yields \( v = 0 \) in \( D \). For a solution \( v \in L^2(D) \) of \( \Delta v = F \) for \( F \in L^2(D) \) with vanishing \( H^{-\frac{3}{2}}(\partial D) \) trace on \( \partial D \) we denote by \( \bar{v} \) the solution of \( \Delta \bar{v} = F \) in \( H^1_0(D) \) and apply the uniqueness result for the difference \( v - \bar{v} \) to obtain that \( v = \bar{v} \in H^1_0(D) \).

The statement of the lemma now follows from the observation that the unique solution \( w \in H_1(D) \) of the Laplace equation \( \Delta w = 0 \) with boundary condition \( w = g \) on \( \partial D \) is in \( H^{m+2}(D) \) and that the mapping taking \( g \) into \( w \) is bounded from \( H^{m+\frac{5}{2}}(\partial D) \) into \( H^{m+2}(D) \) as can be observed from the single-layer boundary integral equation approach. \( \square \)
Lemma 3.3 The linear operators

\[ \varphi \mapsto S_k A_k^{-1} \varphi - S_k A_k^{-1} \varphi \]  

(3.11)

from \( H^{-\frac{1}{2}}(\partial D) \) into \( H^2(D) \) and \( M(k; \eta) : H^{-\frac{1}{2}}(\partial D) \rightarrow H^2(\partial D) \) are bounded.

Proof. By definition \( M(k) \varphi \) is the normal derivative trace on the boundary \( \partial D \) of

\[ u := S_k A_k^{-1} \varphi - S_k A_k^{-1} \varphi, \quad \varphi \in H^{-\frac{1}{2}}(\partial D). \]

Then,

\[ \Delta u = -k^2 S_k A_k^{-1} \varphi + k^2_n S_k A_k^{-1} \varphi \]

is in \( L^2(\partial D) \) and the mapping \( \varphi \rightarrow \Delta u \) is bounded from \( H^{-\frac{1}{2}}(\partial D) \) into \( L^2(\partial D) \). Furthermore from

\[ [S_k A_k^{-1} \varphi]_{\partial D} - i\eta P^3 \frac{\partial}{\partial \nu} S_k A_k^{-1} \varphi = [S_k - i\eta P^3 (I + K'_k)] A_k^{-1} \varphi = \varphi \]

we have that

\[ u = g \quad \text{on} \quad \partial D \]

with

\[ g = iP^3 \frac{\partial u}{\partial \nu}. \]

Since \( \varphi \mapsto u \) is bounded from \( H^{-\frac{1}{2}}(\partial D) \) into \( L^2(\partial D) \) we have that \( \varphi \mapsto \partial_\nu u \) is bounded from \( H^{-\frac{1}{2}}(\partial D) \) to \( H^{-\frac{3}{2}}(\partial D) \) and our assumption on the operator \( P \) finally ensures that the mapping \( \varphi \rightarrow g \) is bounded from \( H^{-\frac{1}{2}}(\partial D) \) into \( H^\frac{3}{2}(\partial D) \).

From this, Lemma 3.2 for \( m = 0 \) implies the first statement and the second follows by taking the normal trace. \( \square \)

Theorem 3.4 Let \( \kappa > 0 \) and \( \kappa_n := \kappa \sqrt{n} \). Then

\[ (\kappa^2 - \kappa^2_n) M(i\kappa; i) : H^{-\frac{1}{2}}(\partial D) \rightarrow H^\frac{3}{2}(\partial D) \]

is coercive.
Proof. For \( u, v \in H^2(D) \) we can transform
\[
\int_D v(\Delta - \kappa^2)(\Delta - \kappa_n^2) u \, dx \\
- \int_D [\Delta u \Delta v + (\kappa^2 + \kappa_n^2) \text{grad} \, u \cdot \text{grad} \, v + \kappa^2 \kappa_n^2 uv] \, dx
\]
\[
= \int_D (v \Delta u - \Delta v \Delta u) \, dx - (\kappa^2 + \kappa_n^2) \int_D (v \Delta u + \text{grad} \, u \cdot \text{grad} \, v) \, dx
\]
From this, by Green’s theorem we obtain
\[
\int_D v(\Delta - \kappa^2)(\Delta - \kappa_n^2) u \, dx \\
- \int_D [\Delta u \Delta v + (\kappa^2 + \kappa_n^2) \text{grad} \, u \cdot \text{grad} \, v + \kappa^2 \kappa_n^2 uv] \, dx = \int_{\partial D} (v \partial \Delta u \partial \nu - \Delta u \partial v \partial \nu) \, ds - (\kappa^2 + \kappa_n^2) \int_{\partial D} v \partial u \partial \nu \, ds.
\]
\[
(3.12)
\]
For \( v = \bar{u} \) the second domain integral is equivalent to the \( \| \cdot \|_{H^2} \) norm as can be seen with the aid of Green’s representation formula, that is,
\[
\int_D [||\Delta u||^2 + (\kappa^2 + \kappa_n^2)||\text{grad} \, u||^2 + \kappa^2 \kappa_n^2 |u|^2] \, dx \geq c \|u\|_{H^2(D)}^2
\]
for all \( u \in H^2(D) \) and some constant \( c > 0 \).

Now, for \( \varphi \in H^{-1/2}(\partial D) \) as above we define
\[
u := S_{\kappa} A_{\kappa}^{-1} \varphi - S_{\kappa_n} A_{\kappa_n}^{-1} \varphi
\]
which belongs to \( H^2(D) \) by Lemma 3.3. Then
\[
(\Delta - \kappa^2)(\Delta - \kappa_n^2) u = 0 \quad (3.14)
\]
and
\[
\Delta u = \kappa^2 S_{\kappa} A_{\kappa}^{-1} \varphi - \kappa_n^2 S_{\kappa_n} A_{\kappa_n}^{-1} \varphi.
\]
From this, as in the proof of Lemma 3.3, we obtain the boundary conditions
\[
u + P^3 \partial \nu = 0 \quad \text{and} \quad \Delta u + P^3 \partial \Delta u \partial \nu = (\kappa^2 - \kappa_n^2) \varphi \quad \text{on} \ \partial D. \quad (3.15)
\]
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We set $v = \bar{u}$ in (3.12) and use (3.15) and the self-adjointness of $P$ to find that

$$\int_D \left[ |\Delta u|^2 + (\kappa^2 + \kappa_n^2) |\text{grad} u|^2 + \kappa^2 \kappa_n^2 |u|^2 \right] \, dx$$

$$= \left( \kappa^2 - \kappa_n^2 \right) \int_{\partial D} \varphi \frac{\partial \bar{u}}{\partial \nu} \, ds - \left( \kappa^2 + \kappa_n^2 \right) \int_{\partial D} P_3 \frac{\partial u}{\partial \nu} P_3 \frac{\partial \bar{u}}{\partial \nu} \, ds.$$ 

Inserting $\partial_{\nu} u = M(ik, i) \varphi$ and using the positive definiteness of $P$ and (3.13) we get the coercivity estimate

$$\left( \kappa^2 - \kappa_n^2 \right) \int_{\partial D} \varphi M(ik) \varphi \, ds \geq \tilde{C} \|\varphi\|^2_{H^{1/2}(\partial D)} \geq C \|\varphi\|^2_{H^{1/2}(\partial D)} \quad (3.16)$$

for $\varphi \in H^{-\frac{1}{2}}(\partial D)$ and some constants $\tilde{C}, C > 0$, where for the latter inequality we used (3.15) and the definition of the trace of $\varphi$ by duality. □

**Theorem 3.5** The operator

$$M(k; \eta) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} M(ik; i) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$$

is compact.

**Proof.** For $\varphi \in H^{-\frac{1}{2}}(\partial D)$ we define

$$u := S_k A_k^{-1} \varphi - S_{k_n} A_{k_n}^{-1} \varphi \quad \text{and} \quad u_i := S_i|k| A_i^{-1} \varphi - S_i|k_n| A_i|k_n|^{-1} \varphi$$

and let

$$U := u + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} u_i. \quad (3.17)$$

Then, by Lemma 3.3 we have $u, U \in H^2(D)$ with the mappings $\varphi \mapsto U$ and $\varphi \mapsto u$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$. Further $U$ satisfies the boundary conditions (see (3.15))

$$U = -P_3 \frac{\partial U}{\partial \nu} + (1 - i\eta) P_3 \frac{\partial u}{\partial \nu} \quad (3.18)$$

and

$$\Delta U = -P_3 \frac{\partial \Delta U}{\partial \nu} + (1 - i\eta) P_3 \frac{\partial \Delta u}{\partial \nu} \quad (3.19)$$

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on $\partial D$. (We note that the coefficient in the definition of $U$ in (3.17) is chosen such that we obtain (3.19).) By Lemma 3.3 the mappings $\varphi \mapsto U$ and $\varphi \mapsto u$ are bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$. Therefore, in view of our assumption on $P$ the right hand side $g_1$ of (3.18) is in $H^\frac{7}{2}(\partial D)$ with the mapping $\varphi \mapsto g_1$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^\frac{7}{2}(\partial D)$. The right hand side $g_2$ of (3.19) is in $H^\frac{3}{2}(\partial D)$ with the mapping $\varphi \mapsto g_1$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^\frac{3}{2}(\partial D)$.

Furthermore, it is straightforward to check that

$$\Delta \Delta U = F(u, u_i)$$

where

$$F(u, u_i) := -k^2 k_n^2 u - (k^2 + k_n^2) \Delta u - \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} [ |k|^2 |k_n|^2 u_i - (|k|^2 + |k_n|^2) \Delta u_i ]$$

belongs to $L^2(D)$ with the mapping $\varphi \mapsto F$ bounded from $H^{-\frac{1}{2}}(\partial D)$ to $L^2(D)$.

Now, we can use Lemma 3.2 again. Applying it first first for $\Delta U$ we obtain that $\Delta U \in H^2(D)$ with the mapping $\varphi \mapsto \Delta U$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$. Applying the lemma then for $U$ shows that $U \in H^4(D)$ with the mapping $\varphi \mapsto U$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^4(D)$. Therefore, the mapping $\varphi \mapsto \partial \nu U$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^\frac{5}{2}(\partial D)$. Now, in view of

$$\frac{\partial U}{\partial \nu} = M(k; \eta) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} M(|k|; i)$$

the statement of the theorem follows from the compact embedding of $H^\frac{5}{2}(\partial D)$ into $H^\frac{1}{2}(\partial D)$.

Noting that $M(k; \eta)$ is analytic in $k$ since the kernels of $S_k$ and $K'_k$ are analytic in $k$, now Theorems 3.4 and 3.5 imply the following final result. From this, in particular, we can reestablish the discreteness of the set of transmission eigenvalues for the special case of a constant refractive index and the finite multiplicity of the transmission eigenvalues.

**Theorem 3.6** $M(k; \eta) : H^{-\frac{1}{2}}(\partial D) \rightarrow H^\frac{1}{2}(\partial D)$ is a Fredholm operator with index zero and analytic in $\{k \in \mathbb{C} : \text{Re} k > 0 \text{ and } \text{Im} k \geq 0\}$.
Cakoni and Kress [12] also used their boundary integral formulations for actual computations of transmission eigenvalues with the aid of the attractive new algorithm for solving non-linear eigenvalue problems for large sized matrices $A$ that are analytic with respect to the eigenvalue parameter as proposed by Beyn [2]. So far, in the literature, the majority of numerical methods were based on finite element methods applied after a transformation of the homogeneous interior transmission problem to a generalized eigenvalue problem for a fourth order partial differential equation. Boundary integral equations had been employed for the computation of transmission eigenvalues only by Cossonnière [18] and Kleefeld [30] using a two by two system of boundary integral equations proposed by Cossonnière and Haddar [19]. Comparing the computational costs for Beyn’s algorithm as applied to Cossonnière and Haddar’s two by two system it can be shown that the approach presented here reduces the costs in the application of Beyn’s algorithm by about 50 percent. For details of the implementation and numerical results we refer to [12] and for a very recent extension of this approach to the Maxwell equations including numerical results for transmission eigenvalues we refer to [10].

References


