Looking Back on Inverse Scattering Theory

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"History will be kind to me for I intend to write it"

Winston Churchill

Abstract

We present an essay on the mathematical development of inverse scattering theory for time-harmonic waves during the past fifty years together with some personal memories of our participation in these events.

1 Introduction

The purpose of this paper is to discuss some of the highlights of the mathematical theory of inverse scattering since 1970. However, we must first issue a caveat to the reader. Our aim is not to provide an overview of all of the developments of inverse scattering theory in the past fifty years. Such a task would require a book not a paper. Instead, we have adopted a far narrower goal of writing about those areas of inverse scattering theory that we have either been involved in ourselves or that are closely related to our research interests. In particular, we will focus on the mathematical theory of the inverse scattering problem for acoustic and electromagnetic waves with the basic themes of nonlinearity and ill–posedness holding center stage. Although these themes have been the main focus of our research efforts, we believe that it is also true that these two topics are in fact central to the entire field of inverse scattering as it has developed over the past fifty years.

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This rather bold statement is a reflection of our view that scattering theory, and in particular inverse scattering theory, should not only be viewed as an area of applied mathematics but also as a branch of mathematical analysis where theorems and their proofs play a central role.

Having assigned such modest goals, we hasten to further mention that we have no intention of discussing all the main results in a given area of interest. This paper it not intended to be an inclusive survey. We also have made no effort to present the weakest regularity assumptions on the domain and the material properties of the scatterer.

In our view, the mathematical theory of inverse scattering theory began in the 1970s and has continued on to the present. There has been amazing progress with numerous successes both theoretically and numerically. We hope that our brief survey will capture the enthusiasm that we both have for the field as well as encouraging newcomers to find out more about a fascinating area of applied mathematics.

2 Uniqueness in inverse obstacle scattering

Scattering theory is concerned with the effects that obstacles and inhomogeneities have on the propagation of waves and in particular time-harmonic waves. For reasons of brevity, we focus our attention on the case of acoustic waves and only give passing references to the case of electromagnetic waves. Throughout the paper we will consider scattering objects within a homogeneous background that are described by a bounded domain $D \subset \mathbb{R}^3$ with a connected C^2 boundary ∂D and can be either impenetrable or penetrable, i.e, an obstacle or an inhomogeneity.

Although our presentation of inverse scattering will be in $\mathbb{R}^3$, all our results remain valid in $\mathbb{R}^2$ unless otherwise stated.

Given a plane wave $u^i(x) = e^{ik\,x\cdot d}$ propagating in the direction $d \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$, the simplest obstacle scattering problem is to find the total field $u \in H^1_{\rm loc}(\mathbb{R}^3 \setminus \bar{D})$ such that $u = u^i + u^s$ satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{2.1}$$

with positive wave number k, the boundary condition

$$u = 0 \quad \text{on } \partial D, \tag{2.2}$$

and the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|, \tag{2.3}$$

uniformly for all directions. The solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3\setminus\bar{D})$ is to be interpreted in the variational sense. The homogeneous Dirichlet boundary condition (2.2) corresponds to a *sound-soft* obstacle. Boundary conditions other than (2.2) are, for example, the homogeneous Neumann or *sound-hard* boundary condition or the impedance boundary condition

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \quad \text{on } \partial D$$

where ν is the unit outward normal to ∂D and λ is a given non-negative continuous function.

The equation (2.1) carries the name of Helmholtz (1821–1894) for his contributions to mathematical acoustics. The radiation condition (2.3) was introduced by Sommerfeld in 1912 to characterize an outward energy flux. It is equivalent to the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\, u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

uniformly for all directions $\hat{x} = x/|x|$ and where u_∞ is defined for $\hat{x}, d \in \mathbb{S}^2$ and is called the *far field pattern* of u^s. Solutions to the Helmholtz equation satisfying (2.3) are called radiating.

Uniqueness of a solution to the obstacle scattering problem is a consequence of the following fundamental lemma which is due to Rellich (1943) and Vekua (1943) and is known as Rellich's lemma.

Lemma 2.1 *Any radiating solution $u^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3\setminus\bar{D})$ to the Helmholtz equation with far field pattern $u_\infty = 0$ vanishes identically in $\mathbb{R}^3 \setminus \bar{D}$.*

Existence of a solution was first established by Vekua, Weyl and Müller in the 1950s via a double-layer boundary integral equation approach and the non-uniqueness deficiency of the latter were remedied independently by Leis, Brakhage and Panich in the 1960s. For references we refer the reader to our monographs [22, 23].

Given the incident field $u^i(x) = e^{ik\,x\cdot d}$, the basic *inverse obstacle scattering problem* is to determine D from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$

for some observation directions $\hat{x}$ and some incident directions d on $\mathbb{S}^2$ and a fixed wave number k. The history of inverse obstacle scattering theory starts with the following uniqueness result due to Schiffer.

Theorem 2.2 *Assume that D_1 and D_2 are two sound-soft scatterers such that their far field patterns coincide for all $\hat{x}, d \in \mathbb{S}^2$ and one fixed wave number k. Then $D_1 = D_2$.*

Proof. Assume that $D_1 \neq D_2$. By Rellich's lemma, for each incident wave u^i the scattered waves u_1^s and u_2^s for the obstacles D_1 and D_2 coincide in the unbounded component G of the complement of $D_1 \cup D_2$. Without loss of generality, one can assume that $D^* := (\mathbb{R}^3 \setminus G) \setminus \bar{D}_2$ is nonempty. Then u_2^s is defined in D^*, and the total wave $u = u^i + u_2^s$ satisfies the Helmholtz equation in D^* and the homogeneous boundary condition $u = 0$ on ∂D^*. Hence, u is a Dirichlet eigenfunction of $-\Delta$ in the domain D^* with eigenvalue k^2. The proof is now completed by showing that the total fields for distinct incident plane waves are linearly independent, and this contradicts the fact that for a fixed eigenvalue the Dirichlet eigenspace of $-\Delta$ in $H_0^1(D^*)$ has finite dimension. □

Schiffer's uniqueness result was obtained around 1960 and was never published by Schiffer himself. It appeared as a private communication in the monograph by Lax and Philipps [66]. This is notable since nowadays in a time of permanent evaluation and competition for grants nobody would want to give away such a valuable result as a private communication. Noting that the proof presented in [66] contains a slight technical fault since the fact that the complement of $D_1 \cup D_2$ might be disconnected was overlooked, it is comforting to observe that even eminent authors can have errors in their books.

A challenging open problem is to determine whether uniqueness with one incident plane wave at one single wave number is guaranteed. So far, only under additional geometric assumptions on the size or the shape of the scatterer uniqueness for one incident plane wave has been established.

Using the strong monotonicity property of the Dirichlet eigenvalues of $-\Delta$, extending Schiffer's ideas in 1983 Colton and Sleeman [27] showed that a sound-soft scatterer is uniquely determined by the far field pattern for one incident wave under the a priori assumption that it is contained in a ball of radius R such that $kR < \pi$. Note that the smallest eigenvalue for the unit ball is given by the smallest zero π of the spherical Bessel function j_0 and that this is a simple eigenvalue. Hence, exploiting the fact that the wave functions are complex valued with linearly independent real and imaginary parts, in 2005 Gintides [31] improved this bound to $kR < 4.49$ where $4.49\ldots$ is the smallest positive zero of the spherical Bessel function j_1. (In the two dimensional case, the two bounds have to be replaced by the smallest zeros of the Bessel functions J_0 and J_1, respectively.) We note that for other than the Dirichlet boundary condition there is no analogue to the results in [27, 31] since there is no monotonicity property for the eigenvalues of $-\Delta$ for other boundary conditions.

In 1997 Liu [68] showed that a sound-soft ball is uniquely determined by the far field pattern for one incident plane wave. A simpler proof of this result than the one in [68] can be found in [23]. Starting in 2003 in a series of papers by Alessandrini, Cheng, Liu, Rondi and Yamamoto [1, 18, 69, 70] it was established that one incident plane wave is sufficient to uniquely determine a sound-soft polyhedron. Both the results for the ball and the polyhedron have analogs for other boundary conditions and also for electromagnetic waves.

The finiteness of the dimension of the eigenspaces for eigenvalues of $-\Delta$ for the Neumann or impedance boundary condition requires the boundary of the intersection D^* from the proof of Theorem 2.2 to be sufficiently smooth which, in general, is not the case. Therefore, there does not exist an immediate extension of Schiffer's approach to other boundary conditions.

Assuming that two different scatterers have the same far field patterns for all incident directions, in 1990 Isakov [46] obtained a contradiction by considering a sequence of solutions with a singularity moving towards a boundary point of one scatterer that is not contained in the other scatterer. He used weak solutions and the proofs are technically involved. During a hike in the Dolomites, on a long downhill walk from Rifugio Treviso to Passo Cereda, in 1993 Kirsch and Kress [57] realized that these proofs can be simplified by using classical solutions rather than weak solutions and by obtaining the contradiction by considering point wise limits of the singular solutions rather than limits of L^2 norms. Later on it was also observed that simultaneously both the shape ∂D and the boundary condition of the scatterer are uniquely determined by the far field pattern for infinitely many incident plane waves. For boundary conditions of the form $Bu = 0$ on ∂D, where $Bu = u$ for a sound-soft scatterer and $Bu = \partial u/\partial \nu + ik\lambda u$ for the impedance boundary condition one can state the following theorem. For its proof and for later use throughout the remainder of the paper we introduce the notation

$$\Phi(x,y) := \frac{1}{4\pi}\,\frac{e^{ik|x-y|}}{|x-y|}\,, \quad x \neq y, \tag{2.4}$$

for the fundamental solution of the Helmholtz equation in three dimensions.

Theorem 2.3 *Let two scatterers D_1 and D_2 with boundary conditions B_1 and B_2 have the same far field patterns for all $\hat{x}, d \in \mathbb{S}^2$ and one fixed wave number k. Then $D_1 = D_2$ and $B_1 = B_2$.*

Proof. In addition to scattering of plane waves, we also need to consider scattering of point sources $\Phi(\cdot\,, z)$ with source location z in $\mathbb{R}^3 \setminus \bar{D}$. Using Rellich's lemma and the mixed reciprocity relation

$$4\pi w^s_\infty(-d,z) = u^s(z,d), \quad z \in \mathbb{R}^3 \setminus \bar{D},\ d \in \mathbb{S}^2, \tag{2.5}$$

which, for scattering of a point source located at z, connects the far field pattern w_∞ of the scattered wave in observation direction $-d$ with the scattered wave u^s for plane wave incidence in direction d evaluated at z (see Theorem 3.16 in [23]), we can deduce that $w^s_1(x,z) = w^s_2(x,z)$ for all $x, z \in G$. Here, assuming $D_1 \neq D_2$, G is defined as in the proof of Theorem 2.2 and w_1 and w_2 are the scattered waves for point source incidence for the obstacles D_1 and D_2, respectively. Now a contradiction can be obtained choosing $x \in \partial G$ such that $x \in \partial D_1$ and $x \notin \partial D_2$ and a sequence $z_n \in G$ such that $z_n \to x$ as $n \to \infty$. The use of the mixed reciprocity principle in this proof was suggested in 2001 by Potthast [83] whereas in [57] an approximation argument was used. □

The idea of the proof for Theorem 2.3 has been applied to different boundary conditions and also to the Maxwell equations for electromagnetic waves.

3 Solution of the inverse obstacle problem

As previously mentioned, the main features of the inverse scattering problem that make its investigation challenging are its nonlinearity and ill-posedness. In the case of obstacle scattering it is nonlinear since the solution to the scattering problem depends nonlinearly on the boundary of the scatterer and it is ill-posed since the corresponding solution operator is extremely smoothing, i.e., in functional analytic terms the operator mapping the boundary of the scatterer onto the far field pattern of the scattered wave is compact. In particular, due to the ill-posedness, given a measured far field pattern u_∞ the question of existence of a solution to the inverse problem is the wrong question to ask since, due to errors in the data, in general no solution will exist for the given data. Assuming that the given far field data u_∞ is close to the correct far field of some scatterer D the appropriate task is to design stable methods for finding an approximation for ∂D from the perturbed data u_∞.

For testing the accuracy of numerical algorithms of the forward obstacle scattering problem it is straightforward to set up tests using explicit solutions, for example, by considering point sources located in the interior of the scatterer. However, there are no closed form solutions of the direct scattering problem available for plane wave incidence. Therefore numerical tests of approximate methods for the inverse problem usually rely on synthetic far field data obtained by numerically solving the forward scattering problem. In this context, in the late 1980s and early 1990s, when a lively development of inverse algorithms started, it was noted that in order to avoid trivial inversions of finite dimensional problems, for reliably testing the performance of an inverse problem solver it is crucial that the synthetic data are obtained by a forward algorithm which has no connection to the inverse algorithm under consideration. For numerical tests violating this requirement the term *inverse crimes* was created and, to our knowledge, appeared for the first time in printing in 1993 in the first edition of our monograph [23].

3.1 Early attempts and decomposition methods

The first attempts to solve the inverse obstacle problem dealt with the nonlinearity issue by linearizing the problem with the aid of the *physical optics approximation.* For a convex sound-soft scatterer D for large wave numbers k, by Huygens' formula the far field pattern in the backscattering direction is approximately given by

$$u_\infty(-d,d) \approx -\frac{1}{4\pi}\int_{\nu(y)\cdot d<0} \frac{\partial}{\partial\nu(y)}\, e^{2ik\,d\cdot y}\,ds(y), \quad d\in\mathbb{S}^2,$$

and from that Bojarski [10] in 1967 derived the identity

$$\int_{\mathbb{R}^3} \chi(y) e^{2ik\, d\cdot y}\, dy = \frac{\pi}{k^2}\, \left\{ u_\infty(-d,d) + \overline{u_\infty(d,-d)} \right\}, \quad d \in \mathbb{S}^2, \tag{3.1}$$

where χ denotes the characteristic function of D. Hence, in principle, the inverse obstacle scattering problem with backscattering data for all incident directions and all positive wave numbers is reduced to an inversion of the Fourier transform. However, there are two drawbacks to this approach. Firstly, since the physical optics approximation is valid only for large wave numbers, in practice one has to invert the Fourier transform with incomplete data which leads to uniqueness ambiguities and to severe ill-posedness. Secondly, the physical optics approximation will not work at all in situations where far field data are available only for frequencies in the *resonance region* when the wave length is comparable to the size of the scatterer. For these reasons in our further discussion we concentrate our attention to solutions of the full nonlinear inverse obstacle scattering problem.

The earliest attempts to treat the inverse scattering problem without linearizing were made by Imbriale and Mittra [45] in 1970 and were based on expansions of the scattered wave with respect to radiating spherical wave functions and analytic continuation with only little attention being given to issues of stabilization. The basic idea of the approach is to expand the given far field pattern in a Fourier series

$$u_\infty(\hat{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m Y_n^m(\hat{x}) \tag{3.2}$$

where Y_n^m is a spherical harmonic of order n. Then the scattered wave is given by

$$u^s(x) = k \sum_{n=0}^{\infty} i^{n+1} \sum_{m=-n}^{n} a_n^m\, h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right) \tag{3.3}$$

where $h_n^{(1)}$ is a spherical Hankel function order n and this series converges outside the smallest ball centered at the origin that contains the unknown scatterer (see [23, Theorem 2.17]). Now if we assume a convex sound-soft scatterer D, computing the total field $u(R\hat{x}) = u^i(R\hat{x}) + u^s(R\hat{x})$ for successively decreasing values of the radius R and testing whether it becomes zero for some $\hat{x} \in \mathbb{S}^2$ we can, in principle, determine at least one point on ∂D.

In order to obtain further points of ∂D we move to another origin of coordinates and redo the expansions with respect to the new origin to capture more boundary points. This procedure is continued until enough points are determined. However, this approach is time consuming, not very systematic and difficult to be kept stable.

In principle, the Impriale–Mittra approach belongs to a group of methods that are called *decomposition methods*. The main idea of these methods is to break up the inverse obstacle scattering problem into two parts: the first part deals with the ill-posedness by constructing the scattered wave from its far field pattern and the second part deals with the nonlinearity by determining the unknown boundary of the scatterer as the location where the boundary condition for the total field is satisfied. However, these methods face the difficulty that in the first step the domain of definition of the scattered wave is not known. Hence, mathematically satisfying formulations of decomposition methods need to combine both parts into an optimization reformulation of the inverse scattering problem.

A prime example for a decomposition method was proposed in 1986 by Kirsch and Kress [56]. Assuming a priori that enough information is available to place a closed surface Γ inside the unknown scatterer D, the scattered wave is approximated by a single-layer potential

$$u_s(x) := \int_\Gamma \Phi(x,y)\varphi(y)\,ds(y). \tag{3.4}$$

The density $\varphi \in L^2(\Gamma)$ is obtained by solving the ill-posed integral equation of the first kind

$$\frac{1}{4\pi}\int_\Gamma e^{-ik\,\hat{x}\cdot y}\varphi(y)\,ds(y) = u_\infty(\hat{x}), \quad \hat{x} \in \mathbb{S}^2, \tag{3.5}$$

via Tikhonov regularization. If we denote the integral operator on the left hand side of (3.5) by S_∞, the Tikhonov solution corresponds to minimizing the Tikhonov functional

$$\|S_\infty\varphi - u_\infty\|^2_{L^2(\mathbb{S}^2)} + \alpha\|\varphi\|^2_{L^2(\Gamma)} \tag{3.6}$$

with a positive regularization parameter α. The unknown boundary ∂D is then determined by requiring (3.4) to satisfy the sound-soft boundary condition in a least squares sense, i.e., by minimizing the defect

$$\|u^i + S\varphi\|^2_{L^2(\Lambda)} \tag{3.7}$$

over some suitable class of admissible surfaces Λ where S is the single-layer potential operator defined by the right hand side of (3.4). To turn this approach into a stable regularization procedure for the ill-posed inverse obstacle scattering problem it has to be recast into a simultaneous minimization of a weighted sum of the two functionals (3.6) and (3.7).

This method has been extended to other boundary conditions and successfully tested numerically both in two and three dimensions by a large number of researchers. Its extension to the case of the electromagnetic inverse obstacle problem with the perfect conductor boundary condition was carried out in 1987 by Blöhbaum [9], a joint PhD student of the authors.

In principle, one can replace the approximation (3.4) of the scattered field u^s by any other convenient approximation. For example, in 1987 Angell, Kleinman, and Roach [2] suggested using an expansion with respect to radiating spherical wave functions, i.e., by a series of the form (3.3), and in this way sort of revisited the approach of Imbriale and Mittra to give it a more solid basis.

In 1996, Potthast [81] developed a method that eventually was denoted as the *point source method* and which mimics the uniqueness proof of Theorem 2.3 by letting point sources tend to the boundary and by approximating the far field patterns for point source incidence through the far field patterns for plane wave incidence. In view of the mixed reciprocity relation (2.5), Potthast's point source method may be viewed as another alternative for approximating the scattered wave.

Although the decomposition methods described above have been revived through more recent papers and extensions to related inverse scattering problems, they are less popular than the iterative methods described in the next section since the latter in general yield more accurate reconstructions. However, they will keep their importance as instructive examples for the idea of separating the ill-posedness and the nonlinearity in inverse scattering and related areas.

3.2 Iterative methods

The solution to the direct scattering problem with a fixed incident wave u^i defines an operator

$$\mathcal{F} : \partial D \mapsto u_\infty$$

that maps the boundary ∂D of the scatterer D onto the far field pattern u_∞ of the scattered wave. In terms of this operator, given a far field pattern u_∞, the inverse problem consists in solving the nonlinear and ill-posed equation

$$\mathcal{F}(\partial D) = u_\infty \tag{3.8}$$

for the unknown boundary ∂D. Hence, one can try one of the regularized Newton iterations such as the Levenberg–Marquardt algorithm or iteratively regularized Gauss–Newton iterations that have been developed and analyzed since the 1980s for the approximate solution of nonlinear and ill-posed operator equations. Of course, these methods also can be understood as an optimization approach for the minimization of a penalized norm of the residual in (3.8).

For the application of these methods, the Fréchet derivative of $\mathcal{F}$ is required and for that we need to select a suitable domain for a proper definition of $\mathcal{F}$. Restricting ourselves to scatterers with boundaries that are diffeomorphic to the unit sphere, let X be the space of C^2 functions $p : \mathbb{S}^2 \to \mathbb{R}^3$ such that $p : \mathbb{S}^2 \to p(\mathbb{S}^2)$ is a diffeomorphism. Then we may view X as an open subset of the Banach space $C^2(\mathbb{S}^2; \mathbb{R}^3)$ and identify boundary surfaces ∂D from $Y := \{p(\mathbb{S}^2) : p \in X\}$ with their parameterization p. In this setting, the operator $\mathcal{F}$ can be understood as an operator from X into $L^2(\mathbb{S}^2)$ and can be shown to be compact (see [23, Theorem 5.9]), i.e., the equation (3.8) is ill-posed.

Using a variational approach to characterize the solution of the direct scattering problem, in 1993 Kirsch [51] rigorously established Fréchet differentiability of $\mathcal{F}$ with the derivative at ∂D in the direction h given by

$$\mathcal{F}'_{\partial D} h = v_\infty$$

where v_∞ denotes the far field pattern of the radiating solution v to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the boundary condition

$$v = -\nu \cdot h \, \frac{\partial u}{\partial \nu} \quad \text{on } \partial D \tag{3.9}$$

and u is the solution to the scattering problem (2.1)–(2.3). In particular, Fréchet differentiability means that

$$\lim_{h \to 0} \frac{1}{\|h\|_{C^2(\partial D)}} \, \|\mathcal{F}(\partial D_h) - \mathcal{F}(\partial D) - \mathcal{F}'_{\partial D} h\|_{L^2(\mathbb{S}^2)} = 0$$

with the perturbed boundary $\partial D_h := \{x + h(x) : x \in \partial D\}$ being well defined for sufficiently small h. A hand waving proof for the boundary condition (3.9) is obtained by formally differentiating the boundary condition (2.2) with respect to h. As always the case for compact nonlinear operators, the Fréchet derivative $\mathcal{F}'_{\partial D}$ turns out to inherit the compactness from $\mathcal{F}$. Therefore, the Newton equations are linear ill-posed equations and the classical Newton–Kantorovitch convergence theory does not apply. In order to apply regularized Newton iterations injectivity of $\mathcal{F}'_{\partial D}$ is important. From the boundary condition (3.9) with the aid of Rellich's lemma and Holmgren's uniqueness theorem (see [23, Theorem 2.3]) we observe injectivity of $\mathcal{F}'_{\partial D}$ provided we confine the perturbations to the normal direction, i.e., to perturbations of the form $h = q\,\nu$ with a scalar function q.

A year later in 1994, Potthast [80] was also able to arrive at (3.9) via the traditional boundary integral equation approach for the solution of the direct scattering problem. In particular Potthast established the Fréchet differentiability of the single- and double-layer boundary integral operators with respect to the boundary ∂D and that the derivatives are obtained by taking the Fréchet derivatives of their kernels with respect to ∂D. In doing this he observed that the type of singularity of the kernels and their derivatives remains the same. In 1980 Roger [85] had been the first to employ Newton iterations for the solution of the inverse obstacle scattering problem and he also had used the boundary integral equation approach to obtain the derivative. However, his analysis had been only via formal differentiation. An alternative approach to differentiation of the far field pattern with respect to the boundary was contributed in 1999 by Kress and Päivärinta [61] based on Green's theorems and a factorization of the difference $\mathcal{F}(\partial D_h) - \mathcal{F}(\partial D)$. This approach was also applied in 2004 for the electromagnetic case [34] and in 2012 for the elastodynamic case [67].

In subsequent years Fréchet differentiability was established for other boundary conditions, for inverse scattering for cracks and for electromagnetic obstacle scattering and Newton iterations were successfully implemented in two and three dimensions. For the latter we mention Farhat et al [29] and Harbrecht and Hohage [39]. Inverse scattering problems for two-dimensional cracks have been solved using Newton's method by Kress and Mönch [59, 71]. We refer to our monograph [23] for references to results of Kress and Rundell for a frozen Newton method, for Newton iterations with the amplitude of the far field as data, for backscattering data, and for the simultaneous reconstruction of shape and impedance.

With the proper choice of the regularization parameter in the Levenberg–Marquardt or iteratively regularized Gauss–Newton iterations, the above approach leads to highly accurate reconstructions of the unknown scatterer from the far field pattern for one (or a few) incident plane waves with reasonable stability against data errors. However, since each Newton step requires the solution of the forward problem for the evaluation of $\mathcal{F}(\partial D)$ and a finite number of derivatives $\mathcal{F}'(\partial D; h)$ according to the dimension of the approximation space for the update h, an efficient forward solver is needed. Furthermore, good a priori information is required in order to be able to choose an initial guess that ensures convergence to a global minimum. In addition, on the theoretical side, although some progress has been made through the work of Hohage [40] and others on logarithmic source conditions in the analysis of Newton iterations for ill-posed operators, the convergence of regularized Newton iterations for the operator $\mathcal{F}$ has not been settled. In particular, it remains an open problem whether the convergence results for the Levenberg–Marquardt algorithm and the iteratively regularized Gauss–Newton iterations that were obtained over the last two decades are applicable to inverse obstacle scattering.

During the last decade modified Newton type iterations were developed based on boundary integral equations for the solution of the forward scattering problem. Either using a potential approach or the direct approach via Green's representation formula, the inverse obstacle scattering problem is equivalently reformulated as a system of two nonlinear integral equations for the unknown boundary ∂D and a density function on ∂D as a sort of slip variable. Due to the results by Potthast mentioned above, the derivatives of the corresponding operators can be expressed explicitly in terms of boundary integral operators which then offers computational advantages as compared with the derivative of $\mathcal{F}$ as expressed by (3.9). For a sound-soft obstacle the inverse scattering problem is equivalent to solving

$$\int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y)\, ds(y) = u^i(x), \quad x \in \partial D, \tag{3.10}$$

and

$$-\frac{1}{4\pi} \int_{\partial D} \frac{\partial u}{\partial \nu}(y)\, e^{-ik\,\hat{x}\cdot y}\, ds(y) = u_\infty(\hat{x}), \quad \hat{x} \in \mathbb{S}^2, \tag{3.11}$$

for the unknown boundary ∂D and the unknown normal derivative $\varphi := \partial_\nu u$ of the total field. Both equations are linear with respect to φ and nonlinear with respect to ∂D. Equation (3.11) is severely ill-posed whereas (3.10) is only mildly ill-posed. (For simplicity we assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D.)

Obviously there are three options for an iterative solution of (3.10) and (3.11). In a first method, given an approximation for the boundary ∂D, one solves the mildly ill-posed integral equation of the first kind (3.10) for φ. Then, keeping φ fixed, equation (3.11) is linearized with respect to ∂D to update the boundary approximation. This approach has been proposed and implemented in 2007 by Johansson and Sleeman [49].

In a second approach, the system (3.10) and (3.11) is solved simultaneously for ∂D and φ by Newton iterations, i.e., by linearizing both equations with respect to both unknowns. Motivated by ideas first developed for the Laplace equation in [62], in 2006 one of the authors together with his PhD student Ivanyshyn [47] has initiated the investigation of this approach including implementations in three dimensions and also for electromagnetic scattering. Quite recently, this method has also been successfully employed by Ivanyshyn and LeLouër [48] for electromagnetic obstacle scattering together with an innovative application of the Piola transform of the boundary parametrisation to transport the integral operators on a fixed reference boundary.

Whereas in the Johansson–Sleeman method the burden of both the ill-posedness and nonlinearity is put on one equation, in a third method a more even distribution of the difficulties is obtained by reversing the roles of (3.10) and (3.11), i.e., by solving the severely ill-posed equation (3.11) by Tikhonov regularization for φ and then linearizing (3.10) to obtain the boundary update. With a slight modification this leads to a variant of the Kirsch–Kress method where the interior auxiliary surface Γ is considered as an approximation for ∂D and updated in each iteration by solving the equation (3.11) and linearizing the boundary condition. This approach was initiated by one of us in 2003 and further developed together with his PhD student Serranho [60, 88], again including implementations in three dimensions.

As to be expected, a more detailed investigation reveals a close relation of all these three approaches based on (3.10) and (3.11) with the Newton iterations for $\mathcal{F}$. Hence, the quality of the reconstructions obtained be them can compete with those of Newton iterations for (3.8) based on (3.9) with the benefit of reduced computational costs. However, as for the case of the Newton iterations, the convergence issue remains unresolved.

Level set methods were introduced in the 1980s by Osher and Sethian [78] as a numerical method for the approximation of surfaces in $\mathbb{R}^3$ (or curves in $\mathbb{R}^2$) and their movement. Instead of relying on parameterizations, level set methods represent the surfaces as the set of zeros $\Gamma = \{x \in \mathbb{R}^3 : \Psi(x) = 0\}$ of a function Ψ defined in $\mathbb{R}^3$ which has positive values on one side of the surface and negative values on the other side. For a surface changing with time a partial differential equation, the so-called Hamilton–Jacobi equation, is used for the evolution of Ψ and, in principle, all computations are performed in Cartesian coordinates in $\mathbb{R}^3$. In particular, in the level set methods the topology, i.e., the number of connected components, need not be known in advance and may change during the computation. On the other hand, by basing the computations on a Cartesian grid, no parameterization is available and therefore the spectral methods for boundary integral equations for smooth surfaces can no longer be used. In 1996 Santosa [87] suggested applying level set methods to the solution of inverse problems, including inverse obstacle scattering problems, and since then level set techniques for inverse problems have continued to be investigated [12, 28].

3.3 Sampling methods

As already pointed out, the iterative methods discussed in the previous subsection require sufficient a priori information to ensure their numerical convergence. In contrast to this, sampling methods do not need any a priori information on the geometry of the obstacle nor on the boundary condition. However, they require the knowledge of the far field pattern for a large number of incident waves, whereas iteration methods, in general, work with one or a few incident fields. Roughly speaking, sampling methods are based on choosing an appropriate indicator function f on $\mathbb{R}^3$ such that its value $f(z)$ decides on whether a point z lies inside or outside the scatterer D. They are called sampling methods since the indicator function f is sampled on a grid of points, i.e., the scatterer is visualized by numerically evaluating f for points on this grid.

The main actors in the sampling methods that we will discuss are the far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d)g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2, \tag{3.12}$$

i.e., the integral operator with the kernel given by the far field pattern u_∞ for plane wave scattering from D with observation directions $\hat{x}$ and incident directions d, and the Herglotz operator $H : L^2(\mathbb{S}^2) \to H^1_{\rm loc}(\mathbb{R}^3)$ defined by

$$(Hg)(x) = \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\,ds(d), \quad x \in \mathbb{R}^3, \tag{3.13}$$

i.e., by a superposition of plane waves. The latter are called *Herglotz wave functions* with kernel g and were first introduced by Herglotz as special solutions to the Helmholtz equation in a lecture in Göttingen in 1945. Clearly, Fg represents the far field pattern for scattering of a Herglotz wave function with kernel g by the obstacle D. The compact operator F is injective and has dense range if and only if there does not exist a Herglotz wave function that vanishes identically on ∂D. This important property was first discovered in 1984 by Colton and Kirsch [19] in the framework of the underlying completeness result on far field patterns.

Before we proceed with discussing the two main examples of sampling methods, namely the linear sampling method and the factorization method, we briefly mention two other sampling methods that can be considered as implementations of the uniqueness proof in Theorem 2.3.

For Potthast's [82, 83] *singular source method* the indicator function is given by $f(z) := w^s(z,z)$ through the value of the scattered wave $w^s(\cdot\,,z)$ for the singular source $\Phi(\cdot\,,z)$ as incident field evaluated at the source point z. These values will blow up when z approaches the boundary from outside D. Approximating the point source $\Phi(x,\cdot)$ for x in the exterior of an auxiliary closed surface Γ containing D in its interior by a Herglotz wave function such that

$$\Phi(x,z) \approx \frac{1}{4\pi} \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g_z(d)\,ds(d), \tag{3.14}$$

in 2000 Potthast showed that

$$w_s(z,z) \approx \frac{1}{4\pi}\,(Fg_z, R\overline{g}_z)_{L^2(\mathbb{S}^2)}$$

with the reflection operator R given by $(Rg)(x) = g(-x)$ for $x \in \mathbb{S}^2$. The approximation (3.14) can be obtained in practice by solving

$$\int_{\mathbb{S}^2} e^{ik\,x\cdot d} g_z(d)\,ds(d) = 4\pi\Phi(x,z), \quad x \in \Gamma,$$

via Tikhonov regularization.

The *probe method* suggested in 1998 by Ikehata [44] follows the uniqueness proof of Isakov and uses as indicator function an energy integral for $w^s(\cdot\,,z)$

instead of the point evaluation $w^s(z,z)$. We note that both the singular source method and the probe method both work for sound-soft and sound-hard scatterers without knowing the boundary condition a priori.

The origins of the linear sampling method go back to the *dual space method* for inverse obstacle scattering that was proposed by Colton and Monk in 1985 and which has a close connection to the method of Kirsch and Kress, reflecting the fact that both methods were developed at the same time in close collaboration between Delaware and Göttingen (c.f. Section 5.5 of [23] and Section 8 of this paper). Given the far field patterns $u_\infty(\cdot\,,d)$ for all incident directions d, in its first step the dual space method looks for a Herglotz wave function for which the far field pattern for scattering from D coincides with the far field pattern

$$\Phi_\infty(\hat{x},z) = \frac{1}{4\pi}\, e^{-ik\,\hat{x}\cdot z}, \quad \hat{x} \in \mathbb{S}^2,$$

of a point source $\Phi(\cdot,z)$ located in some point $z \in D$ which is kept fixed. (Typically, in the dual space method z was taken to be the origin leading to a constant Φ_∞.) Obviously, this is achieved by solving the ill-posed linear equation

$$Fg_z = \Phi_\infty(\cdot,z) \tag{3.15}$$

for the Herglotz kernel $g_z \in L^2(\mathbb{S}^2)$. In connection with the linear sampling method the equation (3.15) eventually got denoted as the *far field equation*. We note that by the reciprocity theorem for plane wave incidence (see [23, Theorem 3.15]) we may also consider the far field operator as a superposition with respect to the observation directions instead of the incident directions. Therefore, we can view this method as one of determining a linear functional having prescribed values on the set of far field patterns and this gave the method its name. In the second step the unknown boundary ∂D is determined as the location where the boundary condition

$$Hg_z + \Phi(\cdot,z) = 0 \quad \text{on } \partial D \tag{3.16}$$

is satisfied in a minimum norm sense analogously to (3.7). As in the method of Kirsch and Kress, for a stable regularization procedure the dual space method was reformulated as an optimization problem by combining the Tikhonov functional for (3.15) and the least squares fit for (3.16) into one cost functional.

The *linear sampling method* was born in 1995 at the John F. Kennedy International Airport in New York City while Andreas Kirsch had several hours to wait for his flight back to Germany after a visit to the University of Delaware. On his laptop he had the dual space method programmed (in $\mathbb{R}^2$) and for amusement he shifted the source location randomly around. In doing this, he noted that $\|g_z\|_{L^2(\mathbb{S}^1)}$ became large as z approached ∂D and, by plotting the level curves of $\|g_z\|_{L^2(\mathbb{S}^1)}$, the shape ∂D miraculously appeared. After this discovery, the linear sampling method was first presented in 1996 by Kirsch in collaboration with David Colton [20]. They proposed to solve (3.15) by Tikhonov regularization with the regularization parameter chosen according to Morozow's discrepancy principle and to use the norm $\|g_z\|_{L^2(\mathbb{S}^2)}$ as an indicator function. From the boundary condition (3.16) it is to be expected that $\|g_z\|_{L^2(\mathbb{S}^2)} \to \infty$ as z approaches ∂D.

Unfortunately the ill-posed integral equation (3.15), in general, does not have a solution and therefore the mathematical foundation of the linear sampling method in [20] had to be based on the denseness properties of Herglotz wave functions as expressed through the dense range of F mentioned above. Roughly speaking, these denseness properties ensure that, given $\varepsilon > 0$ and assuming that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D, for each $z \in D$ there exists a Herglotz kernel $g_z^\varepsilon \in L^2(\mathbb{S}^2)$ such that (3.15) is approximately satisfied in the sense that

$$\|F g_z^\varepsilon - \Phi_\infty(\cdot\,, z)\|_{L^2(\mathbb{S}^2)} < \varepsilon \tag{3.17}$$

and the norms $\|H g_z^\varepsilon\|_{L^2(D)}$ remain bounded as $\varepsilon \to 0$ whereas for $z \notin D$ every $g_z^\varepsilon \in L^2(\mathbb{S}^2)$ that satisfies (3.17) for a given $\varepsilon > 0$ is such that $\|H g_z^\varepsilon\|_{L^2(D)} \to \infty$ as $\varepsilon \to 0$. Since for the inverse problem D is not known, in practice the indicator function $\|H g_z^\varepsilon\|_{L^2(D)}$ is replaced by $\|g_z^\varepsilon\|_{L^2(\mathbb{S}^2)}$. We will give a justification of the linear sampling method later on in Theorem 6.1 when we discuss the linear sampling method for the inverse medium scattering problem. There we also will address the question whether Tikhonov regularization for (3.15) indeed leads to the approximations predicted using the denseness arguments for the Herglotz wave functions (see Theorem 6.3).

We proceed by describing the *factorization method* that resulted from Kirsch's efforts to overcome the mathematical deficiencies of the linear sampling method. These originate from the unsolvability of the far field equation (3.15) since the operator is strongly smoothing. By choosing a slightly less smoothing operator in 1998 Kirsch [52] was able to establish the following characterization of the scatterer via the far field operator.

Theorem 3.1 *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then $z \in D$ if and only if*

$$(F^*F)^{1/4} g_z = \Phi_\infty(\cdot\,, z)$$

is solvable in $L^2(\mathbb{S}^2)$.

The proof of this theorem required some newly developed functional analytic tools. It is based the fact that F is a normal operator, i.e., $FF^* = F^*F$, and on the factorization

$$F = -2\pi A S^* A^*$$

in terms of the single-layer potential operator $S : L^2(\partial D) \to: L^2(\partial D)$ given by

$$(S\varphi)(x) := \int_{\partial D} \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \partial D,$$

and the operator $A : L^2(\partial D) \to L^2(\mathbb{S}^2)$ taking Dirichlet boundary data onto the far field pattern of the radiating solution of the Helmholtz equation together with their L^2 adjoint operators. This factorization gives the method its name and it expresses the operator F which depends on D only implicitly through its kernel by operators that clearly exhibit their dependence on D. We note that Theorem 3.1 also provides an alternative proof of Schiffer's Theorem 2.2.

Theorem 3.1 can be used for a reconstruction of D with the aid of a singular system $(\lambda_n, \psi_n, \psi_n)$ of the operator F. Then, by Picard's Theorem [23, Theorem 4.8] on ill-posed equations of the first kind, we have that $z \in D$ if and only if

$$\sum_{n=1}^{\infty} \frac{|(\psi_n, \Phi_\infty(\cdot\,, z))|^2}{|\lambda_n|} < \infty,$$

i.e., a truncated Picard series assumes the role of the indicator function in the factorization method.

As compared to the original paper [52], the theory of the factorization method has been largely modified and extended in [55]. Comparing the linear sampling and the factorization method, the latter is mathematically more satisfying but lacks flexibility as compared with the linear sampling method. In particular, despite the recently developed variants of the factorization method a proof of an analog of Theorem 3.1 the scattering of electromagnetic waves by a perfect conductor is still open.

Both the linear sampling method and the factorization method require that $-k^2$ is not a Dirichlet eigenvalue for the Laplacian in the unknown obstacle D. At first glance, it seems surprising that eigenvalues of the interior domain have influence on algorithms for scattering problems in the exterior domain. However this becomes understandable by observing that if the corresponding eigenfunctions can be continued as a solution of the Helmholtz equation into all of $\mathbb{R}^3$ then if these fields are used as incident fields the resulting scattered fields are identically zero. This, for example, is the case for scattering from balls.

The concept of the *topological derivative* was proposed in 1999 by Sokowloski and Zochowski [89] as a measure for the sensitivity of a shape functional to removing small balls from a given domain. Assume Ω is a given domain containing the unknown scatterer D and denote by $B_\rho(x)$ a ball of radius ρ centered at $x \in \Omega$ with volume $V(\rho)$. Then formally the topological derivative of a shape functional J defined for subsets of Ω at the point x is given by

$$\partial_T(x, \Omega) = \lim_{\rho \to 0} \frac{J(\Omega \setminus B_\rho(x)) - J(\Omega)}{V(\rho)}\ .$$

This derivative now can serve as an indicator function for the inverse obstacle scattering problem provided the functional J is designed in a way that $\partial_T(x, \Omega) << 0$ implies that $x \in D$. In [30] it was shown that, given the measured scattered total field u_D for scattering from D on a measurement surface Γ surrounding D, the shape functional

$$J(\Omega) := \int_\Gamma |u_\Omega - u_D|^2 ds$$

where u_Ω is the total field for scattering from Ω has this desired property and its topological derivative was successfully employed for numerical reconstructions. Although further contributions using the topological derivative in inverse scattering have appeared in the literature, it seems too early to highlight this approach in more detail. For numerous references and connections to other approaches in inverse scattering see [8].

4 Uniqueness in inverse medium scattering

As with the inverse scattering problem for obstacles, the mathematical theory of the inverse scattering problem for acoustic waves in an inhomogeneous medium begins with the problem of uniqueness. We first consider the case of an isotropic medium with refractive index n and assume that the contrast $m := 1 - n$ is piecewise continuous and has compact support. It is further assumed that $\operatorname{Im} n \geq 0$ and $\operatorname{Re} n > 0$. Then, given the plane wave $u^i(x) = e^{ik\,x\cdot d}$ with positive wave number $k > 0$ propagating in direction $d \in \mathbb{S}^2$, the simplest inhomogeneous medium scattering problem is to find the total field $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ such that $u = u^i + u^s$ satisfies

$$\Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^3 \tag{4.1}$$

and u^s satisfies the Sommerfeld radiation condition (2.3). Uniqueness of a solution to this inverse medium scattering problem again is a consequence of Rellich's lemma together with a unique continuation principle for solutions of (4.1) due to Müller [23]. Existence of a solution follows by rewriting the scattering problem as the Lippmann–Schwinger equation

$$u(x,d) = e^{ik\,x\cdot d} - k^2 \int_{\mathbb{R}^3} \Phi(x,y) m(y) u(y,d)\, dy, \quad x \in \mathbb{R}^3, \tag{4.2}$$

where Φ is the fundamental solution introduced in (2.4) [23]. The Sommerfeld radiation condition again implies that the scattered field u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{|x|} u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty, \tag{4.3}$$

uniformly for all directions $\hat{x} = x/|x|$ with the far field pattern u_∞. The *inverse scattering problem* is to determine the refractive index n from u_∞ and the uniqueness of a solution to this problem was given by Nachman [73], Novikov [76] and Ramm [84] in 1988.

Theorem 4.1 *The index of refraction n is uniquely determined by a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \mathbb{S}^2$ and a fixed wave number k.*

The proof of the uniqueness result of Nachman, Novikov and Ramm is based on the following two lemmas.

Lemma 4.2 *Let B be an open ball containing the support of $m := 1 - n$. Then there exists a constant $C > 0$ such that for each $z \in \mathbb{C}^3$ with $z \cdot z = 0$*

and $|\mathrm{Re}\, z| > 2k^2 \|n\|_\infty$ *there exists a solution* $v \in H^2(B)$ *of* $\Delta v + k^2 n v = 0$ *in* B *of the form*

$$v(x) = e^{i\, z\cdot x}\, [1 - w(x)]$$

where

$$\|w\|_{L^2(B)} \le \frac{C}{|\mathrm{Re}\, z|}\,.$$

The solution v given in Lemma 4.2 is known as a *complex geometric optics solution* to $\Delta v + k^2 n v = 0$ and has a long history in both inverse scattering theory and the impedance tomography problem [73, 91].

Lemma 4.3 *Let* B *and* B_0 *be two concentric balls containing the support of* $m := 1 - n$ *such that* $\bar{B} \subset B_0$*. Then the set of total fields* $\{u(\cdot, d) : \; d \in \mathbb{S}^2\}$ *satisfying* (4.1) *is complete in the closure of*

$$\left\{ v \in H^2(B_0) : \; \Delta v + k^2 n v = 0 \;\, in \;\, B_0 \right\}$$

with respect to the $L^2(B)$ *norm.*

With these two lemmas at our disposal, the uniqueness of the solution to the inverse scattering problem for (4.1) is now quite straightforward. In particular, suppose that n_1 and n_2 are two refractive indices such that

$$u_{1,\infty}(\cdot, d) = u_{2,\infty}(\cdot, d), \quad d \in \mathbb{S}^2,$$

and let B be a ball containing the support of $1 - n_1$ and $1 - n_2$. Then by Rellich's lemma we have that $u_1(\cdot, d) = u_2(\cdot, d)$ in $\mathbb{R}^3 \setminus \bar{B}$ for all $d \in \mathbb{S}^2$. From Green's second identity we can deduce that

$$\int_B u_1(\cdot, \hat{d}) u_2(\cdot, d)(n_2 - n_1)\, dx = 0$$

for all $d, \hat{d} \in \mathbb{S}^2$ and hence from Lemma 4.3 if follows that

$$\int_B v_1 v_2 (n_1 - n_2)\, dx = 0 \tag{4.4}$$

for all solutions $v_1, v_2 \in H^2(B_0)$ of $\Delta v_1 + k^2 n_1 v_1 = 0$ and $\Delta v_2 + k^2 n_2 v_2 = 0$ in $B_0 \supset \overline{B}$. Given $y \in \mathbb{R}^3 \setminus \{0\}$ and $\rho > 0$ we now choose vectors $a, b \in \mathbb{R}^3$ such that $\{y, a, b\}$ is an orthogonal basis in $\mathbb{R}^3$ and $|a| = 1$, $|b|^2 = |y|^2 + \rho^2$.

Then for $z_1 := y + \rho a + ib$, $z_2 := y - \rho a - ib$ we construct the solutions v_1 and v_2 defined in Lemma 4.2, substitute into (4.4) and let $\rho \to \infty$. This gives

$$\int_B e^{2iy\cdot x}\,[n_1(x) - n_2(x)]\;dx = 0$$

and the desired uniqueness result follows from the Fourier integral theorem. For details we refer the reader to [23]. (Note that in $\mathbb{R}^2$ this proof does not work since a basis analogous to $\{y, a, b\}$ above cannot be constructed.) $\square$

The uniqueness Theorem 4.1 assumes that u and its normal derivative vary smoothly across ∂D. For the case when u and the normal derivative of u vary discontinuously across ∂D, i.e., the inverse transmission problem, the uniqueness of a solution to this inverse scattering problem was proven by Isakov [46] in 1990 using ideas similar to those discussed previously in the proof of Theorem 2.3 for the uniqueness of a solution to the inverse obstacle problem for a sound-soft or sound-hard scatterer. Under the assumption that $m := 1 - n \in C_0^3(\mathbb{R}^3)$ the uniqueness result of Nachman, Novikov and Ramm was extended to the electromagnetic case by Colton and Päivärinta [25] in 1992 (see also [77]). The extension of Isakov's result for the inverse transmission problem for acoustic waves to the case of electromagnetic waves was given by Hähner [36] in 1993. Finally, a uniqueness result for the inverse scattering problem for (4.1) in the two dimensional case was established by Bukhgeim [11] in 2008. Uniqueness results using a single incident plane wave to determine the support m under the assumption that the support is a convex polyhedron were presented by Hu, Salo and Vesalainen [43] in 2016.

Uniqueness results for the case of anisotropic media are quite different from that for isotropic media. Analogous to Section 2 we describe the geometry of the scatterer by a bounded domain $D \subset \mathbb{R}^3$ with a connected C^2 boundary ∂D and assume D and n are such that n is piecewise continuous in $\bar{D}$. We now consider the scattering problem to find $v \in H^1(D)$ and $u^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ such that u^s satisfies the Sommerfeld radiation condition (2.3) and

$$\begin{aligned} \nabla\cdot A\nabla v + k^2 n v &= 0 \quad \text{in } D,\\ \Delta u^s + k^2 u^s &= 0 \quad \text{in } \mathbb{R}^3\setminus\bar{D},\\ v &= u^i + u^s \quad \text{on } \partial D,\\ \nu\cdot A\nabla v &= \frac{\partial u^i}{\partial\nu} + \frac{\partial u^s}{\partial \nu} \quad \text{on } \partial D, \end{aligned} \tag{4.5}$$

where ν is the unit outward normal of ∂D and again $u^i(x) = e^{ik\,x\cdot d}$. The matrix valued function A has entries that are piecewise continuous in $\bar{D}$, is symmetric and satisfies

$$\bar{\xi} \cdot \operatorname{Im} A\xi \le 0 \quad \text{and} \quad \bar{\xi} \cdot \operatorname{Re} A\xi \ge \gamma\, |\xi|^2 \tag{4.6}$$

for all $\xi \in C^3$ and $x \in \bar{D}$ where $\gamma > 1$. Under these conditions it is known [14] that (4.5) has a unique solution and u^s again satisfies the asymptotic relation (4.3). The inverse scattering problem for (4.5) is to now determine A and n from the far field pattern u_∞. Unfortunately, as shown by Gylys–Colwell [33], A and n are in general not uniquely determined by u_∞. However, using the ideas of Isakov [46], it was shown by Hähner [37] in 2000 that the support D is uniquely determined.

Theorem 4.4 *Suppose that the far field patterns corresponding to D_1, A_1, n_1 and D_2, A_2, n_2, respectively, coincide for all $\hat{x}, d \in \mathbb{S}^2$. Then $D_1 = D_2$.*

Theorem 4.4 remains valid if the second assumption on A in (4.6) is replaced by $0 < \operatorname{Re} A\xi \le \gamma\, |\xi|^2$ where $\gamma < 1$.

The above result of Hähner was extended to the case of electromagnetic waves by Cakoni and Colton [13] in 2003. However for electromagnetic waves many open problems remain concerning the uniqueness of the solution to the inverse scattering problem. We mention just two of these. In the isotropic case strong conditions are imposed on the index of refraction. In particular Colton and Päivärinta assumed that $1 - n \in C_0^3(\mathbb{R}^3)$ whereas the result of Hähner assumed that the refractive index was constant near the boundary. It would be important to replace these condition by ones that are more physically reasonable. For the case of anisotropic media it is possible that uniqueness for the inverse scattering problem may be restored if further restrictions are made on the coefficients appearing in Maxwell's equations, e.g. that the permeability is constant. Such problems remain to be investigated. For further information we refer the reader to the monographs [14, 15].

5 Reconstruction of $n(x)$ by optimization

Given the fact that the inverse scattering problem both for obstacles and inhomogeneous media is nonlinear and ill-posed, a natural approach to reconstructing the index of refraction from measured far field data is to use constrained nonlinear optimization techniques. Indeed, one of the earliest attempts to construct a solution to the inverse obstacle scattering problem which acknowledged the nonlinearity and ill-posedness of the problem use precisely such an approach [85] as discussed already in Section 3. Since the ill-posedness of a nonlinear problem is inherited by its linearization, whenever one tries to approximately solve an ill-posed nonlinear equation by linearization, for example by a Newton method, one obtains ill-posed linear equations for which a regularization must be enforced. In particular, the existence and injectivity of the Fréchet derivative of the far field mapping that corresponds to the inverse scattering problem is the theoretical basis for any approach using Newton's method. As also already mentioned in Section 3, for the case of acoustic obstacle scattering the differentiability of the far field mapping was first shown by Kirsch [51] in 1993 and Potthast [80] in 1994. For the case of acoustic scattering by an inhomogeneous medium the corresponding result was, to our knowledge, first obtained by Hohage in 2001. We now briefly describe Hohage's result [41, 42].

For the scattering problem (4.1) from the Lippmann–Schwinger equation (4.2) we have that the far field pattern u_∞ has the representation

$$u_\infty(\hat{x}, d) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\,\hat{x}\cdot y} m(y) u(y, d)\, dy, \quad x \in \mathbb{S}^2.$$

We can now define a far field mapping $\mathcal{F} \colon L^2(B) \to L^2(\mathbb{S}^2 \times \mathbb{S}^2)$ for B a ball containing the unknown support of m by

$$(\mathcal{F}m)(\hat{x}, d) := -\frac{k^2}{4\pi} \int_B e^{-ikx\cdot y} m(y) u(y, d)\, dy, \quad \hat{x}, d \in \mathbb{S}^2,$$

and prove the following theorem.

Theorem 5.1 *The operator $\mathcal{F} \colon m \mapsto u_\infty$ is Fréchet differentiable. The derivative is given by*

$$\mathcal{F}'_m q = v_\infty$$

where v_∞ is the far field pattern of the radiating solution $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ to

$$\Delta v + k^2 n v = -k^2 u q \quad \textit{in } \mathbb{R}^3. \tag{5.1}$$

Theorem 5.1 can now be used to establish the injectivity of $\mathcal{F}'_m$.

Theorem 5.2 *For piecewise continuous m the operator*

$$\mathcal{F}'_m \colon L^2(B) \to L^2(\mathbb{S}^2 \times \mathbb{S}^2)$$

is injective.

Proof. Assume that $q \in L^2(B)$ satisfies $\mathcal{F}'_m q = 0$. Then for each $d \in \mathbb{S}^2$ the far field pattern of the solution to (5.1) vanishes and hence by Rellich's Lemma 2.1 we have that $v(\cdot, d) = \partial_\nu v(\cdot, d) = 0$ on ∂B. Therefore by Green's second integral theorem we have that

$$k^2 \int_B q u(\cdot, d) w \, dx = 0$$

for all $d \in \mathbb{S}^2$ and all solutions $w \in H^1(B_0)$ of $\Delta w + k^2 n w = 0$ in $B_0 \supset \overline{B}$. Lemma 4.3 now implies that

$$\int_B q w \widetilde{w} \, dx = 0$$

for all $w, \widetilde{w} \in H^1(B_0)$ satisfying $\Delta w + k^2 n w = 0$ and $\Delta \widetilde{w} + k^2 n \widetilde{w} = 0$ in B_0. The proof can now be completed analogous to that in Theorem 4.1. $\square$

The analogues of Theorems 5.1 and 5.2 for the case of electromagnetic waves have been presented by Hohage [42] in 2006. The numerical implementation of Theorems 5.1 and 5.2 has also been done by Hohage in the above cited references.

There have of course been numerous other optimization methods to solve the inverse scattering problem for (4.1) at fixed values of the wave number for both acoustic and electromagnetic waves and for a sample of these we refer to Gutman and Klibanov [32], Kleinman and van den Berg [58], Natterer and Wübbeling [75], and Vögeler [94].

The use of optimization methods to solve the inverse scattering problem for (4.1) at fixed frequency suffer from two major drawbacks. The first of these is the need to solve a direct scattering problem at each step of the iterative process. This has been partially addressed by either constructing appropriate preconditioners to reduce the number of iterations and combining this with so-called fast methods to solve the direct scattering problems that arise at each step of the iteration process or by using decomposition methods to avoid the need to solve a direct problem altogether (c.f. Chapter 10 of

[23]). The second defect in using nonlinear optimization methods at fixed frequency is the problem of local minima. This problem cannot be avoided if the frequency is kept fixed. However, if it is assumed that the medium is non-dispersive, then starting with Chen [17] in 1997 and further developed by Bao and Li and their co-workers [6, 7] recursive linearization algorithms have been developed which use multi-frequency scattering data and proceed via a continuation procedure with respect to the frequency. Used appropriately, this can alleviate the problem of local minima at high frequencies for which the nonlinear equation becomes extremely oscillatory and possesses many more local minima.

Before proceeding we wish to make a few remarks on stability estimates. As noted in the 2001 paper by Hähner and Hohage [38], the best estimate that can be established, even under appropriate a priori constraints, is logarithmic stability. This fact at first glance seems to contradict the many excellent reconstructions that are presented in the literature for given noisy far field data. However the stability estimates measure the difference between two far field patterns corresponding to two different refractive indices and this is not the problem in practice where one is interested in the difference between the noise-free far field pattern and the noisy far field pattern where the latter does not correspond to any refractive index. We also note that for isotropic media, the degree of ill-posedness for the nonlinear inverse scattering problem is the same as that of the linear problem arrived at by using the Born approximation [54, 72].

Finally, we note that nonlinear optimization procedures to solve the inverse scattering problem for anisotropic media are fraught with difficulties due to the fact that as mentioned in Section 4 the solution of the inverse scattering problem in this case is not uniquely determined.

6 Sampling methods again

We have previously seen in Section 3 how the linear sampling method in obstacle scattering developed from a decomposition method in obstacle scattering, the dual space method of Colton and Monk. In the same way as in its counterpart for obstacle scattering, in a first step the dual space method for inhomogeneous media determines $g_z \in L^2(\mathbb{S}^2)$ as the (regularized) solution of the far field equation

$$Fg_z = \Phi_\infty(\cdot, z) \tag{6.1}$$

where $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is the far field operator

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2, \tag{6.2}$$

and

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi}\, e^{-ik\,\hat{x}\cdot z}, \quad \hat{x} \in \mathbb{S}^2,$$

is the far field pattern of a point source at a fixed point z where typically $z = 0$ is chosen. This requires that F is injective with dense range and it was shown in 1986 by Kirsch [50] that this is the case provided k is not a *transmission eigenvalue*, i.e., the only solution of

$$\begin{aligned} \Delta w + k^2 n w &= 0 \quad \text{in } D \\ \Delta v + k^2 v &= 0 \quad \text{in } D \\ w &= v \quad \text{on } \partial D \\ \frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} \quad \text{on } \partial D \end{aligned} \tag{6.3}$$

is $v = w = 0$. Here D is a bounded domain with a connected C^2 boundary such that n is piecewise continuous in $\bar{D}$ and ν is the unit outward normal to ∂D. The topic of transmission eigenvalues will be the subject matter of the next section of this paper. From the same heuristic reasoning as in the obstacle case that we described in Section 3 it is expected that if a solution g_z to (6.1) exists then $\|g_z\|_{L^2(\mathbb{S}^2)} \to \infty$ as z approaches ∂D. In particular, this formal method for determining D works both for obstacle and medium scatterers, i.e., it requires no a priori knowledge of the nature of the scattering object nor does it make use of any "weak scattering" assumption or of any nonlinear optimization method.

Although this approach to the inverse scattering problem was originally called the "simple method", this term was objected to by Michele Piana who was visiting the University of Delaware as a postdoc from Italy. His objection was based on the concern that he would have difficulties obtaining a position in the tough job market in Italy if a major area of his research was based on a "simple" method for solving inverse scattering problems. He proposed instead to call this approach the *linear sampling method* since 1) it was a linear method to determine the support D (without making any "weak scattering" assumptions) and 2) it determined D by "sampling" $\mathbb{R}^3$

with points z to determine where $\|g_z\|$ was large and small. The name has stuck and this is what this approach to the inverse scattering problem is now called.

The linear sampling method was first extended to the case of electromagnetic waves by Haddar and Monk [35]. For a comprehensive discussion of the linear sampling method for electromagnetic waves we refer the reader to [15].

The linear sampling method for both acoustic and electromagnetic waves is based on three simple observations. The first observation (restricting again our attention to the scattering problem (4.1)) is that a solution of the far field equation (6.1) cannot exist unless $z \in D$. This follows from Rellich's lemma since the left hand side of (6.1) is the far field pattern of a solution of the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ whereas this is true for the right side if and only if $z \in D$. The second observation, as already noted, is that the far field operator is injective with dense range if k is not a transmission eigenvalue. The third observation is that Fg is the far field pattern corresponding to the incident field being a Herglotz wave function

$$v_g(x) := \int_{\mathbb{S}^2} g(d) e^{ik\, x\cdot d}\, ds(d), \quad x \in \mathbb{R}^3.$$

Since in general the far field equation (6.1) has no solution, some work is required to make use of the above three observations to arrive at a mathematically correct statement. The result of such an investigation is the following theorem [20, 26].

Theorem 6.1 *Assume that either*

1) $\operatorname{Im} n > 0$ *in* $\bar{D}$ *or*

2) $\operatorname{Im} n = 0$ *in* $\bar{D}$ *and either* $0 < n < 1$ *or* $n > 1$ *in* $\bar{D}$*.*

Then if k is not a transmission eigenvalue for D the following is true:

1. *For $z \in D$ and a given $\varepsilon > 0$ there exists a function $g_z^\varepsilon \in L^2(\mathbb{S}^2)$ such that*
$$\|Fg_z^\varepsilon - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S}^2)} < \varepsilon \tag{6.4}$$
and the Herglotz wave function $v_{g_z^\varepsilon}$ with kernel g_z^ε satisfies
$$\lim_{\varepsilon\to 0} \left\|v_{g_z^\varepsilon}\right\|_{L^2(D)} < \infty.$$

2. *For $z \notin D$ every $g_z^\varepsilon \in L^2(\mathbb{S}^2)$ that satisfies* (6.4) *for a given $\varepsilon > 0$ is such that*

$$\lim_{\varepsilon \to 0} \left\| v_{g_z^\varepsilon} \right\|_{L^2(D)} = \infty.$$

In practice the function g_z^ε in the above theorem is determined by using Tikhonov regularization. However the problem in doing this is that, as mentioned previously, in general there does not exist a solution of the far field equation for noise-free data u_∞ and hence it is not clear what solution is obtained by using Tikhonov regularization. This question has been addressed and clarified by Arens and Lechleiter [3, 4]. The following theorem due to Kirsch [53] provides the foundation for what is called the factorization method (since it is based on a factorization of the far field operator F (c.f. Section 3.3).

Theorem 6.2 *Assume that n is real valued and satisfies the assumptions of Theorem 6.1 and assume again that k is not a transmission eigenvalue for D. Then $z \in D$ if and only if*

$$(F^\star F)^{1/4} g_z = \Phi_\infty(\cdot, z) \tag{6.5}$$

is solvable in $L^2(\mathbb{S}^2)$.

The factorization method of Andreas Kirsch has played a central role in the mathematical theory of inverse scattering theory and for a full discussion of this approach for solving the inverse scattering problem as well as its extensions to the case when n is no longer real valued we refer the reader to the monograph [55]. In particular, Arens and Lechleiter used this method to provide the following justification for using Tikhonov regularization in the linear sampling method for the case when n is real valued:

Theorem 6.3 *Let F be the far field operator, assume that n is real valued, satisfies the assumptions of Theorem 6.1 and assume again that k is not a transmission eigenvalue. For $z \in D$ denote by g_z the solution to* (6.5) *and for $\alpha > 0$ and $z \in \mathbb{R}^3$ let g_z^α denote the solution of the far field equation* (6.1) *obtained by Tikhonov regularization. Let $v_{g_z^\alpha}$ denote the Herglotz wave function with kernel g_z^α. If $z \in D$ then $\lim_{\alpha \to 0} v_{g_z^\alpha}(z)$ exists and*

$$c \left\| g_z \right\|^2 \leq \lim_{\alpha \to 0} \left| v_{g_z^\alpha}(z) \right| \leq \left\| g_z \right\|^2$$

for some $c > 0$ depending only on D. If $z \notin D$ then $\lim_{\alpha \to 0} v_{g_z^\alpha}(z) = \infty$.

The approach used by Kirsch to justify the linear sampling method was to change the operator F to $(F^{\star}F)^{1/4}$. An alternative approach is to keep the operator F but to change the penalty term in the Tikhonov functional associated with F. This point of view leads to the *generalized linear sampling method* of Audibert and Haddar and for details of this method for solving the inverse scattering problem as well as its connections to the linear sampling and factorization methods we refer the reader to [5, 14]. In addition to the linear sampling, factorization and generalized linear sampling methods for solving the inverse scattering problemfor an inhomogeneous medium, other non-optimization methods have also been derived and, in particular, we note the *singular source method* of Potthast [82], the *probe method* of Ikehata [44] and Sylvester's [63, 90] method of *convex scattering support*. For details we refer the reader to the recent monograph [74] as well as the above quoted literature.

In the above discussion on sampling methods to solve the inverse scattering problem we have focused on the scattering problem (4.1). However, as previously mentioned, the sampling methods that we have discussed also extend to the case of anisotropic media and electromagnetic waves. For details we refer to [14, 15]. Such results are of particular interest for anisotropic media since, due to the non-uniqueness of the inverse scattering problem, optimization methods are in general no longer applicable.

7 Transmission eigenvalues

In the previous section we already mentioned the transmission eigenvalue problem in connection with the issue of injectivity and dense range of the far field operator. More generally, the *interior transmission problem* is to find $(v, w) \in L^2(D) \times L^2(D)$ such that $w - v \in H^2(D)$ and

$$\begin{aligned} \Delta w + k^2 n w &= 0 \quad \text{in } D \\ \Delta v + k^2 v &= 0 \quad \text{in } D \\ w - v &= f \quad \text{on } \partial D \\ \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} &= h \quad \text{on } \partial D \end{aligned} \tag{7.1}$$

for given $(f, h) \in H^{3/2}(\partial D) \times H^{1/2}(\partial D)$. (Note that $w - v \in H^2(D)$ and $n > 1$ or $0 < n < 1$ in $\bar{D}$ imply that $(v, w) \in L^2(D) \times L^2(D)$.) The interior transmission problem (7.1) plays an essential role in inverse scattering theory for inhomogeneous media, in particular in the mathematical justification of the sampling methods as discussed in the previous section. As already mentioned, the homogeneous form of the interior transmission problem is referred to as the *transmission eigenvalue problem* and the corresponding eigenvalues as *transmission eigenvalues.* Of central concern are 1) the *Fredholm property and solvability* of the interior transmission problem, 2) the *discreteness* of the set of transmission eigenvalues, 3) the *existence* of transmission eigenvalues and 4) the *determination* of the transmission eigenvalues from scattering data and the relationship between them and the material properties of the inhomogeneous media.

The transmission eigenvalue problem was first introduced by Kirsch [50] in 1986 in connection with the denseness and injectivity of the far field operator. The solvability of the interior transmission problem (and the name transmission eigenvalue problem) was subsequently considered by Colton and Monk in 1988 in their study of the dual space method to solve the inverse scattering problem [24]. Shortly thereafter Colton, Kirsch and Päivärinta [21] and Rynne and Sleeman [86] showed that the set of transmission eigenvalues is discrete. At that time transmission eigenvalues were viewed as something to avoid since the primary concern was to guarantee that the far field operator was injective with dense range. This perhaps explains the fact that it wasn't until 2008 that Päivärinta and Sylvester [79] finally proved the existence of real transmission eigenvalues. All of the above research was for the case of acoustic waves in an inhomogeneous isotropic medium. The solvability of the interior transmission problem for scalar anisotropic media and for Maxwell's equations was investigated by Cakoni, Colton and Haddar in 2002 and Haddar in 2004 respectively in which these authors also established the discreteness of the set of transmission eigenvalues [14, 15]. Finally, the existence of real transmission eigenvalues for scalar anisotropic media and Maxwell's equations was established by Cakoni, Gintides and Haddar [16] in 2010 in which they also established monotonicity properties for transmission eigenvalues and the role of such properties in determining material properties of the inhomogeneous media from measured scattering data. The problem of determining transmission eigenvalues from the far field data of the scattered field was first investigated by Cakoni, Colton and Haddar in 2010 with a totally different approach to this problem being presented by Kirsch and Lechleiter in 2013 (c.f. Section 4.4 of [14]). Finally, the question of whether or not complex transmission eigenvalues exist is an open question except for the case of a spherically stratified medium and for details in this latter case we refer the reader to Chapter 5 of [14]. It has been recently shown by Vodev that if complex transmission eigenvalues exist, then under appropriate conditions they all lie in a strip [92, 93].

The above brief outline of the history of the development of the theory of transmission eigenvalues only scratches the surface of the development of this area of research. Since the initial results mentioned above, the theory has been further enriched by the contributions of numerous researchers, in particular Lakshtanov and Vainberg [64, 65] and the previously mentioned papers by Vodev. For further references as well as a detailed discussion of the results obtained in the above cited papers we refer the reader to [14, 15].

In order to give a flavor of the analysis used in studying the interior transmission problem we consider the transmission eigenvalue problem (6.3) and outline the proof of the following theorem:

Theorem 7.1 *Assume that $n > 1$ for $0 < 1 < n$ in $\overline{D}$. Then the set of transmission eigenvalues is at most discrete with infinity as the only possible accumulation point. Furthermore, the multiplicity of each eigenvalue is finite.*

Proof. We write (6.3) as an equivalent eigenvalue problem for $u = w - v$ in $H_0^2(D)$ for the fourth order equation

$$(\Delta + k^2 n)\,\frac{1}{n-1}\,(\Delta + k^2)u = 0$$

which in variational form, after integrating by parts, is formulated as finding a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1}\,(\Delta u + k^2 u)(\Delta \overline{v} + k^2 n \overline{v})\,dx = 0$$

for all $v \in H_0^2(D)$. This can be written as

$$u - \tau K_1 u + \tau^2 K_2 u = 0$$

where $\tau := k^2$, the operators $K_1, K_2 : H_0^2(D) \to H_0^2(D)$ can be shown to be compact and K_2 is non-negative and self-adjoint. Hence, setting

$$U := \left(u, \tau K_2^{1/2} u\right),$$

the transmission eigenvalue problem becomes the eigenvalue problem

$$\left(K - \frac{1}{\tau}\,\mathrm{I}\right) U = 0$$

for the compact non-selfadjoint operator

$$K\colon H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$$

given by

$$K := \begin{pmatrix} K_1 & -K^{1/2} \\ K_2^{1/2} & 0 \end{pmatrix}.$$

The theorem now follows from the spectral theory for compact operators. □

8 A personal note

We met each other in Oberwolfach in 1975 and hence it seemed particularly appropriate that we completed this paper in 2017 while we were again together in Oberwolfach participating in the Research in Pairs program. Although we first met in 1975 and began our collaboration in 1976 when David was a professor at the Univerisity of Strathclyde and Rainer was spending his sabbatical there, we did not begin working together on inverse scattering theory until the mid 1980s when we both became involved in decomposition methods for solving the inverse scattering problem (see Sections 3 and 6 of this paper).

It was not an accident that we both began working on the same approach to inverse scattering since at that time there was a particularly active interaction between the Mathematical Sciences Department at the University of Delaware and the Institut für Numerische und Angewandte Mathematik at the University of Göttingen. In particular, under the chairmanship of Ivar Stakgold, the University of Delaware had become a leading center for inverse problems and inverse scattering theory with Thomas Angell, David Colton, George Hsiao, Ralph Kleinman, Peter Monk and Zuhair Nashed being actively involved in this field. At the same time, Rainer Kress was busy building up the field of inverse problems at Göttingen through his seminars and supervision of students. As part of these efforts Rainer would send some of his most promising students to Delaware for a year while at the same time supporting regular visits of Delaware faculty for shorter periods. As a result Joachim Blöhbaum, Peter Hähner, Andreas Kirsch and Roland Potthast spent a year a Delaware whereas faculty members in scattering theory at Delaware visited Göttingen for shorter (or sometimes longer!) periods of time.

It was during this period of intense interaction between Delaware and Göttingen that we wrote our two books [22, 23] together. The results on inverse scattering in in our first book [23] are by now outdated whereas our second book [23] has enjoyed widespread popularity in the inverse scattering community as evidenced by a third edition appearing in 2013. We note that the final version of the first edition was completed in 1992 when we were also in Oberwolfach participating in the Research in Pairs program. Indeed, this book influenced many people to enter the field of inverse scattering as indicated by the steady production of PhD students in Göttingen (in particular Thorsten Hohage, Olha Ivanyshyn and Pedro Serranho) as well as the arrival of more postdoctoral students at Delaware, most notably Fioralba Cakoni, Houssem Haddar and Michele Piana. This long term interaction between the two of us, as well with our students, has been instrumental in forming the view of inverse scattering that we have presented in this paper.

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References

[1] Alessandrini, G. and Rondi, L.: Determining a sound–soft polyhedral scatterer by a single far–field measurement. Proc. Amer. Math. Soc. **133**, 1685–1691 (2005).

[2] Angell, T.S., Kleinman, R.E., and Roach, G.F.: An inverse transmission problem for the Helmholtz equation. Inverse Problems **3**, 149–180 (1987).

[3] Arens, T.: Why linear sampling works. Inverse Problems **20**, 163–173 (2004).

[4] Arens, T. and Lechleiter, A.: The linear sampling method revisited. Jour. Integral Equations and Applications **21**, 179–202 (2009).

[5] Audibert, L. and Haddar, H.: A generalized formulation of the linear sampling method with exact characterization of targets in terms of far field measurements. Inverse Problems **30**, 035011 (2014).

[6] Bao, G. and Li, P.: Inverse medium scattering for the Helmholtz equation at fixed frequency. Inverse Problems **21**, 1621–1641 (2005).

[7] Bao, G. and Li, P.: Inverse medium scattering problems for electromagnetic waves. SIAM J. Appl. Math. **65**, 2049–2066 (2005).

[8] Bellis, C., Bonnet, M. and Cakoni, F.: Acoustic inverse scattering using topological derivative of far-field measurements-based L^2 cost functionals. Inverse Problems **29**, 075012 (2013).

[9] Blöhbaum, J.: Optimisation methods for an inverse problem with time-harmonic electromagnetic waves: an inverse problem in electromagnetic scattering. Inverse Problems **5**, 463–482 (1989).

[10] Bojarski, N.N.: Three dimensional electromagnetic short pulse inverse scattering. Spec. Proj. Lab. Rep. Syracuse Univ. Res. Corp., Syracuse 1967.

[11] Bukhgeim, A. L.: Recovering a potential from Cauchy data in the two dimensional case. Jour. Inverse Ill-Posed Problems **16**, 19–33 (2008).

[12] Burger, M.: A level set method for inverse problems. Inverse Problems **17**, 1327–1355 (2001).

[13] Cakoni, F. and Colton, D.: A uniqueness theorem for an inverse electromagnetic scattering problem in inhomogeneous anisotropic media. Proc. Edinburgh Math. Soc. **46**, 285–299 (2003).

[14] Cakoni, F., Colton, D., and Haddar, H.: *Inverse Scattering Theory and Transmission Eigenvalues.* SIAM, Philadelphia, 2016.

[15] Cakoni, F., Colton, D., and Monk, P.: *The Linear Sampling Method in Inverse Electromagnetic Scattering.* SIAM, Philadelphia, 2011.

[16] Cakoni, F., Gintides, D., and Haddar, H.: The existence of an infinite discrete set of transmission eigenvalues. SIAM J. Math. Anal. **42**, 237–255 (2010).

[17] Chen, Y.: Inverse scattering via Heisenberg's uncertainty principle. Inverse Problems **13**, 253–282 (1997).

[18] Cheng, J. and Yamamoto, M.: Uniqueness in an inverse scattering problem within non-trapping polygonal obstacles with at most two incoming waves. Inverse Problems **19**, 1361–1384 (2003).

[19] Colton, D. and Kirsch, A.: Dense sets and far field patterns in acoustic wave propagation. SIAM J. Math. Anal. **15**, 996–1006 (1984).

[20] Colton, D. and Kirsch, A.: A simple method for solving inverse scattering problems in the resonance region. Inverse Problems **12**, 383–393 (1996).

[21] Colton, D., Kirsch, A., and Päivärinta, L.: Far field patterns for acoustic waves in an inhomogeneous medium. SIAM J. Math. Anal. **20**, 1472–1483 (1989).

[22] Colton, D. and Kress, R.: *Integral Equation Methods in Scattering Theory.* Classics in Applied Mathematics Vol. 72, SIAM, Philadelphia, 2013.

[23] Colton, D. and Kress, R.: *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd. ed. Springer, New York, 2013.

[24] Colton, D. and Monk, P.: The inverse scattering problem for acoustic waves in an inhomogeneous medium. Quart. J. Mech. Appl. Math. **41**, 97–125 (1988).

[25] Colton, D. and Päivärinta, L.: The uniqueness of a solution to an inverse scattering problem for electromagnetic waves. Arch. Rational Mech. Anal. **119**, 59–70 (1992).

[26] Colton, D., Piana, M., and Potthast, R.: A simple method using Morozov's discrepancy principle for solving inverse scattering problems. Inverse Problems **13**, 1477–1493 (1997).

[27] Colton, D. and Sleeman, B.D.: Uniqueness theorems for the inverse problem of acoustic scattering. IMA J. Appl. Math. **31**, 253–259 (1983).

[28] Dorn, O. and Lesselier, D.: Level set methods for inverse scattering – some recent developments Inverse Problems **25**, 125001 (2009).

[29] Farhat, C., Tezaur, R., and Djellouli, R.: On the solution of three-dimensional inverse obstacle acoustic scattering problems by a regularized Newton method. Inverse Problems **18**, 1229–1246 (2002).

[30] Feijoo, G.R.: A new method in inverse scattering based on the topological derivative. Inverse Problems **20**, 1819–1840 (2004).

[31] Gintides, D.: Local uniqueness for the inverse scattering problem in acoustics via the Faber–Krahn inequality. Inverse Problems **21**, 1195–1205 (2005).

[32] Gutman, S. and Klibanov, M.: Iterative method for multidimensional inverse scattering problems at fixed frequencies. Inverse Problems **10**, 573–599 (1994).

[33] Gylys-Colwel, F.: An inverse problem for the Helmholtz equation. Inverse Problems **12**, 139–156 (1996).

[34] Haddar, H. and Kress. R.: On the Fréchet derivative for obstacle scattering with an impedance boundary condition. SIAM J. Appl. Math. **65**, 194–208 (2004).

[35] Haddar, H. and Monk, P.: The linear sampling method for solving the electromagnetic inverse medium problem. Inverse Problems **18**, 891–906 (2002).

[36] Hähner, P.: A uniqueness theorem for a transmission problem in inverse electromagnetic scattering. Inverse Problems **9**, 667–678 (1993).

[37] Hähner, P.: On the uniqueness of the shape of a penetrable anisotropic obstacle-Jour. Comp. Appl. Math. **116**, 167–180 (2000).

[38] Hähner, P. and Hohage, T.: New stability estimates for the inverse acoustic inhomogeneous medium problem and applications. SIAM J. Math. Anal. **33**, 670–685 (2001).

[39] Harbrecht, H. and Hohage. T.: Fast methods for three-dimensional inverse obstacle scattering problems. Jour. Integral Equations and Appl. **19**, 237–260 (2007).

[40] Hohage, T.: *Iterative Methods in Inverse Obstacle Scattering: Regularization Theory of Linear and Nonlinear Exponentially Ill-Posed Problems.* Dissertation, Linz, 1999.

[41] Hohage, T.: On the numerical solution of a three-dimensional inverse medium scattering problem. Inverse Problems **17**, 1743–1763 (2001).

[42] Hohage, T.: Fast numerical solution of the electromagnetic medium scattering problem and applications to the inverse problem. J. Comp. Phys. **214**, 224–238 (2006).

[43] Hu, G., Salo, M., and E. Vesalainen, E.: Shape identification in inverse medium scattering problems with a single far field pattern. SIAM J. Math. Anal. **48**, 152–165 (2016).

[44] Ikehata, M.: Reconstruction of the shape of an obstacle from the scattering amplitude at a fixed frequency. Inverse Problems **14**, 949–954 (1998).

[45] Imbriale, W.A. and Mittra, R.: The two-dimensional inverse scattering problem. IEEE Trans. Ant. Prop. **AP-18**, 633–642 (1970).

[46] Isakov, V.: On uniqueness in the inverse transmission scattering problem. Comm. Part. Diff. Equa. **15**, 1565–1587 (1990).

[47] Ivanyshyn, O. and Kress, R.: Nonlinear integral equations in inverse obstacle scattering. In: *Mathematical Methods in Scattering Theory and Biomedical Engineering*, Fotiatis, Massalas, editors, pp 39–50 World Scientific, Singapore, 2006.

[48] Ivanyshyn, O and Louër, L.: Material derivatives of boundary integral operators in electromagnetism and application to inverse scattering problems. Inverse Problems **32**, 095003 (2016).

[49] Johansson, T. and Sleeman, B.: Reconstruction of an acoustically sound-soft obstacle from one incident field and the far field pattern. IMA J. Appl. Math. **72**, 96–112 (2007).

[50] Kirsch, A.: The denseness of the far field patterns for the transmission problem. IMA J. Appl. Math. **37**, 213–225 (1986).

[51] Kirsch, A.: The domain derivative and two applications in inverse scattering. Inverse Problems **9**, 81–96 (1993).

[52] Kirsch, A.: Characterization of the shape of the scattering obstacle by the spectral data of the far field operator. Inverse Problems **14**, 1489–1512 (1998).

[53] Kirsch, A.: Factorization of the far field operator for the inhomogeneous medium case and an application to inverse scattering theory. Inverse Problems **15**, 413–429 (1999).

[54] Kirsch, A.: Remarks on the Born approximation and the factorization method. Appl. Anal. **96**, 70–84 (2017).

[55] Kirsch, A. and Grinberg, N.: *The Factorization Method for Inverse Problems.* Oxford University Press, Oxford, 2008.

[56] Kirsch, A. and Kress, R.: On an integral equation of the first kind in inverse acoustic scattering. In: *Inverse Problems* (Cannon and Hornung, eds). ISNM **77**, 93–102 (1986).

[57] Kirsch, A. and Kress, R.: Uniqueness in inverse obstacle scattering. Inverse Problems **9**, 285–299 (1993).

[58] Kleinman, R. and van den Berg, P.: A modified gradient method for two dimensional problems in tomography. J. Comp. Appl. Math. **42**, 17–35 (1992).

[59] Kress, R.: Inverse scattering from an open arc. Math. Meth. in the Appl. Sci. **18**, 267–293 (1995).

[60] Kress, R.: Newton's Method for inverse obstacle scattering meets the method of least squares. Inverse Problems **19**, 91–104 (2003).

[61] Kress, R. and Päivärinta, L.: On the far field in obstacle scattering. SIAM J. Appl. Math. **59**, 1413–1426 (1999).

[62] Kress, R. and Rundell, W.: Nonlinear integral equations and the iterative solution for an inverse boundary value problem. Inverse Problems **21**, 1207–1223 (2005).

[63] Kusiak, S. and Sylvester, J.: The scattering support. Comm. Pure Applied Math. **56**, 1525–1548 (2003).

[64] Lakshtanov, E. and Vainberg, B.: Ellipticity in the interior transmission problem in anisotropic media. SIAM J. Math. Anal. **44**, 1165–1174 (2012).

[65] Lakshtanov, E. and Vainberg, B.: Sharp Weyl law for signed counting function of positive interior transmission eigenvalues. SIAM J. Math. Anal. **47**, 3212–3224 (2015).

[66] Lax, P.D. and Phillips, R.S.: *Scattering Theory.* Academic Press, New York 1967.

[67] Le Louër, F.: On the Fréchet derivative in elastic obstacle scattering, SIAM J. Appl. Math. **72**, 1493–1507 (2012).

[68] Liu, C.: Inverse obstacle problem: local uniqueness for rougher obstacles and the identification of a ball. Inverse Problems **13**, 1063–1069 (1997).

[69] Liu, H., Yamamoto, M., and Zou, J.: Reflection principle for the Maxwell equations and its application to inverse electromagnetic scattering. Inverse Problems **23**, 2357–2366 (2007).

[70] Liu, H., and Zou, J.: Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers. Inverse Problems **22**, 515–524 (2006).

[71] Mönch, L.: On the inverse acoustic scattering problem from an open arc: the sound-hard case. Inverse Problems **13**, 1379–1392 (1997).

[72] Moskow, S. and Schotland, J.: Convergence and stability of the inverse Born series for diffuse waves. Inverse Problems **24**, 065004 (2008).

[73] Nachman, A.: Reconstructions from boundary measurements. Annals of Math. **128**, 531–576 (1988).

[74] Nakamura, G and Potthast, R.: *Inverse Modeling.* IOP Publishing, Bristol, 2015.

[75] Natterer, F., and Wübbeling, F.: A propagation-backpropagation method for ultrasound tomography. Inverse Problems **11**, 1225–1232 (1995).

[76] Novikov, R.: Multidimensional inverse spectral problems for the equation $-\Delta\psi + (v(x) - E\,u(x))\,\psi = 0$. Translations in Func. Anal. and its Appl. **22**, 263–272 (1988).

[77] Ola, P., Päivärinta, L., and Somersalo, E.: An inverse boundary value problem in electrodynamics. Duke Math. Jour. **70**, 617–653 (1993).

[78] Osher, S. and Sethian, J. A.: Fronts propagating with curvature-dependent speed: algorithms based on Hamilton–Jacobi formulations. J. Comput. Phys. **79**, 12–49 (1988).

[79] Päivärinta, L. and Sylvester, J.: Transmission eigenvalues. SIAM J. Math. Anal. **40**, 738–758 (2008).

[80] Potthast, R.: Fréchet differentiability of boundary integral operators in inverse acoustic scattering. Inverse Problems **10**, 431–447 (1994).

[81] Potthast, R.: A fast new method to solve inverse scattering problems. Inverse Problems **12**, 731–742 (1996).

[82] Potthast, R.: Stability estimates and reconstructions in inverse acoustic scattering using singular sources. J. Comp. Appl. Math. **114**, 247–274 (2000).

[83] Potthast, R.: *Point-Sources and Multipoles in Inverse Scattering Theory.* Chapman & Hall, London, 2001.

[84] Ramm, A.G.: Recovery of the potential from fixed energy scattering data. Inverse Problems **4**, 877–886 (1988).

[85] Roger, A.: Newton Kantorovich algorithm applied to an electromagnetic inverse problem. IEEE Trans. Ant. Prop. **AP-29**, 232–238 (1981).

[86] Rynne, B.P. and Sleeman, B.D.: The interior transmission problem and inverse scattering from inhomogeneous media. SIAM J. Math. Anal. **22**, 1755–1762 (1991).

[87] Santosa, F.: A level set approach for inverse problems involving obstacles. ESAIM Control Optim. Calculus Variations **1**, 17–33 (1996).

[88] Serranho, P.: A hybrid method for sound-soft obstacles in 3D. Inverse Problems and Imaging **1**, 691–712 (2007).

[89] Sokolowski, J. and Zochowski, A.: On the topological derivative in shape optimization SIAM J. Control Optim., **37**, 1251–1272 (1999).

[90] Sylvester, J.: Notions of support for far fields. Inverse Problems **22**, 1273–1288 (2006).

[91] Sylvester, J. and Uhlmann, G.: A global uniqueness theorem for an inverse boundary value problem. Ann. of Math. **125**, 153–169 (1987).

[92] Vodev, G.: Transmission eigenvalue-free regions. Comm. Math. Physics **336**, 1141–1166 (2015).

[93] Vodev, G.: High-frequency approximation of the interior Dirichlet-to-Neumann map and applications to the transmission eigenvalues. To appear.

[94] Vögeler, M.: Reconstruction of the three-dimensional refractive index in electromagnetic scattering using a propagation-backpropagation method. Inverse Problems **19**, 739–753 (2003).