Minimal stabilization techniques for incompressible flows

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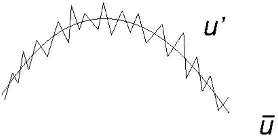

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Outline

F. Brezzi/ M. Fortin: *A minimal stabilisation procedure for mixed finite element methods*, Numer. Math. 89 (2001), 457-492.

- **Goal:** Critical review of stabilization techniques for inf-sup stable pairs (in view of VMS-methods)
- **Acknowledgments:** Thanks to G. Matthies, J. Löwe, T. Heister and X. Zhang.

Outline

Treatment of nonstationary laminar Navier-Stokes problem

Find $U = (\mathbf{u}, p) \in V \times Q := \left(H_0^1(\Omega)\right)^d \times L_0^2(\Omega)$ s.t. for $t \in (0, T)$ a.e.

$$B\big(U, V\big) = (\tilde{\mathbf{f}}, \mathbf{v}) \qquad \forall V = \big(\mathbf{v}, q\big) \in V \times Q \tag{1}$$

$$B\big(U, V\big) := (\partial_t \mathbf{u}, v) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \big((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}\big) - (p, \operatorname{div} \mathbf{v}) + (q, \operatorname{div} \mathbf{u}).$$

- Semidiscretise (1) first in time (e.g., BDF(q) or SDIRK methods).
- Newton-type iteration in each time step leads to Oseen type problem:

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$$a\big(U,V\big) = \big(\mathbf{f}, \mathbf{v}\big) \qquad \forall V = \big(\mathbf{v}, q\big) \in V \times Q.$$

$$a\big(U,V\big) := \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \big((\mathbf{b}\cdot\nabla)\mathbf{u} + \sigma\mathbf{u}, \mathbf{v}\big) - (p, \operatorname{div}\mathbf{v}) + (q, \operatorname{div}\mathbf{u})$$

with given $\mathbf{b} \in H(\operatorname{div},\Omega) \cap \left(L^\infty(\Omega)\right)^d$, $\operatorname{div}\mathbf{b} = 0$, $\quad \nu > 0$ and $\frac{1}{\Delta t} \sim \sigma \geq 0$

Galerkin finite element discretization

- $\mathcal{T}_h$ – shape-regular decomposition of polyhedral domain Ω
- $\mathbb{Y}^{r}_{\mathcal{T}_h} := \{v \in C(\bar{\Omega}) \mid v|_K \in \mathbb{P}_r(K) \text{ or } \mathbb{Q}_r(K) \ \forall K \in \mathcal{T}_h\}, \quad r \in \mathbb{N}$
- FE spaces for velocity/ pressure:

$$\mathbf{V}_h^r := \left[\mathbb{Y}^r_{\mathcal{T}_h} \cap H^1_0(\Omega)\right]^d, \qquad \mathbf{Q}_h^{r-1} := \mathbb{Y}^{r-1}_{\mathcal{T}_h} \cap L^2_0(\Omega)$$

with discrete inf-sup (LBB) compatibility condition

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Galerkin FEM:

$$\text{Find} \quad U = (\mathbf{u}, p) \in \mathbf{W}_h^{r,r-1} := \mathbf{V}_h^r \times \mathbf{Q}_h^{r-1}, \text{ s.t.}$$

$$a(U,V) = (\mathbf{f}, \mathbf{v}) \quad \forall\, V = (\mathbf{v}, q) \in \mathbf{W}_h^{r,r-1}$$

Goal: Robustness w.r.t. ν, σ, h

”Classical” residual-based stabilisation (RBS)

Residual-based scheme:

$$\text{Find } U = (\mathbf{u}, p) \in \mathbf{W}_h^{r,r-1} = \mathbf{V}_h^r \times \mathbf{Q}_h^{r-1}, \text{ s.t.}$$

$$a_{rbs}(U,V) \;=\; l_{rbs}(V) \quad \forall\, V = (\mathbf{v}, q) \in \mathbf{W}_h^{r,r-1}$$

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$$a_{rbs}(U,V) := a(U,V) + \underbrace{\sum_{K\in\mathcal{T}_h} \left(L_{Os}(\mathbf{u},p),\ \tau_K((\mathbf{b}\cdot\nabla)\mathbf{v} + \nabla q)\right)_K}_{SUPG-\ and\ PSPG-stabilisation} + \underbrace{\sum_{K\in\mathcal{T}_h} \left(\gamma_K\ (\nabla\cdot\mathbf{u}), \nabla\cdot\mathbf{v}\right)_K}_{(div-)div-stabilisation}$$

$$l_{rbs}(V) := (\mathbf{f},\mathbf{v})_\Omega + \overbrace{\sum_{K\in\mathcal{T}_h} \left(\mathbf{f}, \tau_K((\mathbf{b}\cdot\nabla)\mathbf{v} + \nabla q)\right)_K}$$

Other variants:

- Galerkin/ Least-squares method (GaLS): Test with $\tau_K L_{Os}(\mathbf{v}, q)$
- Algebraic subgrid-scale method ("unusual" GaLS): Test with $-\tau_K L_{Os}^*(\mathbf{v}, q)$

A-priori analysis on isotropic meshes

$$|[V]|_{rbs}{}^2 := \|\sqrt{\nu}\vec{S}(\vec{v})\|^2_{L^2(\Omega)} + \|\sqrt{c}\vec{v}\|^2_{L^2(\Omega)} + \sum_K \Big(\delta_K \|(\vec{b}\cdot\vec{\nabla})\vec{v} + \vec{\nabla} q\|^2_{L^2(K)} + \gamma_K \|\vec{\nabla}\cdot\vec{v}\|^2_{L^2(K)} \Big)$$

Theorem: see: GL/ G. Rapin M^3AS 16 (2006) 7

- $\delta^u_K = \delta^p_K \sim \frac{h_K^2}{r^2\gamma_0}$ (SUPG/PSPG)
- $\gamma_K \sim \frac{h_K^2}{r^2\delta_K} \sim \gamma_0 \sim 1$ (div-div) **Attention required !?**

$\rightsquigarrow$

- **Stability:** $\mathcal{A}_{rbs}(\vec{b}; V, V) \geq \frac{1}{2} |[V]|_{rbs}{}^2, \quad \forall V = \{\vec{v}, q\} \in \mathbf{V}_h^r \times \mathbf{Q}_h^{r-1}$
- **Consistency:** $\mathcal{A}_{rbs}(\vec{b}; U - U_h, V) = 0, \quad \forall V \in \mathbf{V}_h^r \times \mathbf{Q}_h^{r-1}$
- **A-priori estimate:**

$$\| [U - U_h] \|_{rbs}{}^2 \preceq \sum_{K\in\mathcal{T}_h} \frac{h_K^{2l}}{r^{2k}} \left(\left(1 + \frac{\|\nu\|^2_{W^{k-1,\infty}(K)}}{\|\nu\|_{L^\infty(K)}} \right) \|\vec{u}\|^2_{H^{k+1}(K)} + \|p\|^2_{H^k(K)} \right)$$

Summary: Classical RBS-schemes

Pro's of RBS-schemes

- A-priori analysis available for h- and p-variants
- Robust w.r.t. h and almost w.r.t. r

Con's of RBS-schemes

- Basic drawback: Strong velocity-pressure coupling in SUPG-terms !
  $\Rightarrow$ (Rather) expensive implementation in 3D !
- Sensitive design of parameters τ_K and γ_K
- Non-symmetric form of stabilisation terms
- Construction of efficient preconditioners for mixed algebraic problem !

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Problem:

Is PSPG-stabilization necessary for div-stable pairs ?

Outline

Reduced RBS-scheme

Reduced RBS scheme: Joint work with G. Matthies, L. Röhe (2008)

$$\text{Find } U = (\mathbf{u}, p) \in \mathbf{W}_h^{r,r-1} = \mathbf{V}_h^r \times \mathbf{Q}_h^{r-1}, \text{ s.t.}$$

$$a_{red}(U, V) \; = \; l_{red}(V) \quad \forall\, V = (\mathbf{v}, q) \in \mathbf{W}_h^{r,r-1}$$

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$$a_{red}(U,V) := a(U,V) + \underbrace{\sum_{K\in\mathcal{T}_h} \left(L_{Os}(\mathbf{u},p),\ \tau_K((\mathbf{b}\cdot\nabla)\mathbf{v})\right)_K}_{SUPG-stabilisation} + \underbrace{\sum_{K\in\mathcal{T}_h} \left(\gamma_K\,(\nabla\cdot\mathbf{u}), \nabla\cdot\mathbf{v}\right)_K}_{(div-)div-stabilisation}$$

$$l_{red}(V) := (\mathbf{f},\mathbf{v})_\Omega + \overbrace{\sum_{K\in\mathcal{T}_h} \left(\mathbf{f}, \tau_K((\mathbf{b}\cdot\nabla)\mathbf{v})\right)_K}$$

with Oseen operator $L_{Os}(\mathbf{u},p) := -\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \sigma\mathbf{u} + \nabla p$

Remark: Approach is often used in practice (Turek, Codina etc.) !?

Stability analysis

Seminorm/ norm:

$$|[\mathbf{v}]|_{red} := \Big(\nu|\mathbf{v}|_1^2 + \sigma\|\mathbf{v}\|_0^2 + \sum_K \gamma_K\|\nabla\cdot\mathbf{v}\|_{0,K}^2 + \sum_K \tau_K\|\mathbf{b}\cdot\nabla\mathbf{v}\|_{0,K}^2\Big)^{\frac{1}{2}}$$

$$|||V|||_{red} := \Big(|[\mathbf{v}]|_{red}^2 + \alpha\|q\|_0^2\Big)^{\frac{1}{2}}$$

Conditional stability:

- $0 \le \gamma_K \le \gamma, \qquad 0 \le \tau_K \le \frac{\beta_0^2}{30\mu^2}\,\frac{\mathbf{h_K^2}}{\varphi^2}$

  with $\qquad \varphi^2 := \nu + \sigma C_F^2 + \|\mathbf{b}\|_{L^\infty(\Omega)}^2 \min\left(\frac{1}{\sigma}; \frac{C_F^2}{\nu}\right) + \gamma$

- Pressure control constant: $\qquad \frac{16}{15}\cdot\frac{\beta_0^2}{\varphi^2} \le \alpha \le \frac{26}{15}\cdot\frac{\beta_0^2}{\varphi^2}$

$\rightsquigarrow \quad \exists\, \beta_S \neq \beta_S(\nu,\sigma,h) : \qquad \inf_{V_h}\sup_{W_h} \frac{a_{red}(V_h,W_h)}{|||V_h|||_{red}\ |||W_h|||_{red}} \ge \beta_S > 0$

Sketch of stability proof:

1. Show

$$a_{red}\big((\mathbf{v}_h, q_h), (\mathbf{v}_h, q_h)\big) \geq C_1 \, |[\mathbf{v}_h]|^2_{red} - \overline{\delta} \, \|p\|^2_0$$

with constants C_1 and $\overline{\delta}$. Critical constant $\overline{\delta}$ scales like τ_K / h_K^2

2. Get from discrete inf-sup condition existence of $\mathbf{z}_h \in \mathbf{V}_h^r$ s.t.:

$$a_{red}\big((\mathbf{v}_h, q_h), (-\mathbf{z}_h, 0)\big) \geq \frac{2}{3}\beta_0 \, \|p\|^2_0 - C_2 \, |[\mathbf{v}_h]|^2_{red}$$

with C_2 scaling like φ^2.

3. $(\mathbf{w}_h, r_h) := (\mathbf{v}_h, q_h) + \lambda(-\mathbf{z}_h, 0) \in \mathbf{V}_h^r \times \mathbf{Q}_h^{r-1}$ with appropriate $\lambda > 0$ satisfies

$$a_{red}\big((\mathbf{v}_h, q_h), (\mathbf{w}_h, r_h)\big) \geq C_3 \big|\big|\big|(\mathbf{v}_h, q_h)\big|\big|\big|^2_{red}$$

and

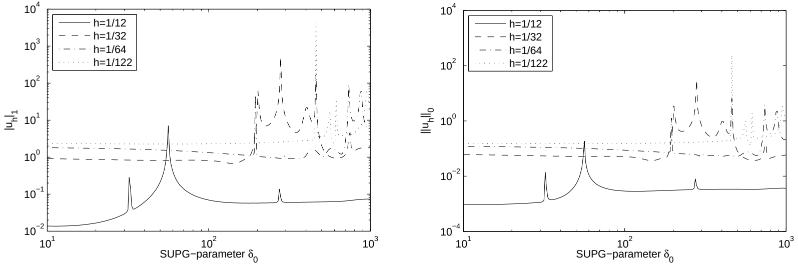
$$\big|\big|\big|(\mathbf{w}_h, r_h)\big|\big|\big|_{red} \leq C_4 \big|\big|\big|(\mathbf{v}_h, q_h)\big|\big|\big|_{red}$$

which together result in the assertion.

There is a conditional stability problem for SUPG !

Vortex flow: $u(x) = (\sin(2\pi x_1)\cos(2\pi x_2), -\cos(2\pi x_1)\sin(2\pi x_2))^T$

- Exact solution of incompressible Euler problem for $\nu = 0$
- Test with Q_2/Q_1-pair for $\nu = 10^{-6}$ and $\sigma = 0$



H^1- and L^2-norms vs. scaling parameter δ_0 of SUPG-stabilization with $\delta_K = \delta_0 h_K^2$ (without div-stabilization) and different values of h

Convergence result. Parameter design

Preliminary a-priori estimate:

- Let $(\mathbf{u}, p) \in [\mathbf{V} \cap H^{r+1}(\Omega)]^d \times [\mathbf{Q} \cap H^r(\Omega)]$.
- Stability assumption implies $\tau_K \leq \frac{Ch_K^2}{\gamma+\nu+\sigma C_F^2}$.

$$|||U - U_h|||_{red}^2 \leq C \sum_K \Big(\frac{1}{\nu + \gamma_K} h_K^{2r} \|p\|_{r,K}^2$$

$$+ \left[\gamma_K + \nu + \sigma h_K^2 + \tau_K \|\mathbf{b}\|_{\infty,K}^2 + \frac{\|\mathbf{b}\|_{\infty,K}^2 h_K^2}{\tau_K \|\mathbf{b}\|_{\infty,K}^2 + \nu + \sigma h_K^2} \right] h_K^{2r} \|\mathbf{u}\|_{r+1,\omega(K)}^2 \Big)$$

Conclusions for stabilization:

- SUPG **not** required if: $\nu \geq \|\mathbf{b}\|_{\infty,K}^2 h_K^2$, i.e. $Re_K := \frac{h_K \|\mathbf{b}\|_{\infty,K}}{\nu} \leq \frac{1}{\sqrt{\nu}}$
  and/ or $\sigma \geq \|\mathbf{b}\|_{\infty,K}^2$ $\rightsquigarrow$ time step restriction: $h_K^2 \lesssim \delta t \sim \frac{1}{\sigma} \lesssim \frac{1}{\|\mathbf{b}\|_{\infty}^2}$
- Div-stabilization useful if: $\|p\|_{r,K} \sim (\nu + \gamma_K) \|\mathbf{u}\|_{r+1,\omega(K)}$

Upper bound of critical Galerkin terms

Upper bound of $a_{red}(\cdot,\cdot)$ requires sharp estimates of Galerkin terms:

- Discrete-divergence preserving interpolant I_h GIRAULT/SCOTT ['03]
- Standard Lagrangian interpolant J_h

$\rightsquigarrow$ For all $W_h = (\mathbf{w}_h, r_h) \in \mathbf{V}_h \times \mathbf{Q}_h$:

$$(r_h, \nabla \cdot (\mathbf{u} - I_h \mathbf{u})) = 0 \qquad \text{avoids negative power of pressure weight } \alpha$$

$$|(p - J_h p, \nabla \cdot \mathbf{w}_h)| \leq C\Big(\sum_K \frac{2}{\nu + \gamma_K} h_K^{2r} \|p\|_{r,K}^2\Big)^{\frac{1}{2}} |||W_h|||_{red}$$

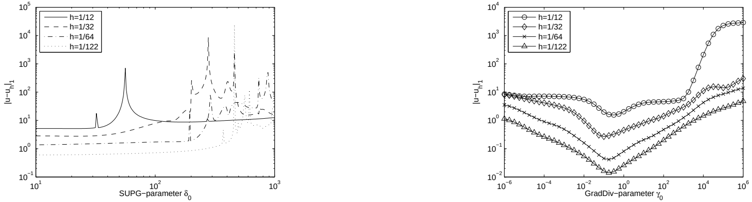
$$|(\mathbf{b} \cdot \nabla(\mathbf{u} - I_h \mathbf{u}), \mathbf{w_h})| \leq C\Big(\sum_K \frac{3\|\mathbf{b}\|_{\infty,K}^2 h_K^2}{\tau_K \|\mathbf{b}\|_{\infty,K}^2 + \nu + \sigma h_K^2} h_K^{2r} \|\mathbf{u}\|_{r+1,\omega(K)}^2\Big)^{\frac{1}{2}} |||W_h|||_{red}$$

$\rightsquigarrow$ Div- resp. SUPG-stabilization avoid negative powers of ν resp. ν, σ

Convergence with SUPG and without div-stabilisation ?

Vortex flow: Problem with rotation and $(b \cdot \nabla)u \not\equiv 0$

- **Data:** $\nu = 10^{-6}$, $\sigma = 0$, $h \in \{\frac{1}{12}, \frac{1}{32}, \frac{1}{64}, \frac{1}{122}\}$
- **SUPG-stabilization:** $\tau_K = \tau_0 h_K^2$, **Div-stabilization:** $\gamma_K = \gamma_0$



Left: H^1-velocity error vs. scaling parameter τ_0 of SUPG-stabilization with $\gamma_K = 0$

Right: H^1-velocity error vs. scaling parameter γ_0 of div-stabilization with $\tau_K = 0$

- "Optimized" div-stabilization outperforms SUPG-stabilization !

Role of grad-div stabilization I

Examples with $\|p\|_{r,K} \sim \|\mathbf{u}\|_{r+1,\omega(K)} \quad \rightsquigarrow \; \gamma \sim 1 \gg \nu$

$$-\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}$$

with solution $\mathbf{u} = \mathbf{b} = (\sin(\pi x_1), -\pi x_2 \cos(\pi x_1))^T, \;\; p = \sin(\pi x_1)\cos(\pi x_2)$

Table: Comparison of different variants of stabilization with Q_2/Q_1 and $\nu = 10^{-6}, \sigma = 1, h = \frac{1}{64}$

SUPG: τ_0	div: γ_0	PSPG: α_0	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_1$	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$\|\nabla\cdot\mathbf{u}_h\|_0$	$\|p-p_h\|_0$
0.000	0.000	0.000	2.56E-1	5.42E-4	2.02E-1	2.31E-4
0.056	0.562	0.010	1.91E-3	6.21E-6	1.82E-4	9.08E-5
0.056	0.562	0.000	1.91E-3	6.20E-6	1.66E-4	8.06E-5
0.000	0.562	0.000	2.61E-3	7.42E-6	1.72E-4	8.05E-5
3.162	0.000	0.000	1.87E-2	7.50E-5	1.56E-2	1.08E-4

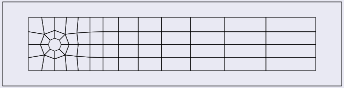
Problem:

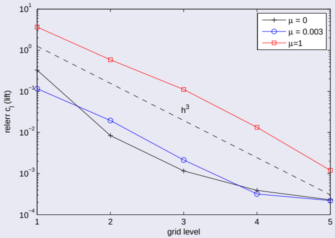
General criterion for div-div stabilization *or* a-posteriori approach ?!

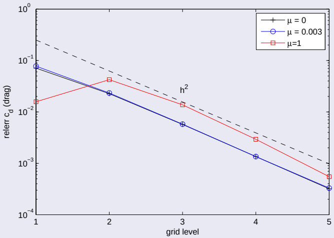
Role of grad-div stabilization II

Examples with $\|p\|_{r,K} \ll \|\mathbf{u}\|_{r+1,\omega(K)}$

- Poiseuille flow: $\nabla p = \nu \Delta u$ $\rightsquigarrow$ div-stabilization superfluous
- Stationary flow around cylinder at $\nu = 0.001$ (corresponds to $Re = 20$)





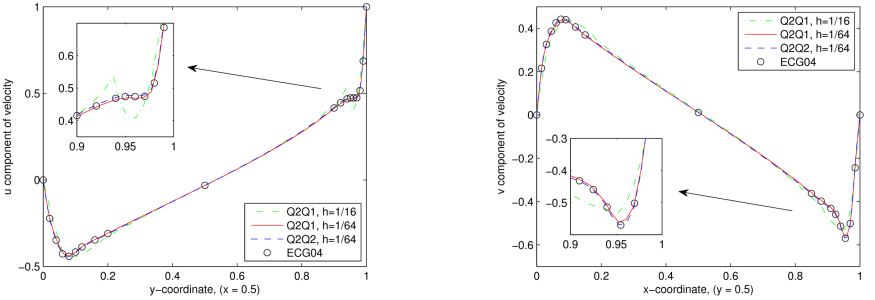


Convergence plots with Q_2/Q_1 for lift and drag coefficients

Can SUPG be avoided for laminar Navier-Stokes flows ?

Example: Driven cavity with stationary solutions

- Non-stationary approach with moderately large time steps
- SUPG is not required up to $Re = 7.500$



Driven-cavity problem with $Re = 5,000$: Cross-sections of the solutions for Q_2/Q_1 without SUPG/PSPG and Q_2/Q_2 with SUPG/PSPG

Summary: Reduced stabilized scheme for laminar flows

- **In practice:**
  Pressure stabilization (PSPG) is often omitted (cf. Turek, Codina et.al.)
- **Problem:** Conditional stability result $\tau_K \leq \tau_0 h_K^2$ for SUPG !
- **But:** SUPG stabilization can be often avoided for laminar flows
  . Div-stabilization outperforms SUPG for rotating flows.
- **Open problem:** Convincing parameter design of div-stabilization.
- **Remedy:** Symmetric stabilization techniques (like LPS) !

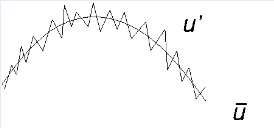
Outline

Variational multiscale (VMS)-decomposition

Decomposition of trial and test spaces:
large (resolved) scales + fine (resolved) scales + fine (unresolved) scales

$$\mathcal{W} = \overline{\mathcal{W}}_{\mathbf{h}} \oplus \tilde{\mathcal{W}}_{\mathbf{h}} \oplus \hat{\mathcal{W}}$$

$$U = \overline{U}_h + \tilde{U}_h + \hat{U}, \quad V = \overline{V}_h + \tilde{V}_h + \hat{V}$$



Decomposition of weak form:

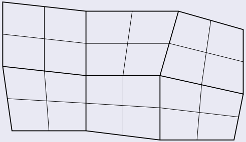
$$a(\overline{U}_h + \tilde{U}_h + \hat{U}, V) = F(V) \;\; \forall\, V = \overline{V}_h + \tilde{V}_h + \hat{V} \in \overline{\mathcal{W}}_h \oplus \tilde{\mathcal{W}}_h \oplus \hat{\mathcal{W}}$$

Scale separation: No direct influence of $\hat{U}$ on $\overline{U}_h$
Subgrid viscosity model: Unresolved scales $\hat{U}$ dissipate energy from $\tilde{U}_h$

Resulting discrete problem: Find $U_h := \overline{U}_h + \tilde{U}_h \in \mathcal{W}_h := \overline{\mathcal{W}}_h \oplus \tilde{\mathcal{W}}_h$

$$a(U_h, V_h) + s_h(U_h, V_h) = F(V_h) \qquad \forall\, V_h := \overline{V}_h + \tilde{V}_h \in \mathcal{W}_h$$

Two-grid setting and local projection



Primal grid $\mathcal{T}_h$ with FE spaces
$\mathbf{V}_h^r$ and $\mathbf{Q}_h^{r-1}$ for velocity and pressure

Macro grid $\mathcal{M}_h = \mathcal{T}_{2h}$
with discontinuous FE spaces of reduced order

- $\mathbf{D}_h^u := \{v \in [L^2(\Omega)]^d : \; v|_M \in \mathbb{Q}_{\mathcal{M}_h}^{r-1}, \; \forall M \in \mathcal{M}_h \}$
- $\mathbf{D}_h^p := \{v \in L^2(\Omega) : \quad v|_M \in \mathbb{Q}_{\mathcal{M}_h}^{r-2}, \; \forall M \in \mathcal{M}_h \},$

- **Local L^2-projection:** $\pi_M^{u/p} : L^2(M) \rightarrow \mathbf{D}_h^{u/p}|_M$
- **Global projection:** $\pi_h^{u/p} : L^2(\Omega) \rightarrow \mathbf{D}_h^{u/p}, \quad (\pi_h^{u/p} w)|_M := \pi_M^{u/p}(w|_M)$

Local projection stabilisation

Fluctuation operators:

- $\kappa_h^{u/p} : [L^2(\Omega)] \to [L^2(\Omega)], \qquad \kappa_h^{u/p} := id - \pi_h^{u/p}$
- $\vec{\kappa}_h^u : [L^2(\Omega)]^d \to [L^2(\Omega)]^d, \qquad \vec{\kappa}_h^u \vec{w} := \big((id - \pi_h^u) w_i\big)_{i=1}^d$

Local projection stabilisation

Fluctuation operators:

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Discrete LPS-problem: Subgrid model as minimal stabilization

Find $U_h = (\mathbf{u}_h, p_h) \in \mathbf{W}_h^{r,r-1} : \quad a_{lps}(U_h, V) = (\mathbf{f}, \mathbf{v}) \ \forall\, V = (\mathbf{v}, q) \in \mathbf{W}_h^{r,r-1}$

$$a_{lps}(U,V) := a(U,V) + s_h(U,V).$$

$$s_h(U,V) = \sum_M \underbrace{\alpha_M (\vec{\kappa}_h^u \nabla p, \vec{\kappa}_h^u \nabla q)_M}_{pressure \quad stab.} + \underbrace{\tau_M \left(\vec{\kappa}_h^u \mathbf{b} \cdot \nabla \mathbf{u}, \vec{\kappa}_h^u \mathbf{b} \cdot \nabla \mathbf{v}\right)_M}_{advection \quad stab.} + \underbrace{\gamma_M (\kappa_h^p \nabla \cdot \mathbf{u}, \kappa_h^p \nabla \cdot \mathbf{v})_M}_{divergence \quad stab.}$$

Stability

Comparison of LPS to RBS-schemes:

- Symmetric, non-consistent form of stabilization terms
- Stabilization (or: subgrid viscosity) term acts only on ”fine” scales (!)

Stability

Comparison of LPS to RBS-schemes:

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$$|||V_h|||_{lps} := \left(|[V_h]|^2_{lps} + \alpha\|q\|^2_0\right)^{\frac{1}{2}}, \qquad \alpha = \alpha(\nu,\sigma) > 0$$

$$|[V_h]|_{lps} := \left(\nu\|\nabla \mathbf{v}_h\|^2_0 + \sigma\|\mathbf{v}_h\|^2_0 + s_h(V_h,V_h)\right)^{\frac{1}{2}}$$

Unconditional (!) stability: $\Longrightarrow$ Existence / uniqueness

$$\inf_{V_h\in\mathbf{W}_h^{r,r-1}} \sup_{W_h\in\mathbf{W}_h^{r,r-1}} \frac{(a+s_h)(V_h,W_h)}{|[V_h]|_{lps}\,|[W_h]|_{lps}} \geq 1$$

$$\exists\beta_S \neq \beta_S(\nu,h): \quad \inf_{V_h\in\mathbf{W}_h^{r,r-1}} \sup_{W_h\in\mathbf{W}_h^{r,r-1}} \frac{(a+s_h)(V_h,W_h)}{|||V_h|||_{lps}\,|||W_h|||_{lps}} \geq \beta_S > 0$$

A-priori error estimate

Technical ingredient: MATTHIES/TOBISKA M^2AS [2007]

Construction of special interpolation operator $\vec{j}_h^u$

- s.t. $\mathbf{v} - \vec{j}_h^u \mathbf{v}$ is L^2-orthogonal to $\mathbf{D}_h^u$ for all $\mathbf{v} \in \mathbf{V}$
- which preserves the discrete divergence constraint.

Preliminary a-priori estimate:

- *Let* $\mathbf{u} \in [H_0^1(\Omega) \cap H^{r+1}(\Omega)]^d, \; p \in L_0^2(\Omega) \cap H^r(\Omega)$.

$\Rightarrow \; \exists C \neq C(\nu, \sigma, h)$:

$$|[U - U_h]|_{lps}^2 \leq C \sum_{M \in \mathcal{M}_h} \Big((\alpha_M + \frac{h_M^2}{\gamma_M}) h_M^{2(r-1)} \|p\|_{r,\omega_M}^2 + \tau_M h_M^{2r} \|\mathbf{b} \cdot \nabla \mathbf{u}\|_{r,\omega_M}^2$$

$$+ \; (\nu + \sigma h_M^2 + \gamma_M + \frac{h_M^2}{\tau_M} + \|\mathbf{b}\|_{\infty,M}^2 \tau_M) \; h_M^{2r} \|\mathbf{u}\|_{r+1,\omega_M}^2 \Big)$$

Parameter design

Conclusions for parameter design:

- **PSPG-type term:** $\alpha_M = 0$ is possible !
- **SUPG-type term:** $\tau_M \sim \frac{h_M}{\|\mathbf{b}\|_{\infty,M}}$

  Careful analysis $\rightsquigarrow$ SUPG avoidable if

$$\sigma \geq \|\mathbf{b}\|_\infty^2 \quad \text{or} \quad Re_M := \frac{h_M \|b\|_{L^\infty(\Omega)}}{\nu} \leq \frac{1}{\sqrt{\nu}}$$

- **Div-type term:**

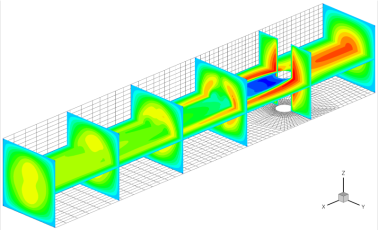
  Equilibration of red velocity and pressure terms with $\alpha_M = 0$ yields

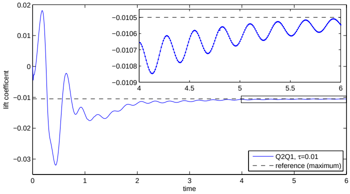
$$\gamma_M \|\mathbf{u}\|_{r+1,\omega_M} \sim \|p\|_{r,\omega_M}$$

$\rightsquigarrow$ Same problem (and strategies) as for reduced RBS-scheme !

Can SUPG be avoided for laminar flows ?

Example: Time-dependent flow around cylinder at $\nu = 0.001$ ($\rightsquigarrow$ $Re = 100$)

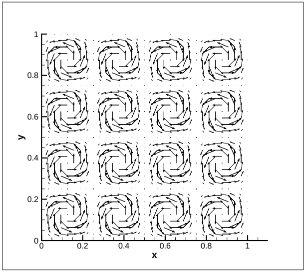




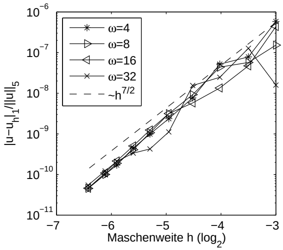
Snapshot of flow (left). Development and convergence of lift coefficient (right)

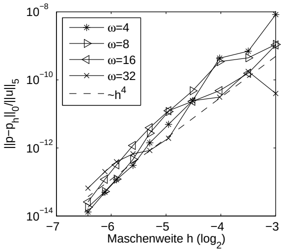
- Q_2/Q_1 elements with no stabilization at all.
- Convergence of lift coefficient compared to solution with 12 million dof's (University of Erlangen)
- Here: 709.592 dof's, time step $\Delta t = 0.01$ for BDF2 with Newton iteration

Full LPS-stabilization for vortex problem



- Fully stabilized simulation with $Q4/Q3$ elements
- Dependence of number of vortices (via ω)
- Velocity gradient strongly increasing with ω
- Convergence rates according to theory (see below)





Outline

VMS-decomposition of Navier-Stokes problem

Navier-Stokes problem :

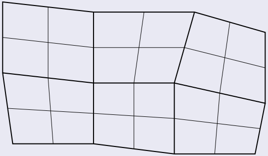
Find $U = (\mathbf{u}, p) \in \mathcal{V}$ s.t. $\mathbf{u}(0) = \mathbf{u}_0$ and

$$a(U,V) = (\mathbf{f}, v) \qquad \forall V = (\mathbf{v}, q) \in \mathcal{W}$$

Decomposition of trial and test spaces:

Two-level setting with FE spaces:

. $\mathcal{W}_H \subseteq \mathcal{W}_h \subset \mathcal{W}$



$$\mathcal{W} = \underbrace{\mathcal{W}_H \oplus (id - \Pi)\mathcal{W}_h}_{=:\mathcal{W}_h} \oplus \hat{\mathcal{W}}$$

$$U = \underbrace{U_H + (id - \Pi)U_h}_{=:U_h} + \hat{U}, \qquad V = \underbrace{V_H + (id - \Pi)V_h}_{=:V_h} + \hat{V}$$

Discrete VMS problem on FEM-level

VMS assumptions:

A.1 Scale separation: No direct influence of $\hat{U}$ on U_H

A.2 Subgrid viscosity model $s_h : \mathcal{W}_h \times \mathcal{W}_h \to \mathbb{R}$
Unresolved scales dissipate energy from small resolved scales

Discrete VMS problem:

$$\begin{aligned} a(U_h, V_H) &= (\mathbf{f}, \mathbf{v}) \qquad \forall V_H \in \mathcal{W}_H \\ a(U_h, \tilde{V}_h) + s_h(\tilde{U}_h, \tilde{V}_h) &= (\mathbf{f}, \tilde{\mathbf{v}}_h) \qquad \forall \tilde{V}_h \in (id - \Pi)\mathcal{W}_h \end{aligned}$$

Assumption: $s_h(\cdot, V_H) = 0 \quad \forall V_H \in \mathcal{W}_H \quad \rightsquigarrow$

Compact discrete VMS problem:

$$\text{Find } U_h \in \mathcal{W}_h : \quad a(U_h, V) + s_h(U_h, V) = (\mathbf{f}, \mathbf{v}) \qquad \forall V \in \mathcal{W}_h$$

Outline

Summary. Outlook

Summary:

- PSPG-type stabilization avoidable for div-stable velocity-pressure pairs
- SUPG-type stabilization less important for laminar flows
- New characterization of div-div stabilization required !
- LPS-type approach as minimal stabilization in LES/DES

Outlook: see Lecture III

- FEM with LPS-type LES: MILES vs. turbulent subgrid model
- Model reduction with DES

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THANKS FOR YOUR ATTENTION !