A Projection-based Variational Multiscale Method for the Incompressible Navier-Stokes/Fourier Model

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1 Introduction

In a bounded domain $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$, we consider the Navier-Stokes/Fourier equations as a model of non-isothermal, incompressible flows

$$\partial_t \mathbf{u} - \nabla \cdot (2\nu \mathbb{D}\mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p + \alpha \mathbf{g}\theta = \mathbf{f} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } (0,T] \times \Omega \tag{2}$$

$$\partial_t \theta - \varkappa \Delta \theta + \mathbf{u} \cdot \nabla \theta = Q \tag{3}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \theta|_{t=0} = \theta_0 \quad \text{in } \{0\} \times \Omega \tag{4}$$

for velocity $\mathbf{u}$, pressure p, and temperature θ with appropriate boundary conditions. The deformation tensor is $\mathbb{D}\mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^t)$. Viscosity ν and diffusivity $\varkappa$, together with reference temperature $\theta_{max} - \theta_{min}$, characteristic length L, thermal expansion coefficient α and gravity vector $\mathbf{g}$ determine the relevant dimensionless Rayleigh number $Ra = \frac{\alpha |\mathbf{g}| L^3 (\theta_{max} - \theta_{min})}{\nu \varkappa}$.

In Section 2, we introduce a projection-based variational multiscale model. Section 3 is concerned with aspects of the numerical analysis of the semidiscrete model. Finally, in Section 4, the approach is applied to a benchmark problem of natural convection.

2 Variational Multiscale Model

Let $\mathcal{T}_h$ be an admissible triangulation of Ω s.t. $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} \overline{K}$. For simplicity, we assume that Dirichlet boundary conditions for velocity and temperature are homogenized. Then, we seek conforming finite element (FE) approximations of velocity, pressure, and temperature in subspaces of

$$V = [H_0^1(\Omega)]^d, \quad Q = L_*^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \; dx = 0\}, \quad \Psi = H_0^1(\Omega).$$

Let us consider inf-sup stable velocity-pressure FE spaces $V_h \times Q_h \subset V \times Q$. The basic Galerkin FE method reads:
find $(\mathbf{u}_h, p_h, \theta_h)\colon [0,T] \to V_h \times Q_h \times \Psi_h$ s.t. for all $(\mathbf{v}_h, q_h, \psi_h) \in V_h \times Q_h \times \Psi_h$

$$\begin{aligned}(\partial_t \mathbf{u}_h, \mathbf{v}_h) + (2\nu \mathbb{D}\mathbf{u_h}, \mathbb{D}\mathbf{v_h}) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + (\alpha \mathbf{g}\theta_h, \mathbf{v}_h)&\\ -(p_h, \nabla \cdot \mathbf{v}_h) + (q_h, \nabla \cdot \mathbf{u}_h) = (\mathbf{f}, \mathbf{v}_h) &\quad (5)\\ (\partial_t \theta_h, \psi_h) + (\varkappa \nabla \theta_h, \nabla \psi_h) + c_S(\mathbf{u}_h, \theta_h, \psi_h) = (Q, \psi_h) &\quad (6)\end{aligned}$$

with the skew-symmetric form of the advective terms

$$\begin{aligned}b_S(\mathbf{u}, \mathbf{v}, \mathbf{w}) &:= [((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}) - ((\mathbf{u} \cdot \nabla)\mathbf{w}, \mathbf{v})]/2,\\ c_S(\mathbf{u}, \theta, \psi) &:= [((\mathbf{u} \cdot \nabla)\theta, \psi) - ((\mathbf{u} \cdot \nabla)\psi, \theta)]/2.\end{aligned}$$

The variational multiscale (VMS) approach, developed by Hughes [5], has been used as a tool for scale separation in turbulence since 2000; for a review see [3]. Consider a three-scale decomposition

$$V \ni \mathbf{v} = \underbrace{\overline{\mathbf{v}}_h + \tilde{\mathbf{v}}_h}_{=\mathbf{v}_h \in V_h} + \hat{\mathbf{v}}_h; \quad Q \ni q = \underbrace{\overline{q}_h + \tilde{q}_h}_{=q_h \in Q_h} + \hat{q}_h; \quad \Psi \ni \psi = \underbrace{\overline{\psi}_h + \tilde{\psi}_h}_{=\psi_h \in \Psi_h} + \hat{\psi}_h$$

with resolved scales $(\mathbf{v}_h, q_h, \psi_h) \in V_h \times Q_h \times \Psi_h \subset V \times Q \times \Psi$. Inspired by [8], for the model influence of $(\hat{\mathbf{v}}_h, \hat{q}_h, \hat{\psi}_h)$ on $(\tilde{\mathbf{v}}_h, \tilde{q}_h, \tilde{\psi}_h)$ define discontinuous FE spaces L_H, M_H for the deformation tensor and temperature gradient

$$\begin{aligned}&\{0\} \subseteq L_H \subseteq L := \left\{\mathbf{L} = (l_{ij}) \in [L^2(\Omega)]^{d \times d} \mid l_{ij} = l_{ji}, 1 \le i, j \le d\right\}\\ &\{0\} \subseteq M_H \subseteq M := [L^2(\Omega)]^d\end{aligned}$$

on $\mathcal{T}_H, H \ge h$ and the L^2-orthogonal projection operators $\Pi_H^u : L \to L_H$ and $\Pi_H^\theta : M \to M_H$ together with the fluctuation operators

$$\kappa_u(\mathbb{D}\mathbf{u}_h) \colon= (Id - \Pi_H^u)(\mathbb{D}\mathbf{u}_h), \qquad \kappa_\theta(\nabla\theta_h) \colon= (Id - \Pi_H^\theta)(\nabla\theta_h).$$

For the calibration of the subgrid models for velocity and temperature, we introduce cellwise constant terms $\nu_S(\mathbf{u}_h, \theta_h)$ and $\varkappa_S(\mathbf{u}_h, \theta_h)$ s.t.

$$\nu_S^K(\mathbf{u}_h, \theta_h) := \nu_S(\mathbf{u}_h, \theta_h)|_K, \qquad \varkappa_S^K(\mathbf{u}_h, \theta_h) := \varkappa_S(\mathbf{u}_h, \theta_h)|_K.$$

As model of the small unresolved pressure scales we add grad-div stabilization with cellwise constant $\gamma_K(\mathbf{u}_h, p_h) := \gamma(\mathbf{u}_h, p_h)|_K$ s.t.

$$(\gamma(\mathbf{u}_h, p_h)(\nabla \cdot \mathbf{u}_h), \nabla \cdot \mathbf{v}_h) := \sum_{K \in \mathcal{T}_h} \gamma_K(\mathbf{u}_h, p_h)(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h)_K.$$

Summarizing, we obtain the following variational multiscale model:
find $(\mathbf{u}_h, p_h, \theta_h)$ s.t. for all $(\mathbf{v}_h, q_h, \psi_h) \in V_h \times Q_h \times \Psi_h$:

$$\begin{aligned}(\partial_t \mathbf{u}_h, \mathbf{v}_h) + 2\nu\,(\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + (\alpha \mathbf{g}\theta_h, \mathbf{v}_h)&\\ + (\nu_S(\mathbf{u}_h, \theta_h)\kappa_u \mathbb{D}(\mathbf{u}_h), \kappa_u \mathbb{D}(\mathbf{v}_h)) + (\gamma(\mathbf{u}_h, p_h)\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h) &\quad (7)\\ + (\nabla \cdot \mathbf{u}_h, q_h) - (\nabla \cdot \mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) &\quad (8)\\ (\partial_t \theta_h, \psi_h) + (\varkappa \nabla \theta_h, \nabla \psi_h) + c_S(\mathbf{u}_h, \theta_h, \psi_h)&\\ + (\varkappa_S(\mathbf{u}_h, \theta_h)\kappa_\theta(\nabla\theta_h), \kappa_\theta(\nabla\psi_h)) = (Q, \psi_h). &\quad (9)\end{aligned}$$

3 A priori error analysis of the semidiscrete model

Following [7], we obtain stability estimates for the VMS scheme (7)-(9).

Lemma 1. *Let* $\mathbf{f} \in L^1(0,T;L^2(\Omega)), Q \in L^1(0,T;L^2(\Omega))$ *and* $\mathbf{u}_0 \in [L^2(\Omega)]^d$, $\theta_0 \in L^2(\Omega)$. *Then we obtain for* $t \in (0,T]$ *control of kinetic and heat energy*

$$\begin{aligned}
\|\theta_h\|_{L^\infty(0,t;L^2(\Omega))} &\leq K_1(Q,\theta_0) &&:= \|\theta_0\|_0 + \|Q\|_{L^1(0,t;L^2(\Omega))} \\
\|\mathbf{u}_h\|_{L^\infty(0,t;L^2(\Omega))} &\leq K_2(\mathbf{f},\mathbf{u}_0,Q,\theta_0) &&:= \|\mathbf{u}_0\|_0 + \|\mathbf{f}\|_{L^1(0,t;L^2(\Omega))} + C\alpha\|\mathbf{g}\|_0 K_1
\end{aligned}$$

and control of dissipation and subgrid terms

$$\varkappa\|\nabla\theta_h\|^2_{L^2(0,t;L^2(\Omega))} + \int_0^t \sum_K \varkappa_S^K(\mathbf{u}_h,\theta_h)\|\kappa_\theta\nabla\theta_h\|^2_{0,K}dt \leq \frac{3}{2}K_1^2$$

$$\begin{aligned}
\nu\|\mathbb{D}\mathbf{u}_h\|^2_{L^2(0,t;L^2(\Omega))} &+ \frac{1}{2}\int_0^t \sum_K \nu_S^K(\mathbf{u}_h,\theta_h)\|\kappa_u\mathbb{D}\mathbf{u}_h\|^2_{0,K}dt \\
&+\frac{1}{2}\int_0^t \sum_K \gamma_K(\mathbf{u}_h,p_h)\|\nabla\cdot\mathbf{u}_h\|^2_{0,K}dt \leq 3K_2^2.
\end{aligned}$$

Now we introduce elementwise multiscale viscosities $\nu^K_{VMS}, \varkappa^K_{VMS}$ via

$$\sum_{K\in\mathcal{T}_h} \nu_S^K(\mathbf{u}_h,\theta_h)\,\|\kappa_u\mathbb{D}\mathbf{v}_h\|^2_{0,K} = \sum_{K\in\mathcal{T}_h} \underbrace{\nu_S^K(\mathbf{u}_h,\theta_h)\left(1-\frac{\|\Pi_H^u\mathbb{D}\mathbf{v}_h\|^2_{0,K}}{\|\mathbb{D}\mathbf{v}_h\|^2_{0,K}}\right)}_{=:\nu^K_{\mathrm{VMS}}(\mathbf{v}_h)\geq 0}\|\mathbb{D}\mathbf{v}_h\|^2_{0,K}\,,$$

$$\sum_{K\in\mathcal{T}_h} \varkappa_S^K(\mathbf{u}_h,\theta_h)\,\|\kappa_\theta\nabla\psi_h\|^2_{0,K} = \sum_{K\in\mathcal{T}_h} \underbrace{\varkappa_S^K(\mathbf{u}_h,\theta_h)\left(1-\frac{\|\Pi_H^\theta\nabla\psi_h\|^2_{0,K}}{\|\nabla\psi_h\|^2_{0,K}}\right)}_{=:\varkappa^K_{\mathrm{VMS}}(\psi_h)\geq 0}\|\nabla\psi_h\|^2_{0,K}$$

where we applied the projector properties of the fluctuation operators.

In the following we will omit the dependency of the parameters on $\mathbf{u}_h, \theta_h$ for better readability. Using the modified elementwise viscosities

$$\nu^K_{\mathrm{mod}}(\mathbf{v}_h) := 2\nu + \nu^K_{\mathrm{VMS}}(\mathbf{v}_h), \quad \varkappa^K_{\mathrm{mod}}(\psi_h) := \varkappa + \varkappa^K_{\mathrm{VMS}}(\psi_h)$$

we define the following mesh-dependent expressions

$$|||\mathbf{u}(t)|||^2 := \|\mathbf{u}(t)\|_0^2 + \sum_{K\in\mathcal{T}_h}\int_0^t \left(\frac{\nu^K_{\mathrm{mod}}(\mathbf{u})}{2}\,\|\mathbb{D}\mathbf{u}\|^2_{0,K} + \gamma_K(\mathbf{u}_h,p_h)\,\|\nabla\cdot\mathbf{u}\|^2_{0,K}\right)dt,$$

$$|[\theta(t)]|^2 := \|\theta(t)\|_0^2 + \sum_{K\in\mathcal{T}_h}\int_0^t \frac{1}{2}\varkappa^K_{\mathrm{mod}}(\theta)\,\|\nabla\theta\|^2_{0,K}\,dt.$$

The following semidiscrete a priori estimate is an extension of a previous result in [6] and [10] for the isothermal case. The proof takes advantage of the fact that, for inf-sup stable FE spaces for velocity/pressure, the space V_h^{div} of discretely divergence free functions is not empty. Thus one can separate estimates for velocity/temperature and pressure and apply an interpolation operator by Girault/Scott [2] in V_h^{div} on isotropic meshes.

Theorem 1. *For a sufficiently smooth solution* $(\mathbf{u},\theta)$ *of the Navier-Stokes/ Fourier model with* $\nabla\mathbf{u} \in L^4(0,t;L^2\Omega)), \partial_t\mathbf{u} \in L^2(0,t;H^{-1}(\Omega))$ *and* $\nabla\theta \in L^4(0,t;L^2\Omega)), \partial_t\theta \in L^2(0,t;H^{-1}(\Omega))$, *it holds for the solution of (7)-(9) for all* $t \in (0,T)$*:*

$$
\begin{aligned}
&|||(\mathbf{u}-\mathbf{u}_h)(t)|||^2 + |[(\theta-\theta_h)(t)]|^2 \\
&\qquad \le 2 \inf_{\substack{\tilde{\mathbf{u}}_h \in L^2(0,t;V_h^{div}) \\ \tilde{\theta}_h \in L^2(0,t;\Psi_h)}} |||(\mathbf{u}-\tilde{\mathbf{u}}_h)(t)|||^2 + |[(\theta-\tilde{\theta}_h)(t)]|^2 \\
&+ \inf_{\substack{\tilde{\mathbf{u}}_h \in L^4(0,t;V_h^{div}) \\ \tilde{p}_h \in L^2(0,t;Q_h) \\ \tilde{\theta}_h \in L^2(0,t;\Psi_h)}} e^{\int_0^t g(s)ds}\left(\|(\mathbf{u}_h-\tilde{\mathbf{u}}_h)(0)\|_0^2 + \|(\theta_h-\tilde{\theta}_h)(0)\|_0^2 + \int_0^t g_2(s)ds \right)
\end{aligned}
$$

with

$$
\begin{aligned}
g(t) :=& \frac{27C_{LT}^4}{2\nu_{\text{mod}}^{\min}(\mathbf{e}_h^u)^3}\|\mathbb{D}\mathbf{u}\|_0^4 + \frac{8C_1^4}{\nu_{\text{mod}}^{\min}(\mathbf{e}_h^u)\,\varkappa_{\text{mod}}^{\min}(e_h^\theta)^2}\|\nabla\theta\|_0^4 + 2\alpha\|\mathbf{g}\|_\infty, \\
g_2(t) :=& 2\sum_{K\in\mathcal{T}_h}\Bigg[\min\left(\frac{9C_{Ko}^2}{\nu_{\text{mod}}^{\min}(\mathbf{e}_h^u)}, \frac{1}{\gamma_K(\mathbf{u}_h)}\right)\Big(\|p-\tilde{p}_h\|_{0,K}^2 + \gamma_K^2(\mathbf{u}_h)\|\nabla\cdot\epsilon^u\|_{0,K}^2\Big) \\
&+ 6\Big(\nu + \nu_{\text{VMS}}^K(\epsilon^u)\Big)\|\mathbb{D}\epsilon^u\|_{0,K}^2 + \Big(2\varkappa + 4\varkappa_{\text{VMS}}^K(\epsilon^\theta)\Big)\left\|\nabla\epsilon^\theta\right\|_{0,K}^2 \\
&+ 6\nu_S^K(\mathbf{u}_h,\theta_h)\|\kappa_u\mathbb{D}\mathbf{u}\|_{0,K}^2 + 4\varkappa_S^K(\mathbf{u}_h,\theta_h)\|\kappa_\theta\nabla\theta\|_{0,K}^2\Bigg] \\
&+ \frac{6C_{Ko}^2}{\nu_{\text{mod}}^{\min}(\mathbf{e}_h^u)}\|\partial_t\epsilon^u\|_{-1,\Omega}^2 + \frac{4}{\varkappa_{\text{mod}}^{\min}(e_h^\theta)}\left\|\partial_t\epsilon^\theta\right\|_{-1,\Omega}^2 + \alpha|\mathbf{g}|_\infty\left\|\epsilon^\theta\right\|_0^2 \\
&+ \frac{6C_{LT}^2}{\nu_{\text{mod}}^{\min}(\mathbf{e}_h^u)}\Big(C_FC_{Ko}\|\mathbb{D}\mathbf{u}\|_0^2 + \|\mathbf{u}_h\|_0\|\mathbb{D}\mathbf{u}_h\|_0\Big)\|\mathbb{D}\epsilon^u\|_0^2 \\
&+ \frac{4C_1^2C_{Ko}}{\varkappa_{\text{mod}}^{\min}(e_h^\theta)}\Big(C_FC_{Ko}\|\nabla\theta\|_0^2\|\mathbb{D}\epsilon^u\|_0^2 + \|\mathbf{u}_h\|_0\|\mathbb{D}\mathbf{u}_h\|_0\left\|\nabla\epsilon^\theta\right\|_0^2\Big)
\end{aligned}
$$

where $\nu_{\text{mod}}^{\min}(\mathbf{e}_h^u) := \min_{K\in\mathcal{T}_h}\nu_{\text{mod}}^K(\mathbf{e}_h^u)$, $\varkappa_{\text{mod}}^{\min}(e_h^\theta) := \min_{K\in\mathcal{T}_h}\varkappa_{\text{mod}}^K(e_h^\theta)$ *and*

$$\mathbf{u}_h - \mathbf{u} = (\mathbf{u}_h-\tilde{\mathbf{u}}_h)-(\mathbf{u}-\tilde{\mathbf{u}}_h) =: \mathbf{e}_h^u - \epsilon^u, \quad \theta_h-\theta = (\theta_h-\tilde{\theta}_h)-(\theta-\tilde{\theta}_h) =: e_h^\theta - \epsilon^\theta.$$

C_F *and* C_{Ko} *are the constants of the inequalities of Friedrichs and Korn.* C_{LT} *and* C_1 *are related to upper bounds of the advective terms.*

Sketch of the proof: Please note that the regularity assumptions imply uniqueness of the continuous solution $(\mathbf{u},p,\theta)$. Starting from the error equations for $\mathbf{e}_h^u$ and e_h^θ, careful estimates of the right hand side terms lead to

$$\partial_t\Big(\|\mathbf{e}_h^u\|_0^2 + \|e_h^\theta\|_0^2\Big) + g_1(t) \le g(t)\left(\|\mathbf{e}_h^u\|_0^2 + \|e_h^\theta\|_0^2\right) + g_2(t)$$

with $g(t)$ and $g_2(t)$ as stated in the Theorem and

$$\begin{aligned} g_1(t) :=& \frac{1}{4}\sum_{K\in\mathcal{T}_h} \nu_{\mathrm{mod}}^K(\mathbf{e}_h^u)\,\|\mathbb{D}\mathbf{e}_h^u\|_{0,K}^2 + \frac{1}{2}\sum_{K\in\mathcal{T}_h} \varkappa_{\mathrm{mod}}^K(e_h^\theta)\,\left\|\nabla e_h^\theta\right\|_{0,K}^2 \\ &+ \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h,p_h)\,\|\nabla\cdot\mathbf{e}_h\|_{0,K}^2 . \end{aligned}$$

Gronwall's Lemma implies for all $t \in [0,T]$

$$\|\mathbf{e}_h^u(t)\|_0^2 + \|e_h^\theta(t)\|_0^2 + \int_0^t g_1(s)ds \le e^{\int_0^t g(s)ds}\big(\,\|\mathbf{e}_h^u(0)\|_0^2 + \|\mathbf{e}_h^\theta(0)\|_0^2 + \int_0^t g_2(s)ds\big).$$

Finally, the triangle inequality concludes the proof. For full details of the proof, we refer to [9]. □

Let us discuss the result for FE spaces $\mathbb{Q}_k/\mathbb{Q}_{k-1}/\mathbb{Q}_k$ or $\mathbb{P}_k/\mathbb{P}_{k-1}/\mathbb{P}_k$ for $(\mathbf{u}_h, p_h, \theta_h)$ on isotropic meshes. Moreover, we formally assume sufficiently smooth solutions $(\mathbf{u}, p, \theta)$ of the continuous model. In particular, it can be shown that the piecewise constants $\nu_{\mathrm{mod}}^{\min}(\mathbf{e}_h^u)$, $\varkappa_{\mathrm{mod}}^{\min}(e_h^\theta)$, and $\gamma_K(\mathbf{u}_h, p_h)$, occuring on the right hand side of the error estimate, remain bounded.

The third line of term $g_2(t)$ consists of model errors. Let us assume $\nu_S^K(\mathbf{u}_h, \theta_h), \varkappa_S^K(\mathbf{u}_h, \theta_h) = \mathcal{O}(h_K^2)$ for the subgrid functions. For the discontinuous spaces $L_H = [\mathbb{Q}_{k-2}^{disc}]^{d\times d}$, $M_H = [\mathbb{Q}_{k-2}^{disc}]^d$, or $\mathbb{Q}_{k-2}^{disc}$ replaced with $\mathbb{P}_{k-2}^{disc}$, the fluctuation operators provide an interplation error of $\mathcal{O}(h_K^{2(k-1)})$. Thus the model error terms are of order $\mathbb{O}(h^{2k})$.

The remaining approximation terms in $g_2(t)$ are formally of order $\mathbb{O}(h^{2k})$, based on the V_h^{div}-interpolation operator on isotropic meshes, see [2]. Properly chosen subgrid parameters improve the estimate. see, e.g., the role of γ_K. Moreover, in term $g(t)$, viscosity ν is replaced with $\nu + \frac{1}{2}\min_K \nu_{VMS}^K(\mathbf{u}_h; \mathbf{u} - \tilde{\mathbf{u}}_h)$, which corresponds to an increased effective Reynolds number. A similar argument holds for $\varkappa$.

Let us finally consider the Gronwall factor $e^{\int_0^t g(s)ds}$. Following [4], this factor is unavoidable for unstable solutions of the Navier-Stokes problem. It can be avoided in case of (quasi-)exponentially stable solutions.

4 Application to natural convection flow

For the spatial discretization we apply quadrilateral meshes with FE spaces $\mathbb{Q}_2/\mathbb{Q}_1/\mathbb{Q}_2$ for velocity/pressure/temperature within the FE package `deal.II`, see [1]. The arising semidiscrete problem of the form

$$\begin{pmatrix} M_u & 0 & 0 \\ 0 & M_\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h'(t) \\ \theta_h'(t) \\ \mathbf{p}_h'(t) \end{pmatrix} = \begin{pmatrix} \mathbf{f}_h(t) \\ \mathbf{q}_h(t) \\ 0 \end{pmatrix} - \begin{pmatrix} A_u(\mathbf{u}_h) & C & B \\ 0 & A_\theta(\mathbf{u}_h) & 0 \\ B^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h(t) \\ \theta_h(t) \\ \mathbf{p}_h(t) \end{pmatrix}$$

is a DAE-system with differentiation index 2 and perturbation index 2. For the time discretization, we apply the BDF(2)-formula for velocity

$$\mathbf{u}_h'(t_{n+1}) \approx [3\mathbf{u}_h(t_{n+1}) - 4\mathbf{u}_h(t_n) + \mathbf{u}_h(t_{n-1})]/(2\tau_n)$$

and similarly for $\theta_h'(t_{n+1})$. This results in favourable stability properties and does not lead to order reduction for the algebraic variable. A fixed-point iteration is performed for the arising non-linear implicit scheme.

In a next step, we have to introduce the non-isothermal viscosity model. We start from the residual stress tensor τ^R and residual temperature flux $\mathbf{h}$

$$\tau^R := \langle \mathbf{u} \otimes \mathbf{u} \rangle - \mathbf{u}_h \otimes \mathbf{u}_h \approx -2\nu_S \mathbb{D}\mathbf{u}_h, \qquad \mathbf{h} := \langle \mathbf{u}\theta \rangle - \mathbf{u}_h\theta_h \approx -\varkappa_S \nabla\theta_h$$

and apply the subgrid model of Smagorinsky-Eidson [12]

$$\nu_S = (C_E \varDelta)^2 \max\left(0\,, \|\mathbb{D}\mathbf{u}_h\|_F^2 + \frac{\beta}{Pr_S}\mathbf{g}\cdot\nabla\theta_h\right)^{1/2}, \qquad \varkappa_S = \nu_S / Pr_S$$

with $C_E = 0.21$ and $Pr_S = 0.4$. As filter width $\varDelta$ we use an anisotropic scaling matrix that takes local mesh anisotropy and orientation into account. This approach gave better results than taking an isotropic filter width (e.g. length of shortest edge). The model reduces to the Smagorinsky model if $\mathbf{g}\cdot\nabla\theta_h = 0$. For wall bounded flows the turbulent viscosities may be multiplied by van Driest-type damping functions for reasonable near wall behavior. It is well known that the Smagorinsky model is over-diffusive. A reduction of model dissipation may then be established by an application of the fluctuation operators

$$\tau^R = -2\nu_S\kappa_u(\mathbb{D}\mathbf{u}_h), \qquad \mathbf{h} = -\varkappa_S\kappa_\theta(\nabla\theta_h).$$

In our implementation, we use an one-level approach with $H = h$ and the discontinuous spaces $L_h = [\mathbb{Q}_0^{disc}]^{d\times d}$ and $M_h = [\mathbb{Q}_0^{disc}]^d$.

Now we apply the method to natural convection in a differentially heated cavity $\varOmega := (0,1)^d$. The numerical simulations in [12] in a three-dimensional cavity show that for appropriate boundary conditions in x_3-direction there appears a statistically two-dimensional flow. This motivates the present restriction to $d = 2$. Heating $\theta = \theta_{max}$ and cooling $\theta = \theta_{min}$ is performed at lateral boundaries, whereas the upper and lower boundaries are highly conducting. As suggested in [11] we use experimental data as boundary conditions on these walls. No-slip conditions $\mathbf{u} = \mathbf{0}$ for velocity are given at the whole boundary $\partial\varOmega$. Computations were done on two meshes with 64 and 32 cells in each dimension. An anisotropic mesh refinement had been performed at all boundaries by transforming an equidistant reference mesh with $x = \hat{x} - \frac{19}{40\pi}\sin(2\pi\hat{x})$ and $y = \hat{y} - \frac{7}{16\pi}\sin(2\pi\hat{y})$. The maximum aspect-ratio of cells at the vertical walls was about 36:1.

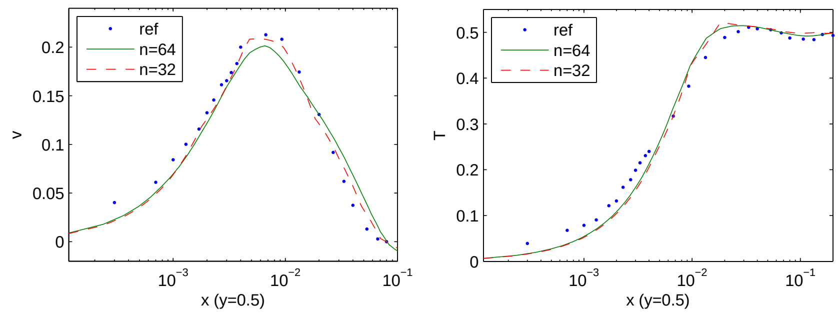

Fig. 1. Boundary layer profiles for horizontal velocity profile $v(x, 0.5)$ (left) and for horizontal temperature profile $T(x, 0.5)$ (right) and experimental data [11]

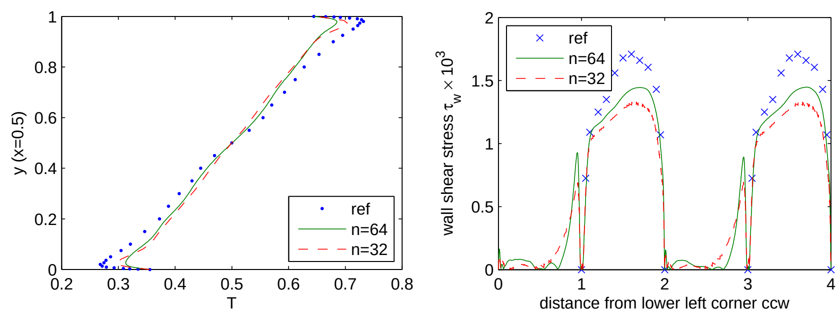

Fig. 2. Temperature profile on vertical centerline $T(0.5, y)$ (left) and wall shear stress (right) and experimental data of [11]

Let us present some first results for time-averaged quantities of a low-turbulence flow at $Ra = 1.58 \times 10^9$. Here we used the projection-based VMS with Smagorinsky-Eidson parametrization of the subgrid model without van Driest damping. On both meshes the results for velocity and temperature profiles, wall shear stress (see Fig. 1 and 2) and Nusselt number (not shown) are in good agreement to experimental data of [11]. Interestingly, we observed (for fixed parameters) no big difference of the solutions on the two grids with exception of wall-shear stress.

We used grad-div stabilization with constant $\gamma_K = 0.3$ to improve the mass conservation properties of the scheme. On the coarse grid with $n = 32$, we obtained $\|\nabla \cdot \mathbf{u}_h\|_0 = 0.0029$ for $\gamma = 0.3$ as opposed to $\|\nabla \cdot \mathbf{u}_h\|_0 = 0.0517$ for $\gamma = 0$, i.e., an improvement by a factor of 18.

One critical point of the simulation is the separation of the flow at the vertical walls and its reattachment at the horizontal walls. Experiments show small counter-rotating vortices in these corners, which we also found in our simulations on the fine mesh.

5 Summary and Outlook

In this paper, we applied a variational multiscale model to the time-dependent Navier-Stokes/Fourier model of incompressible and non-isothermal flows. For the case of piecewise nonlinear subgrid models for the unresolved velocity, temperature, and pressure fluctuations, an a priori analysis of the nonlinear semidiscrete problem was given. Finally, we applied the approach to the standard benchmark problem of natural convection problem in a differentially heated two-dimensional cavity.

Some open problems are the extension to Rayleigh-Benard convection and to mixed convection problems in indoor air-flow simulation. This will be considered in future research.

References

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