Large-Eddy Simulation of Wall-Bounded Turbulent Flows – Layer-adapted Meshes vs. Weak Dirichlet Boundary Conditions –

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1 Introduction

In a bounded domain $\Omega \subseteq \mathbb{R}^3$, we consider the incompressible Navier-Stokes model to determine velocity $\mathbf{u}$ and pressure p s.t.

$$\partial_t \mathbf{u} - \nabla \cdot (2\nu \mathbb{D}\mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } (0,T] \times \Omega \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } [0,T] \times \Omega \tag{2}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}^0 \qquad \text{in } \Omega \tag{3}$$

together with appropriate boundary conditions on the boundary $\partial\Omega$. The deformation tensor is denoted by $\mathbb{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$. The Reynolds number $Re = \frac{\mathbf{U}L}{\nu}$ relies on viscosity ν, a reference length L and velocity $\mathbf{U}$.

In Section 2, we introduce a variational multiscale (VMS) finite element model. Aspects of the numerical analysis of the semidiscrete model are addressed in Section 3. Section 4 is concerned with the application of the approach to a benchmark problem of wall-bounded flows in a channel. In particular, we discuss the problem whether a layer-adapted mesh in the boundary layer regions or a weak implementation of boundary conditions for the velocity at the wall is appropriate.

2 Variational multiscale approach

For simplicity, we consider no-slip boundary conditions and thus, for a weak formulation, the spaces

$$V = [H_0^1(\Omega)]^3, \qquad Q = L_*^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}.$$

By $\|\cdot\|_{0,G}$ we denote the standard L^2-norm on a measurable domain $G \subseteq \Omega$. Moreover, let $\|\cdot\|_0 = \|\cdot\|_{0,\Omega}$.

Let $\mathcal{T}_h$ be an admissible (possibly anisotropic) mesh s.t. $\overline{\Omega} = \cup_{K\in\mathcal{T}_h}\overline{K}$. We consider finite element (FE) spaces $V_h \times Q_h \subset V \times Q$ for velocity/pressure subject to the discrete inf-sup stability condition

$$\exists\beta \neq \beta(h) \quad s.t. \quad \inf_{q_h\in Q_h} \sup_{\mathbf{v}_h\in V_h} \frac{(q_h, \nabla\cdot\mathbf{v_h})}{\|q_h\|_0\|\nabla\mathbf{v}_h\|_0} \geq \beta > 0.$$

The basic Galerkin FE method reads:
find $(\mathbf{u}_h, p_h)\colon [0,T] \to V_h \times Q_h$ s.t. $\forall(\mathbf{v}_h, q_h) \in V_h \times Q_h$

$$\begin{aligned}(\partial_t\mathbf{u}_h, \mathbf{v}_h) + (2\nu\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h)\\ (q_h, \nabla\cdot\mathbf{u}_h) &= 0\end{aligned}$$

with the skew-symmetric advective term

$$b_S(\mathbf{u}, \mathbf{v}, \mathbf{w}) := [((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}) - ((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v})]/2.$$

We consider the following three-scale decomposition

$$V \ni \mathbf{v} = \underbrace{\overline{\mathbf{v}}_h + \tilde{\mathbf{v}}_h}_{=\mathbf{v}_h\in V_h} + \hat{\mathbf{v}}_h; \qquad Q \ni q = \underbrace{\overline{q}_h + \tilde{q}_h}_{=q_h\in Q_h} + \hat{q}_h$$

with resolved scales $(\mathbf{v}_h, q_h) \in V_h \times Q_h \subset V \times Q$. The influence of the small unresolved scales $(\hat{\mathbf{v}}_h, \hat{q}_h)$ on $(\tilde{\mathbf{v}}_h, \tilde{q}_h)$ will be modelled following the variational multiscale approach, see [5]. Define the FE space L_H for the deformation tensor on $\mathcal{T}_H, H \geq h$

$$\{0\} \subseteq L_H \subseteq \mathbb{D}V_h \subseteq L := \left\{\mathbf{L} = (l_{ij}) \mid l_{ij} = l_{ji} \in L^2(\Omega)\ \forall i,j \in \{1,2,3\}\right\}$$

and the L^2-orthogonal projection operator $\Pi_H : L \to L_H$. The model of the small unresolved velocity scales is defined by means of the fluctuation operator

$$\kappa(\mathbb{D}\mathbf{v}_h) \colon = (Id - \Pi_H)(\mathbb{D}\mathbf{v}_h).$$

For the calibration of the subgrid model for velocity, we introduce cellwise constant terms $\nu_T(\mathbf{u}_h)$ s.t. $\nu_T^K(\mathbf{u}_h) := \nu_T(\mathbf{u}_h)|_K$.

As a model of the small unresolved pressure scales, we add the so-called grad-div stabilization [12] with cellwise constant $\gamma_K(\mathbf{u}_h) := \gamma(\mathbf{u}_h)|_K$ s.t.

$$(\gamma(\mathbf{u}_h)(\nabla\cdot\mathbf{u}_h), \nabla\cdot\mathbf{v}_h) := \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h)(\nabla\cdot\mathbf{u}_h, \nabla\cdot\mathbf{v}_h)_K.$$

Finally, the VMS model reads as follows: find $(\mathbf{u}_h, p_h)$ s.t.

$$(\partial_t\mathbf{u}_h, \mathbf{v}_h) + 2\nu\,(\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) \tag{4}$$
$$+ (\nu_T(\mathbf{u}_h)\kappa(\mathbb{D}\mathbf{u}_h), \kappa(\mathbb{D}\mathbf{v}_h)) + (\gamma_T(\mathbf{u}_h)\nabla\cdot\mathbf{u}_h, \nabla\cdot\mathbf{v}_h) \tag{5}$$
$$+ (\nabla\cdot\mathbf{u}_h, q_h) - (\nabla\cdot\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \tag{6}$$

for all $(\mathbf{v}_h, q_h) \in V_h \times Q_h$.

3 Aspects of the numerical analysis

The following sketch of the semidiscrete analysis is an extension of a result in [8]. It takes advantage of the fact that, for inf-sup stable FE spaces for velocity/pressure, the space V_h^{div} of discretely divergence free functions is not empty. In particular, we can separate estimates for velocity and pressure. Moreover, an additional pressure stabilization is not required.

Following the approach in [10], we obtain the following stability estimates which are valid on arbitrary admissible grids.

Lemma 1. *Let* $\mathbf{f} \in L^1(0,T;L^2(\Omega)), \mathbf{u}_0 \in [L^2(\Omega)]^3$. *Then, for all* $t \in (0,T]$, *there is control of kinetic energy*

$$\|\mathbf{u}_h\|_{L^\infty(0,t;L^2(\Omega))} \leq K(\mathbf{f},\mathbf{u}_0) \equiv \|\mathbf{u}_0\|_0 + \|\mathbf{f}\|_{L^1(0,t;L^2(\Omega))}$$

and of the dissipation and subgrid terms

$$\nu\|\mathbb{D}\mathbf{u}_h\|^2_{L^2(0,t;L^2(\Omega))} + \frac{1}{2}\int_0^t \sum_K \nu_T^K(\mathbf{u}_h)\|\kappa(\mathbb{D}\mathbf{u}_h)\|^2_{0,K}dt$$
$$+\frac{1}{2}\int_0^t \sum_K \gamma_K(\mathbf{u}_h)\|\nabla\cdot\mathbf{u}_h\|^2_{0,K}dt \leq 3K^2(\mathbf{f},\mathbf{u}_0).$$

We introduce elementwise multiscale viscosities $\nu^K_{\mathrm{VMS}}(\mathbf{u}_h,\mathbf{v}_h)$ via

$$\sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\,\|\kappa(\mathbb{D}\mathbf{v}_h)\|^2_{0,K} = \sum_{K\in\mathcal{T}_h} \underbrace{\nu_T^K(\mathbf{u}_h)\left(1-\frac{\|\Pi_H\mathbb{D}\mathbf{v}_h\|^2_{0,K}}{\|\mathbb{D}\mathbf{v}_h\|^2_{0,K}}\right)}_{=:\nu^K_{\mathrm{VMS}}(\mathbf{u}_h,\mathbf{v}_h)\geq 0}\|\mathbb{D}\mathbf{v}_h\|^2_{0,K}$$

where we take advantage of the projector properties of the fluctuation operator κ. Then we define the following mesh-dependent norm for the analysis

$$|||\mathbf{u}(t)|||^2 := \|\mathbf{u}(t)\|^2_0 + \sum_{K\in\mathcal{T}_h}\int_0^t \left(\frac{\nu^K_{mod}(\mathbf{u},\mathbf{u}_h)}{2}\,\|\mathbb{D}(\mathbf{u})\|^2_{0,K} + \gamma_K(\mathbf{u}_h)\,\|\nabla\cdot\mathbf{u}\|^2_{0,K}\right)dt$$

with modified elementwise viscosities:

$$\nu^K_{\mathrm{mod}}(\mathbf{u}_h,\mathbf{v}_h) := 2\nu + \nu^K_{\mathrm{VMS}}(\mathbf{u}_h,\mathbf{v}_h).$$

Then we obtain the following semidiscrete a priori estimate.

Theorem 1. *For a sufficiently smooth solution* $\mathbf{u}$ *of the Navier-Stokes model (1)-(3) it holds for the solution of the VMS model (4)-(6) for all* $t \in (0,T)$:

$$|||(\mathbf{u}-\mathbf{u}_h)(t)|||^2 \leq 2 \inf_{\tilde{\mathbf{u}}_h \,\in\, L^2(0,\,t;\,V_h^{div})} |||(\mathbf{u}-\tilde{\mathbf{u}}_h)(t)|||^2$$

$$+\, e^{\int_0^t \frac{27C^4_{LT}}{2\nu^{min}_{mod}(\mathbf{u}_h,\mathbf{e}^u_h)^3}\|\mathbb{D}\mathbf{u}(s)\|^4_0 ds} \inf_{\substack{\tilde{\mathbf{u}}_h \,\in\, L^4(0,\,t;\,V_h^{div}) \\ \tilde{p}_h \,\in\, L^2(0,\,t;\,Q_h)}} \left(\|(\mathbf{u}_h-\tilde{\mathbf{u}}_h)(0)\|^2_0 + \int_0^t A(s)ds\right)$$

with

$$\begin{aligned}
A(t) :=& 2\sum_{K\in\mathcal{T}_h}\Bigg[6\nu^K_{VMS}(\mathbf{u}_h,\mathbf{u})\,\|\mathbb{D}\mathbf{u}\|^2_{0,K}\\
&+6\Big(\nu+\nu^K_{VMS}(\mathbf{u}_h,\epsilon^u)\Big)\,\|\mathbb{D}\epsilon^u\|^2_{0,K}\\
&+\min\left(\frac{9C^2_{Ko}}{\nu^{min}_{mod}(\mathbf{u}_h,\mathbf{e}^u_h)},\frac{1}{\gamma_K(\mathbf{u}_h)}\right)\Big(\|p-\tilde{p}_h\|^2_{0,K}+\gamma^2_K(\mathbf{u}_h)\,\|\nabla\cdot\epsilon^u\|^2_{0,K}\Big)\Bigg]\\
&+\frac{6C^2_{LT}}{\nu^{min}_{mod}(\mathbf{u}_h,\mathbf{e}^u_h)}\Big(C_F C_{Ko}\,\|\mathbb{D}\mathbf{u}\|^2_0+\|\mathbf{u}_h\|_0\,\|\mathbb{D}\mathbf{u}_h\|_0\Big)\,\|\mathbb{D}\epsilon^u\|^2_0\\
&+\frac{6C^2_{Ko}}{\nu^{min}_{mod}(\mathbf{u}_h,\mathbf{e}^u_h)}\,\|\partial_t\epsilon^u\|^2_{-1,\Omega}
\end{aligned}$$

where $\nu^{min}_{mod}(\mathbf{u}_h,\mathbf{e}^u_h) := \min_K \nu^K_{mod}(\mathbf{u}_h(t),\mathbf{v}_h(t))$ *and*

$$\mathbf{u}_h-\mathbf{u}=(\mathbf{u}_h-\tilde{\mathbf{u}}_h)-(\mathbf{u}-\tilde{\mathbf{u}}_h)=:\mathbf{e}^u_h-\epsilon^u.$$

C_F and C_{Ko} are the constants of the inequalities of Friedrichs and Korn. C_{LT} is related to an upper bound of the advective term.

Remark 1. The first r.h.s. term in the first line of term $A(t)$ is related to the VMS-model error. For the remaining approximation terms in A, we can apply the interpolation operator by Girault/Scott [6] in V^{div}_h on isotropic meshes and a standard interpolation operator for the pressure. Then these terms are formally of order $\mathbb{O}(h^k)$ for FE spaces $\mathbb{Q}_k/\mathbb{Q}_{k-1}$ or $\mathbb{P}_k/\mathbb{P}_{k-1}$ for velocity/pressure.

Sketch of the proof: From the weak form of (1)-(3) and (4)-(6), we obtain the error equation

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\mathbf{e}^u_h\|^2_0+\sum_{K\in\mathcal{T}_h}\nu^K_{\rm mod}(\mathbf{u}_h,\mathbf{e}^u_h)\,\|\mathbb{D}\mathbf{e}^u_h\|^2_{L^2(K)}+\sum_{K\in\mathcal{T}_h}\gamma_K(\mathbf{u}_h)\,\|\nabla\cdot\mathbf{e}^u_h\|^2_{L^2(K)}\\
&=(\partial_t\epsilon^u,\mathbf{e}^u_h)+(2\nu\mathbb{D}\epsilon,\mathbb{D}\mathbf{e}^u_h)+b_S(\mathbf{u},\mathbf{u},\mathbf{e}^u_h)-b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{e}^u_h)-(p-\lambda_h,\nabla\cdot\mathbf{e}^u_h)\\
&\quad+\sum_{K\in\mathcal{T}_h}\big[\gamma_K(\mathbf{u}_h)\,(\nabla\cdot\epsilon^u,\nabla\cdot\mathbf{e}^u_h)_K+\nu^K_T(\mathbf{u}_h)\,(\kappa\mathbb{D}\epsilon^u,\kappa\mathbb{D}\mathbf{e}^u_h)_K\\
&\qquad-\nu^K_T(\mathbf{u}_h)\,(\kappa\mathbb{D}\mathbf{u},\kappa\mathbb{D}\mathbf{e}^u_h)_K\big],\quad\forall\lambda_h\in Q_h.
\end{aligned}$$

Careful estimates of the right hand side terms lead to

$$\partial_t\|\mathbf{e}^u_h(t)\|^2_0+d(t)\le g(t)\,\|\mathbf{e}^u_h(t)\|^2_0+A(t)$$

with

$$\begin{aligned}
g(t) &:= \frac{27C^4_{LT}}{2\nu^{\min}_{\rm mod}(\mathbf{u}_h,\mathbf{e}^u_h)^3}\,\|\mathbb{D}\mathbf{u}(t)\|^4_0,\\
d(t) &:= \frac{1}{4}\sum_{K\in\mathcal{T}_h}\nu^K_{\rm mod}(\mathbf{u}_h,\mathbf{e}^u_h)\,\|\mathbb{D}\mathbf{e}^u_h(t)\|^2_{0,K}+\sum_{K\in\mathcal{T}_h}\gamma_K(\mathbf{u}_h)\,\|\nabla\cdot\mathbf{e}_h(t)\|^2_{0,K},
\end{aligned}$$

and $A(t)$ as given in the Theorem. Gronwall's Lemma implies for all $t \in [0,T]$

$$\|\mathbf{e}_h^u(t)\|_0^2 + \int_0^t d(s)ds \leq e^{\int_0^t g(s)ds}\big(\|\mathbf{e}_h^u(0)\|_0^2 \|_0^2 + \int_0^t A(s)ds\big).$$

Finally, the triangle inequality concludes the proof. For full details of the proof, we refer to [13]. □

In the remaining part of paper, we will discuss how the given approach can be adapted to the case of turbulent channel flows.

4 Application to turbulent channel flow

For the spatial discretization, we apply hexahedral meshes with FE spaces $\mathbb{Q}_2/\mathbb{Q}_1$ for velocity/pressure within the FE package `deal.II`, see [2]. The arising semidiscrete problem of the form

$$\begin{pmatrix} M_u & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h'(t) \\ \mathbf{p}_h'(t) \end{pmatrix} = \begin{pmatrix} \mathbf{f}_h(t) \\ 0 \end{pmatrix} - \begin{pmatrix} A_u(\mathbf{u}_h) & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h(t) \\ \mathbf{p}_h(t) \end{pmatrix}$$

is a DAE-system with differentiation index 2 and perturbation index 2. For the time discretization, we apply the BDF(2)-formula

$$\mathbf{u}_h'(t_{n+1}) \approx \frac{1}{2\delta t}[3\mathbf{u}_h(t_{n+1}) - 4\mathbf{u}_h(t_n) + \mathbf{u}_h(t_{n-1})]$$

which provides favourable stability properties and does not lead to order reduction for the pressure. A fixed-point iteration is applied for the resulting non-linear implicit scheme.

The calibration of the viscosity model is motivated by the Boussinesq approximation of the residual stress tensor via

$$\tau^R := \langle \mathbf{u} \otimes \mathbf{u} \rangle - \mathbf{u}_h \otimes \mathbf{u}_h \approx -2\nu_t \mathbb{D}\mathbf{u}_h$$

together with the classical Smagorinsky model and van-Driest damping

$$\nu_t(\mathbb{D}\mathbf{u}_h)|_K = \left[C_S \varDelta_K (1 - \exp{(-\frac{u_\tau \mathrm{dist}(S_K, \partial\varOmega)}{26\nu})} \right]^2 \|\mathbb{D}\mathbf{u}_h(S_K)\|_F.$$

Here, S_K denotes the center of gravity of element K and u_τ denotes the wall friction velocity (see below). A reduction of model dissipation is established by application of the fluctuation operator

$$\tau^R \approx -2\nu_t(\mathbb{D}\mathbf{u}_h)\kappa(\mathbb{D}\mathbf{u}_h).$$

For the fluctuation operator $\kappa := Id - \varPi_h$ with the L^2-orthogonal projection $\varPi_h : L \to L_h$, we apply a one-level approach with $H = h$ and $L_h = {\mathbb{Q}_0}^{d\times d}$. On each element $K \in \mathcal{T}_h$, the filter width is given by $\varDelta_K = \mathrm{meas}(K)/(2(k-1))$ with element order k of V_h. The Smagorinsky constant $C_S^2 = 0.0942$ is taken from Lilly's argument for isotropic homogeneous turbulence, see [13].

4.1 Turbulent channel flow at moderate Reynolds number Re_τ

We start with channel flow at a moderate Reynolds number $Re_\tau = 180$ (corresponding to $Re = 5644$ in channel center) for which an anisotropic grid resolution of the boundary layer regions is feasible. The Reynolds number $Re_\tau = Hu_\tau/\nu$ is defined via the half width H of the channel and wall-friction velocity u_τ satisfying Spalding's form of the law of the wall

$$y^+ = f(u^+) := u^+ + e^{-5.5\chi}\left(e^{\chi u^+} - 1 - \chi u^+ - \frac{1}{2}(\chi u^+)^2 - \frac{1}{6}(\chi u^+)^3\right)$$

with $y^+ := \frac{yu_\tau}{\nu}$, $u^+ := \frac{\|\mathbf{u}_h\|}{u_\tau}$, and $\chi = 0.4$.

A careful description of the set-up of the problem (but with different scaling) is given in [9]. We performed simulations with N^3 grid points, with equidistant distribution of elements in x_1, x_3-directions and anisotropic distribution in x_2-direction according to

$$x_2 = y = \tanh(2(2i/(N) - 1))/\tanh(2), \qquad \text{for } i = 0, ..., N.$$

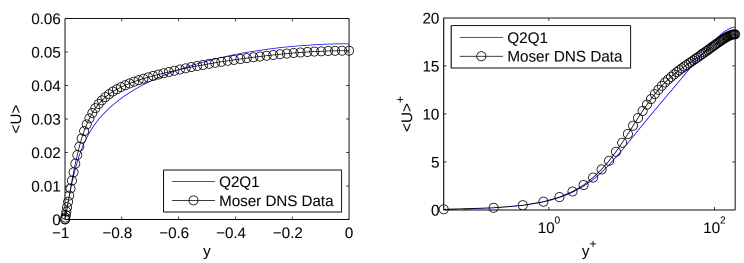

Fig. 1. Channel flow at $Re_\tau = 180$ with 32^3 grid points: Mean streamwise velocity $U = \langle \mathbf{u_h} \rangle \mathbf{e}_1$ (left) and its normalized variant $U^+ = U/u_\tau$ (right)

Statistical averaging $\langle \cdot \rangle$ is performed over all homogeneous directions x_1, x_3, t. As an example of first-order statistics, we present in Fig. 1 the mean streamwise velocity $U = \langle \mathbf{u_h} \rangle \mathbf{e}_1$ and its normalized variant U^+. Compared to direct numerical simulation (DNS) results of [11], we obtain very good agreement in the viscous sub-layer whereas slight deviations can be found in the log-layer and in the center of the channel. As examples of second-order statistics, the normalized fluctuations $\langle u_1', u_2' \rangle^+$ and $\langle u_1', u_1' \rangle^+$ are shown in Fig. 2. The agreement with the DNS data is very good in the vicinity of the wall and (for such relatively coarse grid) resonable in the core of the channel.

Numerical experiments in [1] with lowest-order Taylor-Hood elements on anisotropic grids indicate a potential strong influence of a large aspect ratio $a := \max_K h_{1,K}/h_{2,K}$. In our experiments, we obtained for $N = 32$ grid points in x_2-direction a value of $a \approx 20$. A modification of the V_h^{div}-interpolation

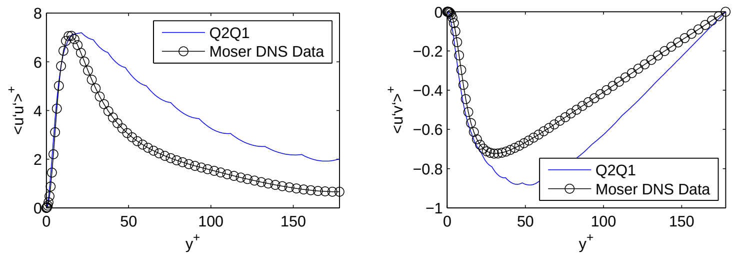

Fig. 2. Channel flow at $Re_\tau = 180$ with 32^3 grid points: Normalized fluctuations $\langle u_1, u_1' \rangle$ (lect) and $\langle u_1', u_2' \rangle$ (right)

results of [6] to Cartesian tensor-product meshes shows the dependence of the aspect ratio on interpolation results. This will be reported elsewhere. This clearly limits the applicability of layer-adapted meshes with Taylor-Hood elements to moderate values of Re_τ.

4.2 Turbulent channel flow at higher Reynolds numbers

A proper anisotropic resolution of near-wall region in LES for higher Re_τ is not feasible. Near-wall modelling with adaption of wall-functions on isotropic grid may be considered as a remedy. Here we follow the approach in [3, 4] with weak implementation of wall boundary conditions on isotropic grids.

The simplest variant [3] is a weak nonsymmetric or symmetric penalty-type implementation of Dirichlet condition $\mathbf{u}_h = \mathbf{0}$ at the wall Γ_W and is performed by adding the following terms

$$-\sum_K (2\nu \mathbb{D}\mathbf{u}_h \cdot \mathbf{n}, \mathbf{v}_h)_{\partial K \cap \Gamma_W} \pm (\mathbf{u}_h, 2\nu \mathbb{D}\mathbf{v}_h \cdot \mathbf{n})_{\partial K \cap \Gamma_W} + (\mathbf{u}_h, \tau_B \mathbf{v}_h)_{\partial K \cap \Gamma_W}) \,.$$

Here the Dirichlet penalty factor is taken as $\tau_B := \frac{u_\tau^2}{\|\mathbf{u}_h\|}$ with tangential velocity $\mathbf{u}_h$ and wall-friction velocity u_τ. A similar approach is realized on inflow and outflow parts of the channel.

In particular, for the non-symmetric case, symmetric testing $\mathbf{v}_h = \mathbf{u}_h$ gives immediately control of $\sum_K \tau_B \|\mathbf{u}_h\|^2_{0,\partial K \cap \Gamma_W}$. A modification of the numerical analysis in Section 3 is possible, but will not considered.

A refined variant of a weak implementation of Dirichlet conditions, including advective and pressure parts of the traction operator, is given in [4]. Convincing numerical results for channel flows on (rather fine) isotropic grids are reported in [3, 4] for Reynolds numbers $Re_\tau \in \{395, 950, 2003\}$.

5 Summary. Outlook

We applied a variational multiscale model to the time-dependent Navier-Stokes model. For wall-bounded flows in a channel, we discussed two variants:

anisotropic mesh resolution in boundary layers for moderate Re_τ and isotropic meshes with near-wall modelling via weak Dirichlet conditions for higher Re_τ. The a priori analysis of the arising nonlinear semidiscrete problem given in [13] can be adapted to both situations.

We believe that the current approach in airbus industry, e.g. at DLR (German Aerospace Center), with delayed detached eddy simulation (DDES) with LES away from layers and RANS in layer regions can be cast into the framework of the proposed projection-based VMS method. In particular, an application of the approach to problems with separation is in order.

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